MONOIDAL BOUSFIELD LOCALIZATION FOR MODEL CATEGORIES OF EQUIVARIANT SPECTRA

DAVID WHITE

1. Motivation from Equivariant Homotopy Theory

A big goal of algebraic topology is to understand how homotopy theory interacts with algebraic structure. For today we use model categories as our language for homotopy theory and operads as our language for algebra. We'll work on the subgoal of understanding when Bousfield localization preserves the structure of algebras over an operad.

This project was motivated by a step in the proof of the Kervaire Invariant One Theorem. The authors needed a 256-periodic $\Omega = D^{-1}MU^{(4)}$ for some D. They were working with MU considered as a commutative equivariant spectrum and needed Ω to be commutative, i.e. have multiplicative norms, i.e. to have $\pi_\star$ forming a Tambara functor. This was needed for reasons related to the spectral sequence computations which occupy the technical details of the proof.

Let G be a finite group and let $\mathscr{S}^G$ be the model category of G-spectra. Recall that for every family of subgroups $\mathscr{F}$ of G there is a model structure on Top^G where weak equivalences and fibrations are maps such that $(-)^H$ is again such a map in Top for all $H \in \mathscr{F}$. For each family there is a universal $\mathscr{F}$-space EF which is a G-CW complex such that $(EF)^H$ is contractible for $H \in \mathscr{F}$ and empty otherwise. The family model structure on Top has generating cofibrations $(G/H \times S^{n-1})_+ \to (G/H \times D^n)_+$ for all n and all $H \in \mathscr{F}$, and the analogous generating trivial cofibrations. These family model structures are also present in $\mathscr{S}^G$, and can be defined by similarly changing the generating (trivial) cofibrations. We will make use of these family model structures at the end of the talk.

Sadly, not every localization of an equivariant commutative ring spectrum is commutative.

Example (Hill, Oberwolfach). Let G be a (non-trivial) finite group.

Consider the reduced real regular representation $\overline{\rho} = \mathbb{R}[G]/\mathbb{R}[e]$, i.e. $\overline{\rho}_G = \rho_G - 1$ where 1 means the trivial representation $\mathbb{R}[e]$. Now, ρ_G takes any subgroup H to $[G : H]$ many copies of H so $\rho_G|_H = [G : H]\rho_H$. Putting these facts together, we see that $\overline{\rho}_G|_H = [G : H]\overline{\rho}_H + ([G : H]1 - 1)$.

Now consider the representation sphere $S^{\overline{\rho}}$ and the inclusion $a_{\overline{\rho}} : S^0 \to S^{\overline{\rho}}$. Thinking of S^0 as $\{0, \infty\}$ we see that the only fixed points of this map are 0 and ∞, so it's not equivariantly trivial. Consider the spectrum $E = S^0[a_{\overline{\rho}}^{-1}]$. We will show that this spectrum does not admit maps from the norms of its restrictions, and hence cannot be commutative. The reason is that for all proper $H < G$, $res_H(E)$ is contractible.

The reason $res_H(E)$ is contractible as an H-spectrum is our computation above regarding $\overline{\rho}_G|_H$. Because $[G : H] - 1$ is a number k greater than 0 we have $res_H S^{\overline{\rho}_G} = (S^{\overline{\rho}_H})^{\#[G:H]} \wedge S^k$. This means that as an H-spectrum it is contractible, because there is enough space in the S^k part to deform it to a point.

Now, a key property of commutative equivariant ring spectra is the existence of multiplicative norms. These functors $N_H^G : \mathscr{S}^H \to \mathscr{S}^G$ are left adjoint to the restriction res_H on the category of commutative ring

Date: November 21, 2013.

spectra. Thus, if E were commutative we would have a ring homomorphism $* \simeq N_H^G res_H E \to E$. This is not a ring map unless E to be contractible, and we know E is not contractible because $a_{\bar{\rho}}$ fixes 0 and ∞.

Here is an equivalent approach, which Hill presented at Oberwolfach. Let $\mathscr{F}$ be the family of proper subgroups of G and let $\widetilde{E\mathscr{F}}$ be the cofiber of the natural map from the classifying space $E\mathscr{F}_+$ to S^0. This $\widetilde{E\mathscr{F}}$ is a localization of S^0 obtained by killing all maps from induced cells. If G is finite then it is our E. It's not contractible because $E\mathscr{F}_+$ is not homotopy equivalent to S^0, because $\mathscr{F}$ doesn't contain G. So while any restriction to a proper subgroup views them to be homotopy equivalent, they are not homotopy equivalent in $\mathscr{S}^G$.

