ULTRAFILTERS

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1. Definitions

Given a set X, an ultrafilter on X is a collection $\mathcal{U}$ of subsets of X such that

(1) The empty set is not an element of $\mathcal{U}$

(2) If $A, B \subset X$, $A \subset B$, and $A \in \mathcal{U}$, then $B \in \mathcal{U}$.

(3) If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.

(4) If $A \subset X$, then either $A \in \mathcal{U}$ or $(X \setminus A) \in \mathcal{U}$.

Note that axioms 1 and 3 imply that A and $X \setminus A$ cannot both be elements of $\mathcal{U}$. Also, the first three axioms tell us that $\mathcal{U}$ is a filter on X. A filter U on a set X is an ultrafilter if one of the following conditions is true.

(1) There is no filter F finer than U, i.e. $U \subset F \Rightarrow U = F$.

(2) $(A \cup B) \in U \Rightarrow A \in U$ or $B \in U$.

(3) For all $A \subset X$, either $A \in U$ or $X \setminus A \in U$.

A principal ultrafilter has a least element. A free ultrafilter is a non-principal ultrafilter. Almost all ultrafilters on an infinite set are free.

2. In Logic

In light of (1) above, ultrafilters on a Boolean algebra can be identified with prime ideals, maximal ideals, and homomorphisms to the 2-element Boolean algebra {true, false}. This is because maximal ideals of a Boolean algebra are the same as prime ideals and because given a homomorphism of a Boolean algebra onto {true, false}, the inverse image of "true" is an ultrafilter, and the inverse image of "false" is a maximal ideal. Also, if you have a maximal ideal, then its complement is an ultrafilter and there is a unique surjective homomorphism taking the maximal ideal to "false." If you have an ultrafilter, then its complement is a maximal ideal and there is a unique surjective homomorphism taking it to true. A filter on a Boolean algebra is an ultrafilter iff for all $a, b \in B$, if $a \wedge b \in F$ then $a \in F$ or $b \in F$.

Ultrafilters are used to define ultraproducts. An ultraproduct is a quotient of the direct product of a family of structures by an ultraproduct, i.e. $\prod_{i \in I} M_i / U$. The equivalence relation is a b iff $\{i \in I \mid a_i = b_i\} \in U$. An ultrapower is an ultraproduct in which all of the factors M_i are equal. The hyperreal numbers are an ultraproduct of the real numbers where you get a copy of $\mathbb{R}$ for each $n \in \mathbb{N}$ with regard to a cofinite ultrafilter over $\mathbb{N}$.

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These appear in model theory, for example to prove that the condition of 3-colorability in a graph is not finitely axiomatizable. To do this proof, construct graphs that are not 3-colorable but whose ultraproduct is. They also give very elegant proofs of the compactness theorem and the completeness theorem. Keisler's ultrapower theorem gives an algebraic characterization of the semantic notion of elementary equivalence. The Robinson-Zakon presentation of the use of superstructures and their monomorphisms to construct nonstandard models of analysis is also simplified with ultraproducts. Los's Theorem (pronounced "Wash") states that any first-order formula is true in the ultraproduct iff $\{i \mid \text{the formula is true in } M_i\} \in U$.

3. In Set Theory

Ultrafilters and finitely additive (0,1)-measures (i.e. a content mapping into $[0, 1]$) are in 1-1 correspondence: any ultrafilter corresponds to $\mu^{-1}(1)$ for a unique μ. Define $\mu(A) = 1$ if $A \in U$ and 0 otherwise.

Every infinite set has a non-trivial ultrafilter. To show this, we first show that any filter on an infinite set extends to an ultrafilter (indeed, every collection with the finite intersection property extends to an ultrafilter). This is accomplished using Zorn's Lemma to get a maximal proper filter containing F. Applying maximality, we can easily prove this is an ultrafilter. Next, every infinite set has a (0,1)-measure. The proof notes that the set of cofinite subsets is a filter, which we can extend to an ultrafilter U. This gives a slick proof of the DeBruijn-Erdos theorem that a graph is k-colorable iff every finite subgraph is k-colorable (for $k \in \mathbb{N}$).

U is an ultrafilter on S iff $P(S) \setminus U$ is an ideal (i.e. $A \subset B, B \in I \Rightarrow A \in I$ and I closed under finite unions). It is possible to put a pre-order on the class of ultrafilters of a set X. This is called the Rudin-Keisler ordering.

4. In topology

A filter is a generalization of a net, which is a generalization of a sequence. They are used to define convergence in messy spaces. Sequences require the space to be first countable. Nets allow more index sets, but still require them to be directed sets. Filters can be thought of as sets build from several nets. To say that filter base B converges to x, means that for every neighbourhood U of x, there is a $B_0 \in B$ such that $B_0 \subset U$. To say that x is a cluster point for a filter base B on X means that for each $B_0 \in B$ and for each neighborhood U of x in X, $B_0 \cap U \neq \varnothing$.

* X is a Hausdorff space if and only if every filter base on X has at most one limit.

* X is compact if and only if every filter base on X clusters.

* X is compact if and only if every filter base on X is a subset of a convergent filter base.

* X is compact if and only if every ultrafilter on X converges.

Thus, every ultrafilter on a compact Hausdorff space converges to exactly one point. Note also that continuous and onto maps preserve ultrafilters.

The set G of all ultrafilters of a poset P can be topologized: for all $a \in P$, let $D_a = \{U \in G \mid a \in U\}$. When P is a Boolean algebra the set of all D_a is a base for a compact Hausdorff topology on G. When P is the power set of a set S, the resulting topological space is the StoneCech compactification of a discrete space of cardinality $|S|$. Thus, points of the Stone-Cech compactification can be thought of as ultrafilters.

5. Extra Section

A non-empty subset F of a partially ordered set $(P, \leq)$ is a filter if the following conditions hold:

(1) For every $x, y \in F$, there is some element $z \in F$, such that $z \leq x$ and $z \leq y$. (F is a filter base)

(2) For every $x \in F$ and $y \in P$, $x \leq y$ implies that $y \in F$. (F is an upper set)

(3) A filter is proper if it is not equal to the whole set P. This is often taken as part of the definition of a filter.

Because of the second axiom, the third is equivalent to $\varnothing \notin F$. Note that these axioms also imply that filters have the finite intersection property.

In the special case of a filter on a set S, non-emptiness is equivalent to $S \in F$ and property 1 is equivalent to closure under intersections. Thus, the only difference between a filter and an ultrafilter on a set is the fourth axiom of ultrafilters regarding A or $A^c \in \mathcal{U}$.