Encoding Equivariant Commutativity via Operads

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Goal for Today

Kervaire Invariant One Problem recently solved using Equivariant Stable Homotopy Theory. Equivariant Commutative Ring Spectra were crucial in the proof (due to multiplicative norms). Commutativity is hard! Equivariantly, strictly commutative is NOT the same as E_∞-ring spectra (i.e. rectification fails).

Blumberg-Hill introduced N_∞-ring spectra to study multiplicative norms. They defined N_∞-operads, gave 4 examples, and proved rectification, but do not construct N_∞-operads in general.

Goal: Construct N_∞-operads, answering the Blumberg-Hill conjecture.

Methods: Model structure on category of G-operads, and cofibrant replacement therein.

Spaces and Spectra

Let G be a group (finite or compact Lie). Work in the category of pointed G-equivariant topological spaces and G-equivariant orthogonal ring spectra. Spectra is the setting for stable homotopy theory; topological objects representing stable π_*^s gps.

Definition

*A G-**spectrum** X is a sequence (X_i) of G-spaces (path conn. CW cpxs) with maps from $\Sigma X_i \to X_{i+1}$ where Σ is reduced suspension.*

Example: $S = (S^n)$ the sphere spectrum. For a spectrum X, $X_* = \pi_k(X) = [\Sigma^k S, X]$. Then S_* is the stable homotopy ring; contains information about smooth manifolds, when they are diffeomorphic, and h-cobordism. Kervaire paper did computation in S_*.

Commutativity

An **orthogonal G-spectrum** X has an action of $O(n)$ on X_n.
Closed symmetric monoidal category under

$$(X \wedge Y)_n = \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} (X_p \wedge Y_q)$$

A **commutative G-ring spectrum** E has $\mu : E \wedge E \to E$,
$\eta : S \to E$, $\tau : E \wedge E \to E \wedge E$, commutative diagrams

$$\begin{tikzcd}
E \wedge E \wedge E \arrow[r] \arrow[d] & E \wedge E \arrow[d] & S \wedge E \arrow[d] \arrow[dl] & & & E \\
E \wedge E \arrow[r] & E & E \wedge E \arrow[l] & E \wedge E \arrow[rr] \arrow[urr] & & E \wedge E \arrow[u]
\end{tikzcd}$$

Morally: $a(bc) = (ab)c$, $1 \cdot a = a$, $ab = ba$

G-operads

A G**-operad** P is a sequence of $G \times \Sigma_n$-spaces $(P(n))_{n\in\mathbb{N}}$, with G-fixed unit $1 \in P(1)$ and composition product

$$\begin{array}{rcl} \circ : P(n) \wedge P(k_1) \wedge \cdots \wedge P(k_n) & \to & P(k_1 + \cdots + k_n) \\ (f, f_1, \ldots, f_n) & \mapsto & f \circ (f_1, \ldots, f_n), \end{array}$$

Satisfying identity, associativity, and composition laws. A P**-algebra** X has a compatible action $P \circ X \to X$.
$\mathrm{Com}(n) \cong S^0$ has Com-alg = commutative G-ring spectra

An N_∞**-operad** has $P(0) \simeq *$, Σ_n acts freely on $P(n)$, and $P(n)$ is the universal space for a family $\mathscr{F}_n(P)$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times 1$.

Note: $\mathrm{Ho}(P\text{-alg}) \cong \mathrm{Ho}(\text{Com-alg})$, and rectification holds.

Constructing N_∞-operads

Theorem (Gutiérrez-W.)

Let $\mathscr{F} = (\mathscr{F}_n)$ be a collection of families of subgroups of $G \times \Sigma_n$, such that whenever $n_1 + n_2 + \cdots + n_k = n$ then $\mathscr{F}_k \wr \mathscr{F}_{n_1} \times \cdots \times \mathscr{F}_{n_k} \subset \mathscr{F}_n$. Then there exists an N_∞-operad P where $P(n)$ is universal for $\mathscr{F}_n$.

Proof: Put a model structure on G-Oper, with weak equivalences (resp. fibrations) f such that the H-fixed point map f_n^H is a weak equivalence (resp. fibration) in Top, for all $H \in \mathscr{F}_n$ and all n.

The model structure is transferred from $\prod_{n\in\mathbb{N}} Top^{G\times\Sigma_n}$. Define P to be the cofibrant replacement of Com. The hypothesis is needed so P is cofibrant in $\prod_{n\in\mathbb{N}} Top^{G\times\Sigma_n}$, hence that $P(n)$ is universal for $\mathscr{F}_n$ and has Σ_n acting freely.

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Model Categories

A **model category** is a setting for abstract homotopy theory. Examples: Top, sSet, Ch(R), R-mod, Spectra, G-spectra, motivic spectra, operads, categories, graphs, flows, ...

Formally, a bicomplete category $\mathcal{M}$ and classes of maps $\mathcal{W}, \mathcal{F}, \mathcal{Q}$ (= weak equivalences, fibrations, cofibrations) satisfying axioms to behave like Top. Lifting, factorization, 2 out of 3, retracts.

An object X is **cofibrant** if $\emptyset \to X$ is a cofibration (where $\emptyset$ is initial). The **cofibrant replacement** QY of Y is the result of factorization. Ex: CW approximation, Projective Resolution.

$$\begin{tikzcd} & QY \arrow[dr, two heads, "\simeq"] & \\ \emptyset \arrow[ur, hook] \arrow[rr] & & Y \end{tikzcd}$$

Applications

Theorem

Let P be a cofibrant N_∞-operad. Then P-alg inherits a model structure from G-spectra, and it's Quillen equivalent to the model category of commutative G-ring spectra.

Theorem

Let C be a set of G-spectra maps. If L_C is a monoidal left Bousfield localization any $\mathrm{Sym}(-)$ takes C to C-local equivalences, then L_C preserves N_∞-algebras.

Future: Preservation of N_∞-algebras under right Bousfield localization.

Summary

- Equivariant commutativity and mult. norms matter.
- N_∞-operads help understand multiplicative norms.
- We can now construct these operads, and understand the homotopy theory of algebras over N_∞-operads and of G-operads.

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