Math 118 – Quiz 7
April 18, 2011

How is it that $0.999\ldots = 1$?
Let us count some ways.

Tool 1: $x^n - 1 = (x-1)(1+x+x^2+\cdots+x^{n-1})$. This is essentially the formula for the sum of a geometric series as given in Chapter 9, Section 2.

App 1: $10^n - 1 = 9 \times (1 + 10 + 10^2 + \cdots + 10^{n-1}) = 999\ldots9$ (n nines). To see this, put $x = 10$ in the formula.

Tool 2: Take Tool 1 and divide both sides by $x-1$. Then multiply the left-hand-side by $(-1)/(-1)$ to change signs in the numerator and denominator. Add $x^n/(1-x)$ to both sides and then multiply by x. Fill in these steps to get the result below:

$$\frac{x}{1-x} = x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x},\ x \neq 1.$$

App 2: Take x to be $1/10$ in this formula to show that

$$\frac{1}{9} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} + \frac{1}{9 \times 10^n}.$$

Way 1

(i) Express the number 0.9 as a fraction; What is 0.99 as a fraction? What about 0.999?

(ii) What is the fractional form of the number $0.999\ldots9$ (n nines)?

(iii) Use (ii) above and App 1 to get another form (using no $9's$) of the fraction for $0.999\ldots9$ (n nines).

(iv) Define a sequence by $s_1 = 0.9$, $s_2 = 0.99$, $s_3 = 0.999$, etc. Use the alternate fractional forms of these numbers, as in (iii) above, to determine the limit of this sequence.

Way 2

Now define another sequence $\{t_n\}$ by

$$t_n = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n}.$$

(i) Write out the first three terms t_1, t_2 and t_3. Is this sequence monotone?

(ii) Give an argument to show that $\lim_{n\to\infty} t_n$ exists. What is this limit? (Hint: App 2 and (i) above)

(iii) Beyond what point in the sequence $\{t_n\}$ can we be sure that every term differs from the limit by less than 1/899? Why? (Hint: App 2 again.)

Way 3

Fact: If $\{s_n\}$ is a sequence with limit L and k is any number, then the sequence $\{ks_n\}$obtained by multiplying each term in $\{s_n\}$ by k has limit kL. That is, $\lim_{n\to\infty} k \times s_n = k \times \lim_{n\to\infty} s_n$.

Applying this fact to the constant $k = 9$ and the sequence defined as above by $t_n = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n}$, whose limit is 1/9, conclude that

$$
\begin{aligned}
1 = 9 \times \frac{1}{9} &= 9 \times \lim_{n\to\infty} \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \cdots + \frac{1}{10^n} \right) \\
&= \lim_{n\to\infty} \left(\frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots + \frac{9}{10^n} \right) \\
&= \lim_{n\to\infty} \left(\underbrace{0.999\ldots9}_{n \text{ nines}} \right).
\end{aligned}
$$

so we see again that the decimal all of whose digits are nines represents the number 1.

Can you explain similarly what the equation

$$\frac{1}{s} = 0.333\ldots$$

means? That is, what sequence is 1/3 the limit of? How do we know that the sequence you propose has the indicated limit? Can you derive this from what we have seen about 1/9, or the Fact above?