In this second approach it becomes clear that this example generalizes to other families of subgroups, proving that in any family model structures (other than $\mathscr{F} = \{e\}$, which recovers naive spectra) one can similarly disprove the preservation of commutativity by localization.

□

The take-away message from this example is that we need a hypothesis on the maps being localized so that equivariant commutativity is preserved. Viewed a certain way, what is failing above is the ability of the localization functor to commute with equivariant suspension with respect to certain representation spheres (namely, those which don't see all the information in G, but only see subgroup information). When localizations kill representation spheres bad things happen. A similar example, due to Carles Casacuberta, proves that not all localizations of spectra preserve ring structure. This is the example of the Postnikov Section:

The n^th Postnikov section functor P_n is a homotopical localization for all n but does not commute with suspension. Furthermore, if R is nonconnective, then $P_1 R$ does not admit a ring spectrum structure (not even the structure of a ring in the homotopy category). The reason is that if it were a ring then multiplication by the unit S would need to be a homotopy equivalence. But the unit map $\nu : S \to P_1 R$ is null since $\pi_0(P_1 R) = 0$. The real issue here is that suspension and localization do not commute, and nonconnective ring spectra can feel the difference. We've chopped off the dimension where the unit is supposed to live.

Casacuberta gets around this by placing hypotheses on the localization (he calls the well-behaved localizations "closed") and similarly Hill and Hopkins get around Hill's example by placing hypotheses on the maps:

Theorem 1 (Hill-Hopkins)**.** *If for all acyclics Z for a localization L and for all subgroups H, $N_H^G Z$ is acyclic, then for all commutative G-ring spectra R, $L(R)$ is a commutative G-ring spectrum.*

Let R be a monoid in the genuine model structure for G-spectra. If the norm functor $N_H^G(-)$ preserves R-acyclicity then the Bousfield localization with respect to R-equivalences preserves commutativity.

Here commutativity can mean either strict commutativity (algebras over the operad Com) or E_∞-structure where E_∞ is the linear isometries operad (a model can be taken with $E_\infty[n] = E_G\Sigma_n$), because in $\mathscr{S}^G$ there is rectification between these operads, as recently proven by Blumberg and Hill in the appendix of their 2013 paper.

This example and theorem open a more general question: find conditions on a general model category $\mathcal{M}$ and on a set of maps C so that the Bousfield localization L_C preserves commutativity. We will answer this question, and when we specialize our machinery to $\mathcal{M} = \mathscr{S}^G$ we'll in fact characterize localizations which preserve commutative structure. This will show us yet another reason why Hill's example is failing. In particular, it will fail because for the set of maps being inverted we have that $C \otimes (G/H)_+$ is not contained in the C-local equivalences (because it contains the zero map, because proper H sees C to be trivial even though it is not). This makes it easy to see the correct condition on C so that L_C viewed in the family model structures preserves commutativity. The condition will be that $C \otimes (G/H)_+ \subset C$-local equivalences for all $H \in \mathscr{F}$.

2. Background: monoidal model categories, operads, Bousfield localization

Recall that we care about model categories $\mathcal{M}$ because the passage to Ho($\mathcal{M}$) works (this functor inverts the weak equivalences $\mathscr{W}$) and we have some control over the resulting maps because of cofibrant and fibrant replacement. Let $\mathscr{Q}$ and $\mathscr{F}$ be the cofibrations and fibrations. If we're going to talk about commutative monoids then we need to have a monoidal structure on $\mathcal{M}$. It turns out that we also need a compatibility hypothesis between $\mathcal{M}$ and the monoidal structure, as explained in chapter 4 of Hovey's book. Let $\otimes$ denote the monoidal product.

Given $f : A \to B$ and $g : X \to Y$, define the *pushout product* $f \,\square\, g$ to be the corner map in

$$\begin{tikzcd} A \otimes X \arrow[r] \arrow[d] & A \otimes Y \arrow[d] \arrow[ddr, bend left] & \\ B \otimes X \arrow[r] \arrow[ur, phantom, "\lrcorner"] \arrow[drr, bend right] & Q_2 \arrow[dr] & \\ & & B \otimes Y \end{tikzcd}$$

A *monoidal model category* is a model category which is also a monoidal category and satisfies:

Pushout product axiom: if $f, g \in \mathscr{Q}$ then $f \,\square\, g \in \mathscr{Q}$. Additionally, if either is in $\mathscr{W}$ then $f \,\square\, g \in \mathscr{W}$.

