TRIANGULATED CATEGORIES AND STABLE MODEL CATEGORIES

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Running examples:

(1) Stable homotopy category
(2) Bousfield localizations of spectra
(3) Comodules over a Hopf algebroid.
(4) Derived category of a ring. Of an abelian category. $K(R)$?
(5) Stable module category of a ring.
(6) Stable module categories coming from Frobenius categories
(7) Equivariant and motivic spectra

1. A comparison of definitions

1.1. **Definition taken from Wikipedia. TR1** For any object X, the following triangle is distinguished:

$X \xrightarrow{\text{id}} X \to 0 \to \cdot$

For any morphism $u : X \to Y$, there is an object Z (called a mapping cone of the morphism u) fitting into a distinguished triangle

$X \xrightarrow{u} Y \to Z \to \cdot$

Any triangle isomorphic to a distinguished triangle is distinguished. This means that if

$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$

is a distinguished triangle, and $f : X \to X, g : Y \to Y$, and $h : Z \to Z$ are isomorphisms, then

$X' \xrightarrow{guf^{-1}} Y' \xrightarrow{hvg^{-1}} Z' \xrightarrow{f[1]wh^{-1}} X'[1]$

is also a distinguished triangle.

TR2 If

$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$

is a distinguished triangle, then so are the two rotated triangles

$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$

and

$Z[-1] \xrightarrow{-w[-1]} X \xrightarrow{u} Y \xrightarrow{v} Z.$

Date: September 10, 2013.

TR3 Given a map between two morphisms, there is a morphism between their mapping cones (which exist by axiom (TR1)), that makes everything commute. This means that in the following diagram (where the two rows are distinguished triangles and f and g form the map of morphisms such that $gu = uf$) there exists some map h (not necessarily unique) making all the squares commute:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dotted] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

TR4: The octahedral axiom

Suppose we have morphisms $u : X \to Y$ and $v : Y \to Z$, so that we also have a composed morphism $vu : X \to Z$. Form distinguished triangles for each of these three morphisms according to TR2. The octahedral axiom states (roughly) that the three mapping cones can be made into the vertices of a distinguished triangle so that "everything commutes".

More formally, given distinguished triangles

$X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{k}$
$Y \xrightarrow{v} Z \xrightarrow{l} X' \xrightarrow{i}$
$X \xrightarrow{vu} Z \xrightarrow{m} Y' \xrightarrow{n}$

there exists a distinguished triangle

$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{h}$

such that $l = gm, \quad k = nf, \quad h = j[1]i, \quad ig = u[1]n, \quad fj = mv.$

This axiom is called the "octahedral axiom" because drawing all the objects and morphisms gives the skeleton of an octahedron, four of whose faces are distinguished triangles. The presentation here is Verdier's own, and appears, complete with octahedral diagram, in (Hartshorne 1966). In the following diagram, u and v are the given morphisms, and the primed letters are the cones of various maps (chosen so that every distinguished triangle has an X, a Y, and a Z letter). Various arrows have been marked with [1] to indicate that they are of "degree 1"; e.g. the map from Z to X is in fact from Z to T(X). The octahedral axiom then asserts the existence of maps f and g forming a distinguished triangle, and so that f and g form commutative triangles in the other faces that contain them:

LOOKS LIKE AN OCTAHEDRON

Two different pictures appear in (Beilinson, Bernstein & Deligne 1982) (Gelfand and Manin (2006) also present the first one). The first presents the upper and lower pyramids of the above octahedron and asserts that given a lower pyramid, we can fill in an upper pyramid so that the two paths from Y to Y, and from Y to Y, are equal (this condition is omitted, perhaps erroneously, from Hartshorne's presentation). The triangles marked + are commutative and those marked "d" are distinguished:

LOOKS LIKE TWO PYRAMIDS

There is also a way to present it where distinguished triangles are presented linearly, and the diagram emphasizes the fact that the four triangles in the "octahedron" are connected by a series of maps of triangles, where three triangles (namely, those completing the morphisms from X to Y, from Y to Z, and from X to Z) are given and the existence of the fourth is claimed. We pass between the first two by "pivoting" about X, to the third by pivoting about Z, and to the fourth by pivoting about X. All enclosures in this diagram are commutative (both trigons and the square) but the other commutative square, expressing the equality of the two paths from Y to Y, is not evident. All the arrows pointing "off the edge" are degree 1.

This last diagram also illustrates a useful intuitive interpretation of the octahedral axiom. Since in triangulated categories, triangles play the role of exact sequences, we can pretend that $Z' = Y/X$, $Y' = Z/X$ in which case the existence of the last triangle expresses on the one hand

$X' = Z/Y$ (looking at the triangle $Y \to Z \to X' \to$), and
$X' = Y'/Z'$ (looking at the triangle $Z' \to Y' \to X' \to$).

