1. Outline

(1) Basic Graph Theory and graph coloring
(2) Pigeonhole Principle
(3) Definition and examples of Ramsey Numbers - $R(3)$, $R(3,3,3)$
(4) Generalized Ramsey Numbers and Ramsey's Theorem
(5) Erdös's famous lower bound on $R(n)$

2. Basic Graph Theory

Definition 1. *A* ***graph*** G *is a pair* (V,E) *where* V *is a set of points, called* ***vertices****, and* E *is a set of pairs of points* (v_i, v_j) *called* ***edges****.*

For us $|V|$ will always be finite. IMAGE: General graph

The complete graph on n vertices has n vertices and edges between all pairs of vertices. IMAGES: K_3, K_4

Graphs are useful all over mathematics and computer science. Much of the world can be modeled through graphs. For example, a group of people can make a friendship graph where there is an edge if two people are friends. Or the internet is a graph with computers as nodes and edges if they're on the same network (or visiting the same websites, or whatever).

3. Graph Coloring

Definition 2. *A* ***coloring*** *of a graph is an assignment of colors (living in some finite set* $\{c_1, \dots, c_r\}$*) to the edges of the graph.*

You may have seen this defined before and probably with the nodes being colored such that no edge connects two nodes of the same color. But for this talk I will need to be coloring the edges of graphs and I will not insist that two adjacent edges are of different colors.

IMAGE: Color a K_5 so it DOES HAVE a monochromatic triangle.

4. Pigeonhole Principle

Proposition 1 (Pigeonhole Principle)**.** *If you are placing* $n+1$ *pigeons into* n *holes, then some hole will end up containing at least two pigeons (obviously this holds for placing* m *pigeons into* n *holes whenever* $m > n$*).*

If you are placing $2n-1$ *pigeons into 2 holes then some hole will end up containing at least* n *balls. So if you have* $2n-1$ *people at a party then at least* n *are of the same gender.*

Because combinatorics studies discrete objects (mostly finite), this is **crucial** to the field and many proofs rely on it. The notion of placing pigeons into 2 holes is exactly the same as 2-coloring the pigeons.

Here do the example with K_6 and show that every vertex has three edges of one color coming out of it.

5. Ramsey

Ramsey Theory generalizes the Pigeonhole Principle and solves the party problem above more generally. The question is: what is the minimum number of guests that must be invited so that at least n will know each other?

Definition 3. *$R(n)$ is the smallest integer m such that in any 2-coloring of K_m there is a monochromatic K_n.*

In the most general sense, Ramsey Theory asks how many elements are necessary to insure that some property is met.

6. Known Ramsey numbers

Theorem 1 (Theorem on Friends and Strangers)**.** *$R(3) = 6$, i.e. 6 is the smallest number such that any 2-coloring of K_6 has a monochromatic K_3 (i.e. a monochromatic triangle)*

This means at any party with at least six people, there are either three people who are all mutual acquaintances (each one knows the other two) or mutual strangers (each one does not know either of the other two).

Proof. First, $R(3) \geq 6$ because here is a 2-coloring of K_5 with no monochromatic triangle. It's always easier to find lower bounds on Ramsey numbers because it's constructive. IMAGES.

Second, $R(3) \leq 6$ because if a is a vertex of K_6 then a has 5 edges touching it. By the **Pigeonhole Principle**, three of them are the same color (e.g. as shown above) without loss of generality red. Consider the three vertices b, c, d those edges connect to. If any edges between them are red (say (b, c) is) then we're done because $\triangle abc$ is red. So none of these edges are red and they must all be blue. This means $\triangle bcd$ is blue. So we have a monochromatic triangle. $\square$

So we know $R(3) = 6$ and it's trivial that $R(1) = 1$ and $R(2) = 2$ since any coloring of a K_2 (i.e. 1 edge) has a monochromatic K_2.