Unit Axiom: If Z is cofibrant then $QS \otimes Z \to S \times Z \cong Z$ is a weak equivalence.

These axioms assure you that $Ho(\mathcal{M})$ is a monoidal category. We'll be studying objects in $\mathcal{M}$ which carry the additional algebraic structure encoded by an operad, e.g. monoids, commutative monoids, A_∞ or E_∞ algebras, Lie algebras, etc. Recall that an operad in $\mathcal{M}$ is a symmetric sequence $P = (P(n))_{n\in\mathbb{N}}$ of objects in $\mathcal{M}$ (i.e. each $P(n)$ has an action of the symmetric group Σ_n) satisfying some axioms. The object $P(n)$ can be thought of as parameterizing maps of arity n. There is a notion for cofibrancy of an operad which comes down to requiring the left lifting property of $\varnothing \to P$ with respect to maps which are levelwise trivial cofibrations in $\prod_{n\in\mathbb{N}} \mathcal{M}^{\Sigma_n}$ where $\mathcal{M}^{\Sigma_n}$ is the model category of objects in $\mathcal{M}$ with a Σ_n action.

A common strengthening of the unit axiom is the Resolution Axiom, which states that cofibrant objects are flat, i.e. whenever $f \in \mathscr{W}$ and X is cofibrant, then $X \otimes f \in \mathscr{W}$.

What if I want to invert some maps $C \not\subset \mathscr{W}$? Because the homotopy category is nice (admits a calculus of fractions), we can do:

$$\begin{tikzcd} \mathcal{M} \arrow[r, dotted] \arrow[d] & ????? \arrow[d, dotted] \\ \mathrm{Ho}(\mathcal{M}) \arrow[r] & \mathrm{Ho}(\mathcal{M})[C^{-1}] \end{tikzcd}$$

We'd like a model category $L_C\mathcal{M}$ which actually sits above Ho($\mathcal{M}$)$[C^{-1}]$. Because all three categories above have the same objects, its objects are determined. It's morphisms will be the same as those in $\mathcal{M}$, but we want maps in C to become isomorphisms in Ho($\mathcal{M}$)$[C^{-1}]$ so we need them to be weak equivalences in $L_C\mathcal{M}$. So this category must have a different model structure, where $\mathcal{W}' = \langle C \cup \mathcal{W} \rangle$ and clearly $\mathcal{W} \subset \mathcal{W}'$. You can't change only $\mathcal{W}$ because it'll screw up the axioms. We want to keep the cofibrations fixed so we can build things out of them and have the two model structures related, so we have to shrink the fibrations: $\mathcal{F} \supset \mathcal{F}'$. **Bousfield's Theorem** (1978) says you can do this and you still get a model structure, but you have to be careful with how you generate $\mathcal{W}'$ from C. Details are in Hirschhorn's book.

The identity maps $\mathcal{M} \rightleftarrows L_f\mathcal{M}$ are a Quillen adjoint pair and prove that $L_C\mathcal{M}$ satisfies a universal property as the "closest" model category to $\mathcal{M}$ in which C is contained in the weak equivalences. The fibrant objects in $L_C\mathcal{M}$ are the C-local objects, and local equivalences between local objects are weak equivalences in the original model category. Bousfield localization gives a Quillen pair (L_C, U_C), which are both the identity functors on objects and morphisms, and these induce (L_C^H, U_C^H) on the homotopy level.

Our goal is to find conditions on $\mathcal{M}$ and C under which Bousfield localization preserves $\mathbb{P}$-algebra structure, i.e. if $[E] \in \mathrm{Ho}(\mathcal{M})$ has a representative $E \in \mathbb{P}$-alg then we're asking for $(U_f^H \circ L_f^H)([E])$ to have a representative in $\mathbb{P}$-alg. Because we know that Bousfield localization works via the derived functors of the identity, this is asking for some P-algebra $\widetilde{E}$ which is weakly equivalent to $R_C QE$

3. Results

Theorem 2. *Let $\mathcal{M}$ be a monoidal model category and let P be an operad valued in $\mathcal{M}$. If P-algebras in $\mathcal{M}$ and in $L_C(\mathcal{M})$ inherit model structures such that the forgetful functors back to $\mathcal{M}$ and $L_C(\mathcal{M})$ are right Quillen functors, then L_C preserves P-algebras up to weak equivalence. For well-behaved P there is a list of easy to check conditions on $\mathcal{M}$ and C guaranteeing these hypotheses hold.*

Proof. Here "inherit" means that a map of P-algebras f is a weak equivalence (resp fibration) iff f is a weak equivalence (resp fibration) in M. Preservation of strict commutative monoids means that there is some P-algebra $\widetilde{E}$ which is homotopy equivalent in M to $L_C(E)$. A similar problem is addressed in [**?**] and there the authors required the map $E \to \widetilde{E}$ to be a map of P-algebras. This worked because they assumed E to be cofibrant as a P-algebra, and we do not. We remark below that if we add this assumption then we can add their conclusion as well.