Putting these together, the octahedral axiom asserts the "third isomorphism theorem":

$(Z/X)/(Y/X) = Z/Y$

When the triangulated category is K(A) for some abelian category A, and when X, Y, Z are objects of A placed in degree 0 in their eponymous complexes, and when the maps $X \to Y, Y \to Z$ are injections in A, then the cones are literally the above quotients, and the pretense becomes truth.

Finally, Neeman (2001) gives a way of expressing the octahedral axiom using a two dimensional commutative diagram with 4 rows and 4 columns. Beilinson, Bernstein, and Deligne (1982) also give generalizations of the octahedral axiom.

1.2. **Definition from Neeman's book.** Need additive and endofunctor Σ is invertible. To be pre-triangulated need Σ additive too, plus a distinguished class of triangles s.t.

TR0: $X = X \to 0 \to \Sigma X$
TR1: All $f : X \to Y$ give $X \to Y \to Z \to \Sigma X$
TR2: If either of $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ or $Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is a distinguished triangle then so is the other.
TR3: Given 2x4 rectangle with two distinguished triangles missing the $Z \to Z'$ arrow, that arrow exists but not necessarily unique.

Rotation strengthens TR1 to let you put f into the second part of a triangle too, i.e. $W \to X \to Y \to \Sigma W$. Rotation can also be iterated, giving you the Puppe Sequence.

It follows from these axioms that the class of triangles is closed under isomorphism.

It follows that two hops in a triangle is 0, i.e. $X \to Y \to Z$ has composite $0 : X \to Z$.

Given a 2x4 rectangle you can form the mapping cone

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} \Sigma X \oplus Z' \xrightarrow{\begin{pmatrix} -\Sigma u & 0 \\ \Sigma f & w' \end{pmatrix}} \Sigma Y \oplus \Sigma X'$$

A homotopy of triangles is a collection of down and left diagonals (Θ, Φ, Ψ) s.t. $f - f' = \Theta u + \Sigma^{-1}\{w'\Psi\}$, $g - g' = \Phi v + u'\Theta$, $h - h' = \Psi w + v'\Phi$.

Mapping cone is homotopy invariant. A candidate triangle C is contractible if $1 : C \to C$ is homotopic to $0 : C \to C$

To be triangulated, need

TR4': Given any 2x4 diagram missing the Z vertical arrow, you can choose h so that the mapping cone is a triangle.

Note: even now, Hovey would argue that pre-triangulated should mean you can only rotate in one direction, i.e. the difference is not the octahedral axiom but rather that in TR2 you have rotation both ways (and hence in TR1 you can fit f into either the first map in the triangle or the second). This is because Hovey wants pre-triangulated to be the unstable analog of triangulated. With Neeman's definition there are no known examples of categories which are pretriangulated but not triangulated.

These axioms imply the 3x3 lemma (Hovey says it is equivalent to the octahedral axiom in the presence of the others). This basically says that if you have 2 triangles then you can build a third from their pieces. Formally, given

$$\begin{array}{ccccccc} & & \Sigma^{-1}X' & & \Sigma^{-1}Y' & & \\ & & \downarrow & & \downarrow & & \\ \Sigma^{-1}Z'' & \to & X'' & \to & Y'' & \to & Z'' \\ & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1}Z & \to & X & \to & Y & \to & Z \\ & & \downarrow & & \downarrow & & \\ & & X' & \to & Y' & & \end{array}$$

There exists Z' and $Y' \to Z' \leftarrow Z$ such that the following commutes and the rows and columns are exact.

$$\begin{array}{ccccccc} \Sigma^{-2}Z' & \to & \Sigma^{-1}X' & \to & \Sigma^{-1}Y' & \to & \Sigma^{-1}Z' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1}Z'' & \to & X'' & \to & Y'' & \to & Z'' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1}Z & \to & X & \to & Y & \to & Z \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-1}Z' & \to & X' & \to & Y' & \to & Z' \end{array}$$

However, the new objects and maps which are asserted to exist need not be unique. This is the same failure as in TR3. There the fill is not unique. In both places this is saying that the map of categories defined on the arrow category via the passage to the new map is not a functor. The octahedral axiom and the 3x3 lemma are the only tools we have to get at the cofiber of a composite.

1.3. **Definition from Hovey's book.** A cofiber sequence is $X \to Y \to Z$ in $Ho(M)$ with a right coaction of ΣX on Z. It must be isomorphic to a diagram $A \to B \to C$ with $A \to B$ a cofibration of cofibrant objects and where the cofiber and action are determined by that data.

A pretriangulated category is a non-trivial right closed HoSSet$_*$-module with distinguished triangles called cofiber sequences (or left triangles) and fiber sequences (or right triangles) s.t. in a cofiber sequence the cogroup ΣX right coacts on Z and in a fiber sequence the group ΩZ right acts on X. The triangles are closed under isomorphism of diagrams (taking into account the action). The identity triangles. Extending maps both ways to triangles. Shifting cofiber sequences right and fiber sequences left. Fill-in maps exist. Octahedral axiom and its dual hold. Cofiber and fiber sequences are compatible (i.e. in a 2x4 where top has Σ and bottom Ω can fill either of the middle arrows). Smash product preserves cofiber sequences in each variable; Hom(-,-) preserves fiber sequences in the second variable and converts cofiber sequences in the first variable to fiber sequences; Map(-,-) preserves fiber sequences in the second variable and converts cofiber sequences in the first variable into fiber sequences.