Fact: $R(4) = 18$. You'll need to find a 2-coloring of a K_{17} without a monochromatic K_4 and then prove that any 2-coloring of K_{18} has a monochromatic K_4. Indeed, the 2-coloring works by coloring an edge (i, j) red if $i - j$ is a square modulo 17 and coloring it blue otherwise (this red subgraph is called the 17-Paley graph). So $R(4) \geq 18$. There has been much beautiful work done here putting lower bounds on Ramsey numbers using elementary number theory. Perhaps more **advanced number theory** would give better bounds! DEBT: $R(4) \leq 18$.

Fact: $43 \leq R(5) \leq 49$ and $102 \leq R(6) \leq 165$. If you find better bounds you will have a very nice paper. If you nail either one down you'll probably get a PhD immediately!

For a K_{48} there $\binom{48}{2} = 1128$ edges. Each can get one of two possible colors. So there are 2^{1128} colorings to consider, which is totally infeasible for a computer.

Paraphrased from Paul Erdös: Aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

Erdös was one of the most prolific publishers of papers in mathematical history, second only to Leonhard Euler; Erdös published more papers, while Euler published more pages.

Proposition 2. $(n-1)^2 < R(n) \leq 4^n$

Proof. Clearly for $k = 1$, $R(k) \leq 4^k$. Suppose this holds for all $k < n$. Let $m = 4^n = 2^{2n}$, and consider any 2-coloring of K_m. IMAGE with flags and neighbors. For any vertex x_1 the degree of x_1 is $2^{2n} - 1$ and so the **pigeonhole principle** assures us that there are 2^{2n-1} neighbors of the same color, call it blue. So **place a blue flag** on x_1. Next, select x_2 from among these blue neighbors of x_1 and note that by the pigeonhole principle it has 2^{2n-2} neighbors of the same color (perhaps it is red and needs a red flag). Continue in this way till you have a sequence $x_1, x_2, \ldots, x_{2n-1}$. By the pigeonhole principle some n of these have the same color flag (perhaps red). Selecting those vertices gives a red K_n. IMAGE with blocks and interior edges.

As for the lower bound, partition $K_{(n-1)^2}$ into $n-1$ sets and color all edges in each set red. So I have $n-1$ groups of red K_{n-1}'s and so every vertex is in one of these groups. Color all remaining edges blue. Clearly there is no red K_n because all the red groups are one vertex short. If there was a blue K_n then consider where its n vertices are. There are only $n-1$ red groups so two of these bad vertices must be in the same red group. But this means the edge between them is red, so in fact this all blue K_n wasn't all blue!

□

7. Generalizing Ramsey numbers

Define $R(s,t)$ and note that $R(s,t) = R(t,s)$. Existence below:

Proposition 3.

1. *Relatively easy to show and very useful:* $R(s,t) \leq R(s-1,t) + R(s,t-1)$.
2. *Theorem of Erdös-Szekeres:* $R(s,t) < \binom{s+t-2}{t-1}$. *Corollary of (1).* *IMAGE*

Proof.

1. $R(s,t) \leq R(s-1,t) + R(s,t-1)$. We may assume by induction that $n_1 = R(s-1,t)$ and $n_2 = R(s,t-1)$ are finite. Let n be their sum and consider any 2-coloring of K_n. Let x be a vertex, so its degree is $n-1 = n_1 + n_2 - 1$. By the **pigeonhole principle** there are either n_1 red edges or n_2 blue edges coming out of x. Assume the first holds (the other case is symmetric) and note that these neighbors form a K_{n_1}. If this graph has a blue K_t we are done. Otherwise, $R(s-1,t)$ gives a red K_{s-1} and so with x this makes a red K_s.
2. $R(s,t) < \binom{s+t-2}{t-1}$ is a corollary the above by induction. Assume it holds for all $2 \leq s' + t' < s+t$. Then by the above we have

$$R(s,t) \leq R(s-1,t) + R(s,t-1) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}$$

