Bousfield Localization and Commutative Monoids

David White

Wesleyan University

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Goal for Today

Big goal: understand how homotopy theory interacts with algebraic structure.We'll use model categories and operads as our language. Notation: $\mathcal{M}=$ monoidal model category, $P=$ operad.

Subgoal: when does Bousfield localization preserve the structure of algebras over an operad?

Motivation: Hill-Hopkins-Ravenel's resolution of the Kervaire Invariant One Problem relied on an equivariant spectrum $\Omega = D^{-1}MU^{(4)}$. When is the localization of a commutative equivariant ring spectrum commutative?

Example: Localization Can Break Commutativity

$G =$ finite group, $\mathscr{S}^G =$ category of equivariant spectra.

Example presented by Mike Hill at Oberwolfach 2011:

The reduced real regular representation $\overline{\rho} = \mathbb{R}[G]/\mathbb{R}[e]$ has representation sphere $S^{\overline{\rho}}$. Let $a_{\overline{\rho}} : S^0 \to S^{\overline{\rho}}$ be inclusion.

Let $E = S^0[a_{\overline{\rho}}^{-1}]$. This is not a commutative ring spectrum! If it were, $H < G$ would give $N_H^G res_H E \to E$ by adjointness. But $res_H(E) \simeq *$ so this would imply $* \simeq E$, contradiction.

Hill-Hopkins (2013) prohibit this behavior via hypotheses on the maps being inverted. Foreshadow: What's failing is that $a_{\overline{\rho}} \otimes (G/H)_+$ is not an $a_{\overline{\rho}}$-equivalence.

Monoidal Model Categories

1. Pushout Product Axiom: Given $f : A \to B$ and $g : X \to Y$ cofibrations, $f \square g$ is a cofibration. If $f \in \mathcal{W}$ then $f \square g \in \mathcal{W}$.

$$\begin{tikzcd}
A \otimes X \arrow[r] \arrow[d] & B \otimes X \arrow[d] \arrow[ddr, bend left] & \\
A \otimes Y \arrow[r] \arrow[drr, bend right] \arrow[ur, phantom, "\searrow"] & P \arrow[dr, "f \square g"] & \\
& & B \otimes Y
\end{tikzcd}$$

2. Unit Axiom: For cofibrant X, $QS \otimes X \to S \otimes X \cong X$ is in $\mathcal{W}$.
3. Resolution Axiom: for all cofibrant X, $X \otimes \mathcal{W} \subset \mathcal{W}$.

Model Categories and Bousfield Localization

$\mathcal{M}$ = model category, $\mathcal{W}$ = weak equivalences, $f \notin \mathcal{W}$.

When $\mathcal{M}$ is left proper and either combinatorial or cellular, there is a model category $L_f\mathcal{M}$ called the *Bousfield localization* of $\mathcal{M}$ with respect to f, with $\mathcal{W}_f = \langle f \cup \mathcal{W} \rangle \supset \mathcal{W}$, $\mathcal{Q}_f = \mathcal{Q}$, $\mathcal{F}_f \subset \mathcal{F}$

Definition

We say L_f preserves P−algebras if for all cofibrant $E \in P$−alg, $L_f(E) \in P$−alg and $E \to L_f(E)$ is a P−alg homomorphism.

More generally: given $E \in P$−alg, we need $\widetilde{E} \in P$−alg with $L_f(E) \simeq \widetilde{E}$.

Preservation of P-algebras

When P−alg inherits a model structure via $P : \mathcal{M} \rightleftarrows P\text{−alg} : U$, then fibrations and weak equivalences are created by forgetful U.

Theorem (W.)

Let $\mathcal{M}$ be a monoidal model category and let P be an operad valued in $\mathcal{M}$. If P-algebras in $\mathcal{M}$ and in $L_f(\mathcal{M})$ inherit (semi) model structures, then L_f preserves P-algebras.

Proof: $L_f(E) \simeq R_f QE$. We prove $R_f QE \simeq R_{f,P} Q_P E$ in $\mathcal{M}$.

$$\begin{tikzcd}
Q_P E \arrow[r, "\simeq_f"] \arrow[d, hook, "\simeq_f"'] & R_{f,P} Q_P E \arrow[d, two heads] \\
R_f Q_P E \arrow[r] \arrow[ur, dotted, "\therefore \simeq_f"'] & *
\end{tikzcd}
\qquad
\begin{tikzcd}
QE \arrow[r, "\simeq"] \arrow[d] & Q_P E \arrow[d] \arrow[dr] & \\
R_f QE \arrow[r, "\simeq_f"'] & R_f Q_P E \arrow[r, "\simeq_f"'] & R_{f,P} Q_P E
\end{tikzcd}$$

When do $P-$algebras inherit a model structure?

Theorem (Spitzweck, 2000)

Suppose P is a Σ-cofibrant operad and $\mathcal{M}$ is a monoidal model category. Then P-alg is a semi-model category which is a model category if P is cofibrant and $\mathcal{M}$ satisfies the monoid axiom.