Let R_C be fibrant replacement in $L_C(M)$, $R_{C,m}$ be fibrant replacement in $P-alg(L_C(M))$, and Q_m be cofibrant replacement in $P-alg(M)$. In our proof, $\widetilde{E}$ will be $R_{C,m}Q_m(E)$. Because Q is the left derived functor of the identity adjunction between M and $L_C(M)$, and R_C is the right derived functor of the identity, we know that $L_C(E) \simeq R_C Q(E)$. We must therefore show $R_C Q(E) \simeq R_{C,m}Q_m(E)$.

The map $Q_m E \to E$ is a weak equivalence in $P-alg(\mathcal{M})$, hence in $\mathcal{M}$. The map $QE \to E$ is also a weak equivalence in $\mathcal{M}$ and lifting gives a map from $QE \to Q_m E$ (necessarily a weak equivalence in $\mathcal{M}$ by the 2 out of 3 property).

Since $Q_m E$ is a P-algebra in $\mathcal{M}$ it must also be a P-algebra in $L_C\mathcal{M}$, since the monoidal structure of the two categories is the same. We may therefore construct a lift:

$$\begin{tikzcd}
Q_m E \arrow[r] \arrow[d, hook] & R_{C,m}Q_m E \arrow[d, two heads] \\
R_C Q_m E \arrow[r] \arrow[ur, dotted] & *
\end{tikzcd}$$

In this diagram the left vertical map is a weak equivalence in $L_C\mathcal{M}$ and the top map is a weak equivalence in $P-alg(L_C\mathcal{M})$. Because this model category $P-alg(L_C\mathcal{M})$ inherits weak equivalences from $L_C\mathcal{M}$ this map is a weak equivalence in $L_C\mathcal{M}$. Therefore, by the 2 out of 3 property, the lift is a weak equivalence in $L_C\mathcal{M}$. Using this lift we can draw the following diagram:

$$\begin{tikzcd}
QE \arrow[rr] \arrow[dd] & & Q_mE \arrow[dl] \arrow[dd] \\
& R_CQ_mE \arrow[dr] & \\
R_CQE \arrow[ur] \arrow[rr] & & R_{C,m}Q_mE
\end{tikzcd}$$

We showed above that $QE \to Q_mE$ is a weak equivalence in $\mathcal{M}$. Thus, $R_CQE \to R_CQ_mE$ is a weak equivalence in $L_C\mathcal{M}$. We then proved $R_CQ_mE \to R_{C,m}Q_mE$ is a weak equivalence in $L_C\mathcal{M}$. Thus, by the 2 out of 3 property, $R_CQE \to R_{C,m}Q_mE$ is a weak equivalence in $L_C\mathcal{M}$. All the objects in the triangle are fibrant in $L_C\mathcal{M}$ so these C-local equivalences are actually weak equivalences in $\mathcal{M}$.

The triangle commutes because the bottom map is defined as the composite. The square commutes in Ho $\mathcal{M}$ and demonstrates that R_CQE is isomorphic in Ho $\mathcal{M}$ to the P-algebra $R_{C,m}Q_mE$. □

This proof also holds if P-algebras only form a semi-model category. In a semi-model category all objects admit cofibrant replacement, but only cofibrant objects admit fibrant replacement. Lifting of a trivial cofibration against a fibration only holds if the domain is cofibrant. Everywhere we've applied fibrant replacement it's been to an object which is cofibrant in the underlying category, so that's no problem. The lifting argument is for a map which has cofibrant domain, so that's fine too. Thus, even if the monoid axiom is not preserved we can still say P-alg is a semi-model category.