A triangulated category then is a pretriangulated category s.t. Σ is an equivalence of categories. Define a pointed model category to be stable if its homotopy category is triangulated. Nowadays this definition is rejected because it assumes HoSSet acts. People might call these things simplicial triangulated categories. Under the more general definition not every triangulated category is the homotopy category of some model category. For example, Muro-Schwede-Strickland. This example also demonstrates that not every triangulated category comes from an infinity category. We're not sure whether or not this example has a $HoSSet_*$ enrichment or not.

2. Stable model category gives a triangulated homotopy category

Every pointed model category gives a pretriangulated homotopy category. You can define the cokernel of a map $f : X \to Y$ as the coequalizer of f and the map $X \to *$. You can define the fiber of f as the equalizer of f with $* \to X$. To define ΣX you need to do a bit more work. Quillen defined it as the cokernel of $X \vee X \to X \times I$, but this only works if X is cofibrant. Better is to define it as the cokernel of $QX \vee QX \to QX \times I$, or as $QX \wedge S^1$ using the simplicial enrichment (or framings). This demonstrates that $\Sigma X = X \wedge^L S^1$ in $Ho(M)$. Dually, $\Omega(X) = RHom_*(S^1, X)$ and on the model category level this is $Hom_*(S^1, RX)$ or the kernel of $(RX)^I \to RX \times RX$. As in Top, $\pi_t Map(A, Y) = [\Sigma^t A, Y] = [A, \Omega^t Y]$. Note that framings allow this to work for all pointed model categories, not just the simplicial ones.

To get the map from $Z \to \Sigma X$ requires a coaction of a cogroup. You use $Z \to Z \coprod \Sigma X \to \Sigma X$. A cogroup structure on X is a lift of the functor $Hom(X, -)$ from $M \to Set$ to Grp. A group structure is a lift of $Hom(-, X)$. That $\Sigma^t X$ is an abelian cogroup object and $\Omega^t Y$ is an abelian group object (for $t > 1$) relies on this fact in Top plus the $HoSSet_*$-enrichment.

To understand the coaction of ΣA on the cofiber C of $A \to B$, note that this coaction is equivalent to an action of $[\Sigma A, X]$ on $[C, X]$ for all (fibrant) X. To construct this, let $g : B \to C$ and $h : A \to X^I$ representing $[h] \in [\Sigma A, X]$. Given $u : C \to X$, look at

$$\begin{array}{ccc} A & \xrightarrow{h} & X^I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_0} \\ B & \xrightarrow{ug} & X \end{array}$$

It's a cofibration-trivial fibration diagram so there's a lift $\alpha : B \to X^I$. Map to X via p_1 and note that $A \to B \to X^I \to X$ is zero by construction (since $p_1 h = 0$ and $p_1 h_1 f = 0$). Thus, there is an induced map $w : C \to X$ and we define $[u] \cdot [h] = [w]$. Dually, there is a coaction of $[A, \Omega B]$ on $[A, F]$. To understand $[w]$ look at $M = Top_*$ and $A = S^0$. In this case h is a loop in B and u is a point in F. The element $[u] \cdot [h]$ is defined by taking a lift of h to a path α starting at u and then taking its other endpoint w. Proving this is a natural action takes some real work. It's Theorem 6.2.1 in Hovey's book. First he proves it's well-defined (via homotopy between homotopies), then that it's natural for maps of fibrant objects (i.e. $[qu] \cdot [q^I h] = q([u] \cdot [h])$), then associative and unital. Next the induced map on cofibers from a map between cofibrant objects is equivariant in $Ho(M)$ with respect to the cogroup homomorphism Σq_1.

From all these facts we can conclude that cofiber (resp fiber) sequences are preserved by left (resp right) Quillen functors (6.4.1 and 6.4.2). Hovey uses this idea to define **exact adjunctions** of pretriangulated categories in 6.5.4. It makes the category of triangulated categories into a 2-category with 2-morphisms being the natural transformations of $HoSSet_*$-module functors. With this language, the homotopy pseudo 2-functor from pointed model categories to closed $HoSSet_*$-modules lifts to a pseudo 2-functor from pointed model categories to pretriangulated categories.

We can also conclude that cofiber and fiber sequences are preserved by the $HoSSet_*$-enrichment from the framing (6.4.5). This is nice because it doesn't seem to require any axioms other than the normal axioms of a triangulated category and the fact that in this case the triangulated category is simplicial. So a **simplicial triangulated category** should be a module over $HoSSet_*$ with a triangulated structure, and you don't need any further compatibility conditions.

Hovey proves triangulated categories are additive. This does not appear to be true of his pretriangulated (unstable) categories.