□

Theorem 2 (Ramsey's Theorem)**.** *Given integers $n_1, \ldots, n_r$ there is a number $m = R(n_1, \ldots, n_r)$ such that for any r-coloring of the edges of K_m there exists an i such that $1 \leq i \leq r$ and there exists a complete K_{n_i} monochromatic in color i.*

That $R(n_1, \ldots, n_r)$ is the **Generalized Ramsey Number**. For example, we've already shown $R(3,3) = 6$ and $R(4,4) = 18$. Clearly $R(s,2) = s$ for all s because either you have a red K_s or you have some blue edge. It's also true that $R(3,4) = 9$, i.e. 9 is the minimal m for which any 2-coloring of K_m contains a red triangle or a blue K_4. Easy exercises: $R(3,5) = 14, R(3,6) = 18, R(3,7) = 23, R(3,8) = 28$

Proposition 4. $R(n_1, \ldots, n_r) \leq R(R(n_1, n_2), n_3, \ldots, n_r)$

(1) and this give an inductive proof of Ramsey's Theorem.

Proof. Suppose $R(n_1, \ldots, n_r) \leq R(R(n_1, n_2), n_3, \ldots, n_r)$ holds for all $r < k$. Because the RHS relies on 2-color Ramsey numbers and $r-1$ color Ramsey numbers we can assume it exists and is finite by induction. Consider a k-colored K_n. Now replace the first two colors with a new color but **remember your old coloring**. So we have $k-1$ colored a K_n and so by the inductive hypothesis this graph contains either a K_{n_i} monochromatic in color i (for some $i \geq 3$) or it has a $K_{R(n_1,n_2)}$ monochromatic in my new color. In the former case we are finished. In the latter case, the definition of $R(n_1, n_2)$ and our memory of the old coloring assures use that we must have either a monochromatic K_{n_1} in color 1 or a K_{n_2} in color 2. In either case the proof is complete.

With this result, **Ramsey's theorem follows** easily. By (1) it holds for the 2-color case because $R(s,t)$ must be finite and we can take it to be the minimum such value. Then, by induction we assume Ramsey's Theorem holds $r-1$ colors and use (3) to bound the r-color case by the $r-1$ color case. So the r-color case is finite and we take the minimum over all n for which any r coloring of K_n contains a monochromatic K_{n_i} in color i. This minimum is $R(n_1, \ldots, n_r)$.

□

We can use (1) to prove some of our earlier claims. For instance: $R(4) \leq R(3,4) + R(4,3) = 9 + 9 = 18$ and $R(3,5) \leq R(2,5) + R(3,4) = 14$. DEBT PAID

These are tools which get things a bit tighter, but don't rely on any advanced mathematics. The beauty of Ramsey theory is that people can get even tighter bounds using random graphs and other fields. These random graphs and tricks get us nice asymptotics too. But even with all these bounds floating around some problems require tricks specific to the numbers involved. SKIP NEXT

For example, the tools give us that $R(3,4) \leq 6 + 4 = 10$. But in fact it's 9 and the K_8 with red perimeter and red diagonals proves $R(3,4) \geq 9$. To get $R(3,4) \leq 9$ we deal with cases. If any vertex has six blue edges coming out then in the set of its neighbors I can find a blue triangle (giving a blue K_4) or a red triangle. Either way I'm done. If x has 4 reds going out I look in the K_4 they make and if there's any red edge in there I have a red triangle with x. If not, then that K_4 is blue. In the only other case I have 5 blue and 3 red going out of each vertex. When counting blue edges we should get an even number since each is counted from both ends. But $9 * 5 = 45$ is not even, contradiction. This is how we show $R(s,t) \leq R(s-1,t) + R(s,t-1) - 1$ if both smaller Ramsey numbers are even.