It's a bit unfair to just assume P-algebras form a model category. After all, it can be very difficult to get your hands on $L_C\mathcal{M}$. We'd rather have hypotheses on $\mathcal{M}$ and C to make sure this situation happens. For cofibrant operads P we can use the following theorem due to Spitzweck:

Theorem 3. *Suppose P is a Σ-cofibrant operad and $\mathcal{M}$ is a monoidal model category. Then P-alg is a semi-model category which is a model category if P is cofibrant and $\mathcal{M}$ satisfies the monoid axiom.*

We see then that if we only care about preserving structure over a cofibrant operad P then we only need to know when $L_C(\mathcal{M})$ is a monoidal model category. We can characterize when this occurs.

Theorem 4. *Assume $\mathcal{M}$ is a left proper, monoidal model category satisfying the resolution axiom. Then*

$L_C(\mathcal{M})$ satisfies the resolution axiom and pushout product axiom if and only if for all cofibrant K, the maps $C \otimes id_K$ are weak equivalences in $L_C(\mathcal{M})$

If $\mathcal{M}$ is tractable then it suffices to check this on K running through the domains and codomains of the generating (trivial) cofibrations

Thus, we have characterized monoidal localizations and there are examples of localizations which fail to be monoidal, e.g. in $Ch(R[G])$ when a localization kills a representation sphere.

Corollary 5. *If $\mathscr{S}^G$ a Bousfield localization preserves genuine commutativity iff $C \otimes (G/H)_+$ is a C-local equivalence for all H.*

We see where our opening example failed: applying $C \otimes -$ killed all G/H and we would have ended up inverting the zero map if the condition of the corollary was satisfied.

What if you don't care about all the norms, but rather only some of them?

Recall that non-equivariantly an operad P is said to be E_∞ if $P(n)$ is contractible and Σ_n acts freely. So the linear isometries operad and little cubes operad are both E_∞. Note that this does not mean the operad is cofibrant as an operad, only Σ-cofibrant. The conventional wisdom that E_∞ means cofibrant replacement for

Com is only true in the model category of collections, not the model category of operads. If you want an E_∞ operad which is honestly cofibrant you need to use the Fulton-MacPherson operad. Thanks to Spitzweck's result, we don't need to be overly careful about the difference between cofibrant and Σ-cofibrant in order to conclude preservation of algebra structure by localization, so we'll choose the E_∞ operad with $E_\infty[n] = E\Sigma_n$.

Equivariantly, this operad encodes naive E_∞ structure. Genuine E_∞ structure is encoded by any operad P where $P(n)$ is an $E_G\Sigma_n$, i.e. a space with a $G\times\Sigma_n$-action which is characterized up to $G\times\Sigma_n$-weak equivalence by the property that for $H < G\times\Sigma_n$, we have $(E_G\Sigma_n)^H = \varnothing$ if $H\cap\Sigma_n \neq \{e\}$ and $(E_G\Sigma_n)^H \simeq *$ otherwise. This space $E_G\Sigma_n$ can be defined as the total space of the universal G-equivariant principle Σ_n-bundle. The following result is joint with Javier Gutierrez, but may have been known previously:

Theorem 6. *The category of simplicial (resp. topological) G-operads can be given a model structure via transfer from the category of collections on G-spaces. Neither Com nor the naive E_∞ operads are cofibrant. Their Σ-cofibrant replacement E_∞^G can be described by $E_\infty^G[n] = E_G\Sigma_n$.*

Returning to the question of handling only some of the norms, the recent paper of Blumberg-Hill proves that this type of algebraic structure is captured by the class of N_∞ operads. Independently, Javier and I were studying operads based on families. There is a collection $E_{\mathscr{F}}\Sigma_n$ whose n^{th} space is the total space of a universal $\mathscr{F}$-equivariant principle Σ_n-bundle. When using the family model structure on G-spaces this becomes a cofibrant collection, and it's equivalent as a collection to an Blumberg-Hill N_∞ operad because of the universal property of the n^{th} space.

Thus, the general preservation theorem specializes to tell us that Bousfield localization preserves this N_∞ structure iff $C\otimes(G/H)_+$ is a C-local equivalence for all $H\in\mathscr{F}$. The proof is to work in the family model structure on spaces and the corresponding semi-model structure on $Op(Top^{\mathscr{F}})$. Mike's example can be generalized to a collection of examples demonstrating necessity of this hypothesis at each level in the tower of $E_\infty^{\mathscr{F}}$'s interpolating between E_∞ and E_∞^G. His original example is maximally bad, i.e. drops from any norm structure all the way down to E_∞, but there are examples which make any drop you like, e.g. from some $E_\infty^{\mathscr{F}}$ to some other $E_\infty^{\mathscr{F}'}$.