Monoid Axiom (Schwede-Shipley): Transfinite compositions of pushouts of maps in $\{\text{Trivial-Cofibrations } \otimes id_X\}$ are in $\mathcal{W}$.

Genuine commutativity in $\mathscr{S}^G$ is encoded by the cofibrant operad E_∞^G with $E_\infty^G[n] = E_G\Sigma_n$ characterized by $(E_G\Sigma_n)^H = \varnothing$ if $H \cap \Sigma_n \neq \{e\}$ and $(E_G\Sigma_n)^H \simeq *$ otherwise. Lesser commutativity is encoded by $E_\infty^{\mathscr{F}}$

When is $L_f(\mathcal{M})$ a monoidal model category?

Characterization of Monoidal Bousfield Localizations (W.)

$L_f(\mathcal{M})$ satisfies the Pushout Product Axiom and the Resolution Axiom iff $f \otimes K$ is an f-local equivalence for all cofibrant K.

For tractable $\mathcal{M}$ (domains of generating I and J are cofibrant), one need only check $K \in \{(\text{co})\text{domains of } I \cup J\}$

Corollary

In $\mathscr{S}^G$ a Bousfield localization preserves genuine commutativity iff $f \otimes (G/H)_+$ is an f-local equivalence for all subgroups H.

Intermediate equivariant commutative structures

Javier Gutiérrez and I found semi-$\mathscr{F}$-model structures on $G-$operads. The cofibrant replacements of Com are Blumberg-Hill N_∞ operads which we denote by $E_\infty^{\mathscr{F}}$. Hill's example is maximally bad, taking E_∞^G-structure down to naive E_∞. The example generalizes to give any drop required.

$$\begin{tikzcd}
E_\infty^G \arrow[r, "L"] \arrow[d, dash] \arrow[dr] \arrow[dddr] \arrow[ddddr] & E_\infty^G \arrow[d, dash] \\
E_\infty^{\mathscr{F}} \arrow[r] \arrow[d, dash] \arrow[dddr] & E_\infty^{\mathscr{F}} \arrow[d, dash] \\
\cdots \arrow[d, dash] & \cdots \arrow[d, dash] \\
E_\infty^{\mathscr{F}'} \arrow[r] \arrow[d, dash] \arrow[dr] & E_\infty^{\mathscr{F}'} \arrow[d, dash] \\
E_\infty \arrow[r] & E_\infty
\end{tikzcd}$$

Corollary

For X above $E_\infty^{\mathscr{F}}$-structure, $L(X)$ has $E_\infty^{\mathscr{F}}$-structure iff $f \otimes (G/H)_+$ is an f-local equivalence for all subgroups $H \in \mathscr{F}$.

Model Structure on Strict Commutative Monoids

Commutative monoid axiom: If g is a (trivial) cofibration then $g^{\square n}/\Sigma_n$ is (trivial) cofibration. Suff. to check on generators. Stronger (Lurie, HTT): $g^{\square n}$ is a Σ_n-projective cofibration.

Theorem (W.)

If a monoidal model category satisfies the monoid axiom and the commutative monoid axiom then commutative monoids inherit a model structure. Without monoid axiom it's a semi-model structure.

Corollary: Preservation of Strict Commutative Monoids (W.)

If $\mathrm{Sym}^n(f)$ is a weak equivalence in $L_f(\mathcal{M})$ for all n then $L_f(\mathcal{M})$ satisfies the commutative monoid axiom.
Here $\mathrm{Sym}(X) = S \coprod X \coprod X^{\otimes 2}/\Sigma_2 \coprod \cdots$

Examples: g (triv) cofib $\Rightarrow g^{\square n}/\Sigma_n$ (triv) cofib

Ch(k) where $char(k) = 0$. Lurie's hypothesis holds.

sSet & Top, though they fail Lurie's hypothesis.

Positive (Flat) model structure on symmetric spectra.

Positive orthogonal (equivariant) spectra

Positive motivic symmetric spectra - joint with M. Spitzweck.

Corollary

Any monoidal localization in sSet preserves commutative monoids, e.g. L_E for a homology theory E. Truncations in sSet, Top, and Ch(k) all preserve strict commutative monoids.

Other Non-Cofibrant Operads

Harper (2010): If all symmetric sequences in $\mathcal{M}$ are projectively cofibrant then for *any* P, P–alg inherits a model structure.

Theorem (W.)

Each row in the following table yields a semi-model structure on P-algebras, under a strengthened monoid axiom.

Hypothesis on $\mathcal{M}$	Class of operad
$\forall X \in \mathcal{M}^{\Sigma_n}$ projectively cofibrant, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration (this follows from the pushout product axiom)	Cofibrant or Σ-Cofibrant
$\forall X \in \mathcal{M}^{\Sigma_n}$ cofibrant in $\mathcal{M}$, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration Note: $X = *$ is the Σ_n-equivariant monoid axiom	Levelwise cofibrant Special case: $P = \mathrm{Com}$
$\forall X \in \mathcal{M}^{\Sigma_n}$, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration	Arbitrary