The only non-trivial case known exactly for the generalized Ramsey numbers with $r = 3$ is $R(3,3,3) = 17$. It's the smallest non-trivial case because any time 2 appears we reduce to $r = 2$ (e.g. $R(3,3,2) = R(3,3)$) since the third color can't be used at all. SKIP NEXT

I'm not going to draw a 3-colored graph with 16 vertices and no monochromatic triangle. The proof that $R(3,3,3) \leq 17$ is that any x has 16 neighbors which is more than $5+5+5$ so there must be some six in the same color, say green. Among those six you can't have a green edge because that would give a green triangle with x. But $R(3,3) = 6$ now forces there to be a red or blue triangle in that set of six vertices.

8. Erdös Probability Method

All the proofs in the last slide were **upper bounds**. There are many creative constructive lower bounds (e.g. giving polynomial lower bounds of any fixed degree), but nothing reaching c^n for any $c > 1$. This is achieved only by the **Erdös Probability Method** and the following two facts:

(1) $P(\cup E_n) \leq \sum P(E_n)$ with equality iff events are pairwise disjoint.

(2) $P(A) = 0$ iff $A = \varnothing$. So $P(E) > 0$ iff there is some model in which E occurs. This is clear when probability space is finite.

Theorem 3. *If* $\binom{m}{n} < 2^{\binom{n}{2}-1}$ *then* $m < R(n)$

Corollary 1. *For* $n \geq 3$, $2^{\frac{n}{2}} < R(n)$

The best asymptotics for m come easily from the corollary and we get

$$m = 2^{\frac{n}{2}} \cdot \frac{n}{e\sqrt{2}} < R(n)$$

PROOF The probability space is the **space of 2-coloring** of the edges of K_m, which has $N = 2^{\binom{m}{2}}$ elements. A subset of n vertices yield a monochromatic K_n with probability $p = \frac{2}{2^{\binom{n}{2}}}$ since only two of all the colorings has this property (the all red and the all blue). Therefore the probability of having some monochromatic K_n is at most $p \cdot \binom{m}{n} < 1$ by hypothesis on m. Thus, the probability of having no monochromatic K_n is strictly positive. Since the **probability space is finite**, this means there **must exist a 2-coloring** with no monochromatic K_n which implies m is a lower bound on $R(n)$ without actually having to find the coloring!

This method generalizes to prove many useful things in graph theory, and more advanced topics from probability theory gave the following bounds: SKIP

$$c' \left(\frac{t}{\log(t)} \right)^2 < R(3,t) < c'' \frac{\log(\log(t))}{\log(t)} t^2$$

NOTE: there is a similar theorem for $R(s,t)$ with basically the same proof.

45 MIN

9. Extensions of Ramsey Numbers

They extend to **directed graphs**: let $R(n)$ be the smallest number m such that any K_m with singly-directed arcs (also called a tournament) contains an acyclic (also called transitive) n-node subtournament. For directed graphs, $R(n)$ is computed up to $n = 6$ and there are sharper bounds above.

They extend to **hypergraphs** (where edges can have more than 2 nodes) and you get $R_t(n_1, \ldots, n_r)$ to be the smallest m such that if the edges of K_m^t are r-colored in any way, then for some i you have a monochromatic K_i^t in color i. Note that K_m^t has m vertices and every t element set is a hyperedge.

They extend to **infinite graphs**, but you need to talk about when colorings are equivalent, irreducible, canonical, etc. There are some beautiful results here which rely on theorems from logic to prove. There are others which are simple statements about finite sets but which cannot be proven using the **Peano Axioms** and seem to require infinite Ramsey numbers.

Fact: In any k-coloring of $\mathbb{N}$ there is an infinite $A \subset \mathbb{N}$ such that all sums $\sum_{x \in X} x$ over $\varnothing \neq X \subset A$ have the same color. The proof uses the **Stone-Cech Compactification** (a semi-group) and **Principal Ultrafilters**. It's a result in **Partition Calculus** which is **SET THEORY**

Fact: if you 2-color K_∞ then there will be a monochromatic K_∞ by the **flag argument**.