4. Strict Commutativity

For a general model category $\mathcal{M}$ it's not true that *Com* and E_∞ encode the same algebras. That fact is special to spectra or other situations where rectification occurs. Checking the hypotheses of Theorem 2 for the operad $Com = (*)_{n\in\mathbb{N}}$ requires a general theorem for when commutative monoids inherit a model structure. For monoids this is done by Schwede-Shipley and the hypothesis needed on $\mathcal{M}$ is the *monoid axiom*, which says that for all objects X, $(id_X\otimes(\mathscr{Q}\cap\mathscr{W})) - cell \subset \mathscr{W}$. Here applying cell to a class of maps means taking its closure under transfinite compositions and pushouts. For commutative monoids the correct hypothesis is the *commutative monoid axiom*: If g is a (trivial) cofibration then $g^{\square n}/\Sigma_n$ is a (trivial) cofibration.

Theorem 7. *If a monoidal model category satisfies the monoid axiom and the commutative monoid axiom then commutative monoids form a model category and the forgetful functor is right Quillen.*

This result generalizes a theorem of Lurie's from DAGIII, i.e. my hypothesis is weaker.

Examples:

(1) Ch(k) where $char(k) = 0$. Lurie had this too. More generally, can get any $\mathbb{Q}$-algebra
(2) sSet - this fails Lurie's hypothesis. My proof uses the fact that cofibrations are monomorphisms to get the bit about cofibrations. For the weak equivalences part we rely on a clever trick of Dror Farjoun.

(3) Positive (Flat) model structure on symmetric spectra. Lurie doesn't apply here. He acknowledges his error in DAGIII 4.3.25 in Math Overflow post 146438. My proof needed a technical lemma that it was sufficient to check the commutative monoid axiom on the generators. Luis Pereira proved the same for Lurie's hypothesis

(4) Top - this fails for Lurie. It works for me because the proof of Farjoun generalizes to any Cartesian concrete category, and with a bit more care we don't need cofibrations to be monomorphisms either, because we have our hands on the generators.

(5) Positive orthogonal (equivariant) spectra - using again that it's sufficient to check it on the generators

(6) Positive motivic symmetric spectra - I'm developing this category with Markus Spitzweck.

If we drop the monoid axiom we only get a semi-model structure on *Com*-alg, but that is enough for preservation by localization. I have a theorem about when localization preserves the monoid axiom but it's unnecessary here for this reason. Anyway, in a combinatorial model category this this result adds no hypotheses at all to the maps being inverted. It simply needs that $\mathcal{M}$ is h-monoidal and satisfies a compactness hypothesis on the generating cofibrations I. This hypothesis holds in all the examples.

Turning now to when localization preserves the commutative monoid axiom, recall that commutative monoids are built via the functor $Sym(X) = S \wedge X \wedge X^2/\Sigma_2 \wedge \dots$. For monoidal structure we needed localization to play well with tensoring. Now we'll need it to work with Sym:

Theorem 8. *Suppose $\mathcal{M}$ satisfies the commutative monoid axiom. If $Sym(-)$ preserves weak equivalences in $L_C(\mathcal{M})$ then $L_C(\mathcal{M})$ satisfies the commutative monoid axiom.*

Combining this with our general preservation result gives:

Corollary 9. *Truncations in sSet, Top, and Ch(k) all preserve strict commutative monoids. Via Farjoun's trick, any monoidal localization in sSet will also preserve, e.g. L_E for a homology theory E.*

I hope to investigate these L_E further and recover classical theorems of Bousfield using this general machinery.

Remark: the commutative monoid axiom generalizes to other non-cofibrant P. We saw already that if P is cofibrant then basically no hypotheses are needed on $\mathcal{M}$ to get admissibility. Harper has a result that if all symmetric sequences in $\mathcal{M}$ are projectively cofibrant then all operads are admissible. This is a strong hypothesis. I don't know of any examples other than Ch(k) which satisfy it. My result shows that you can pay the cofibrancy price partially on $\mathcal{M}$ and partially on P, e.g. to get levelwise cofibrant P you need for all $X \in \mathcal{M}^{\Sigma_n}$ which are cofibrant in $\mathcal{M}$ one has $X \otimes_{\Sigma_n} f^{\square n}$ is a trivial cofibration. There is also a generalizes version of the regular monoid axiom, which requires that applying cell to a certain class of maps results in a weak equivalence. For details see my research statement. In the future I hope to study when localization preserves these axioms.