Traversals of Infinite Graphs with Random Local Orientations

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January 16, 2014

Graph Exploration by a Mobile Entity

Big Problem: graph exploration by a mobile entity using only constant memory, e.g. software moving on a network, web-crawler on the internet, a robot exploring an unfamiliar terrain.

Let's focus on a simple case: $G = [n] \times [n]$ or $\mathbb{Z} \times \mathbb{Z}$.

From any vertex, how does agent choose where to go next?
One method: simple random walk.
Cover time is $O(|V|^2)$. Duplicating work.

Basic Walk

The *basic walk* is an alternative to the simple random walk proposed by Leszek Gąsieniec.

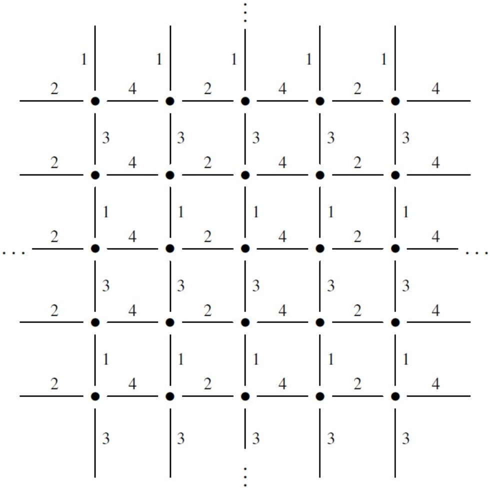
Make all edges in G bidirectional and label both directions. At every v, the outgoing labels are $1, 2, \ldots, deg(v)$

Give the agent 2 bits of memory. If it enters a vertex v along edge labeled i then it exits by label $i + 1 \mod deg(v)$.

There is a labeling with cover time $O(|V|)$.

The *random basic walk* labels all edges uniformly at random at the start, then chooses a random starting point.

Example: Basic Walk



Pólya's Theorem for Simple Random Walk

The simple random walk on G is *recurrent* if it returns to its starting vertex with probability 1. Otherwise it is *transient*. Transient walks escape to infinite distance from the starting vertex.

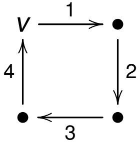
Theorem (Pólya's Theorem)

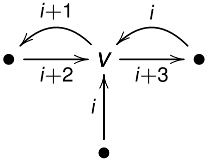
The simple random walk on $\mathbb{Z}^d$ is recurrent for $d \leq 2$, but transient for all $d > 2$.

Conjecture: The random basic walk will behave similarly.

Transience vs. Cycling for Random Basic Walk

The random basic walk need not return to its starting vertex at all. It could instead get trapped in a cycle:





If this occurs with probability 1 we say the random basic walk *cycles* (note: it only explores a finite subgraph). Otherwise the random basic walk is *transient* (it escapes to infinity).

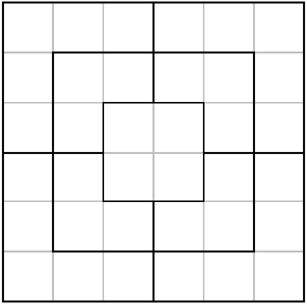
Theorem (W.)

In $\mathbb{Z}^2$, the random basic walk cycles with probability 1.

Proof of cycling: Shells Method

Theorem (W.)

In $\mathbb{Z}^2$, the random basic walk cycles with probability 1.



When any shell is first reached, Pr(trap)=constant $c > 0$. So Pr(escape)= $(1-c) * (1-c) * \cdots = 0$.

Corollary (Random Basic Walk behaves very differently from SRW)

The random basic walk cycles on any $\mathbb{Z}^d$, due to hypercube shells.

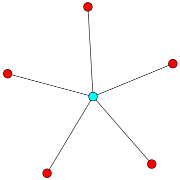
Generalizations

The shells method works for expander graphs, but more is true:

Theorem (W.)

On any graph G with all vertex degrees bounded by a constant D, the random basic walk cycles with probability 1.

Proof: Label as you go.
Look for star-shaped graphs.
Take care with independence.
P(trap)= $c > 0$, so
P(escape)= $(1 - c)^{\infty} = 0$.



Theorem (W.)

Let T be the tree where every vertex in level n has 2^n children.

Results on Finite Graphs

Our proof methods provide upper bounds on the expected number of vertices a random basic walk will visit on finite grids.

The class of complete graphs has random basic walks asymptotically visit a constant fraction of the nodes:

Theorem (joint with Danny Krizanc)

As $n \to \infty$, a random basic walk on K_n is expected to visit at least $(1 - 1/e) * n$ nodes.

Conjecture

The expected number of arcs traversed by a random basic walk on K_n is $1.8 * n$ as $n \to \infty$.

Details are in my thesis, at arXiv:1308.1041