Theorem (infinite hypergraphs): Let X be some countably infinite set and color the elements of $X(n)$ (the subsets of X of size n) in r different colors. Then there exists some infinite subset M of X such that the size n subsets of M all have the same color.

They extend to $\mathbf{R(G_1, G_2, \ldots, G_r)}$ which is the smallest number such that an r-colored K_n has no monochromatic G_i in color i. These are well-studied for the following:

cycles (solved for 2-colorings), stars, wheels, fan with ℓ blades, bipartite graphs, sparse graphs,

trees ($R(K_s, T_t) = (s-1)(t-1)+1$), paths, K_n's with one edge removed,

multiple disjoint copies of graphs: $ps + (q-1)t - 1 \leq r(sK_p, tK_q) \leq ps + (q-1)t + C$

In the plane we have **Happy Ending Problem**: Any set of five points in the plane in general position has a subset of four points that form the vertices of a convex quadrilateral.

More generally: $f(n)$ is the minimal number such that any $f(n)$ points in $\mathbb{R}^2$ in general position contain a convex n-gon. Because $f(n) \leq R_4(5, n)$ and $f(n) \leq R_3(n, n)$, this always exists.

Even more generally: If you have more than $\binom{k+\ell-4}{k-2}+1$ points then you have either a k-cup (convex) or an ℓ-cap (concave)

10. An alternate construction of Ramsey numbers

$R(s,t)$ is the smallest n such that for every graph G with $|G| = n$ either G contains K_s or $\overline{G}$ contains K_t. Here $\overline{G}$ has as vertices the edges of G and connects two if they were adjacent (shared a vertex) in G. Note that $\overline{\overline{G}} = G$ for all graphs. Ramsey used this construction and some set theory to originally prove his theorem.

11. Other attacks on Ramsey Numbers

Linear Algebra Method is viewing edges or vertices as vectors in a vector space and using independence to formulate a bound based on dimension.

Evolutionary Algorithms have been used to evolve constructive lower bounds on some Ramsey numbers of the $R(G_1, \ldots, G_s)$ type. But human-found constructive solutions have already been left in the dust so this may not work for improved lower bounds of $R(5)$ and $R(6)$.

Bounds based on clique number, independence number, chromatic number, discrepancy, and other graph invariants.

Example: $R(G,H) \geq (\chi(G)-1)(c(H)-1)+1$ where $c(H)$ is the size of the largest connected component of H.

$R(G,G) \leq c_d n(G)$ where c_d is a constant depending only on the maximum degree d of G.

Using the generalized Schur number as a lower bound. Or the maximal order of any cyclic coloring.

12. More probability theory

Local Lemma Setup: We say events $\{A_1, \ldots, A_n\}$ have **Dependency Bound** D if for all i, A_i depends on at most D other events A_j. Equivalently, for each i there is some $S_i \subset [n]$ such that $|S_i| \leq D$, $i \notin S_i$, and A_i is independent of any boolean combination of events A_j from $j \notin S_i$. Two events are independent if $P(A \cap B) = P(A)P(B)$.

Local Lemma: Assume $A_1, \ldots, A_n$ have dependency bound D and $P(A_i) \leq \frac{1}{4D}$. Then $P(\cap \overline{A_i}) > 0$. This capitalizes on independent events and gets even better bounds. Also applies widely in graph theory.

Random graphs can be used to find bounds on $R_\alpha(m)$ the smallest integer n such that for any G of size n either G or $\overline{G}$ has a subgraph H of order $\leq m$ with $\delta(H) \geq \alpha(|H| - 1)$. Use the **semi-random method** (a.k.a. **Rödl's nibble method**) to select small random bits and analyze as we go. This gives really good asympotics on Ramsey Numbers.

In probability theory, a **martingale** is a stochastic process (i.e., a sequence of random variables) such that the conditional expected value of an observation at some time t, given all the observations up to some earlier time s, is equal to the observation at that earlier time s. Formally, it's a sequence of random variables $X_1, X_2, X_3, \ldots$ such that for all n, $\mathbf{E}(|X_n|) < \infty$ and $\mathbf{E}(X_{n+1} \mid X_1, \ldots, X_n) = X_n$. These Martingales have recently been used to get better inequalities in Ramsey Theory and elsewhere in probability theory.

Use of CENTRAL LIMIT THEOREM? LAW OF LARGE NUMBERS? CHEBYCHEV'S INEQUALITY? OTHER THINGS FROM PROBABILITY THEORY?

13. Dessert

Proposition 5. *(1) $R(n) \leq 4R(n-2, n) + 2$*

(2) $R(n) \leq R_3(6, n)$ is a non-trivial exercise due to Kiran Kedlaya.

(3) $R(3, 3, \ldots, 3) \leq \lfloor er! \rfloor + 1$

Proof. (1) See K. Walker, Dichromatic graphs and Ramsey numbers, *Journal of Combinatorial Theory*, 5 (1968) 238-243

(2) Let $m = R_3(R(3), n)$ and consider any 2-coloring of $G = K_m$. We'll show this contains a monochromatic K_n. Use the given coloring to define a 2-coloring of K_m^3. Color hyperedge abc red if a, b, and c do not form a monochromatic triangle in G. Color abc blue otherwise. By definition of m, means you either have a red K_6^3 or a blue K_n^3. If you have a red K_6^3 then those six vertices contain no subset of size three which is monochromatic (by the definition of when you color something red). So we have a K_6 without a monochromatic K_3, contradicting the fact that $R(3) = 6$. Thus, we must be in the second case and have a blue K_n^3. But then you must have had a monochromatic K_n in G because we know all triangles contained in this K_n are the same color, which means all edges in K_n are the same color.

This proof generalizes obviously to show $R(n) \leq R_m(R(m), n)$ and $R_i(n) \leq R_m(R_i(m), n)$.

(3) First, $e = \sum_{i=0}^{\infty} \frac{1}{i!}$ so it's sufficient to truncate this sum to get some f and prove $R(3, 3, \ldots, 3) \leq \lfloor fr! \rfloor + 1$. In particular, $f = \sum_{i=0}^{r} \frac{1}{i!}$ will work. This is a proof by induction:

$R(3, 3) = 6 \leq \lfloor 2!(1 + 1 + \frac{1}{2}) \rfloor + 1 = 6$ is true.

Assume the result holds for $R(3,3,\ldots,3)$ with k entries. For simplicity denote this $r(k)$. Finally,

$$(k+1)!(\sum_{i=0}^{k+1}\frac{1}{i!})+1=(k+1)!(\sum_{i=0}^{k}\frac{1}{i!}+\frac{1}{(k+1)!})+1=(k+1)!(\sum_{i=0}^{k}\frac{1}{i!})+2=(k+1)\left(k!\sum_{i=0}^{k}\frac{1}{i!}\right)+2$$

By the inductive hypothesis, this is $\geq (k+1)(r(k)-1)+2$ so to show it's an upper bound for $r(k+1)$ it's sufficient to show $m=(k+1)(r(k)-1)+2$ is such a bound. Consider a $k+1$ coloring of K_m and we'll show it has a monochromatic K_3. For any vertex x its degree is $(k+1)(r(k)-1)+1$ so the pigeonhole principle tells us x connects to $r(k)-1+1$ vertices of one color, say red. If there are any red edges in there we are done because that edge and the two edges to x will form a red triangle. If there are not any red edges then we have a $K_{r(k)}$ which is k-colored and so by the inductive hypothesis it must contain a monochromatic triangle. Thus,

$$r(k+1)\leq(k+1)(r(k)-1)+2\leq(k+1)!(\sum_{i=0}^{k+1}\frac{1}{i!})+1\leq\lfloor(k+1)!e+1\rfloor \text{ since } |V|\in\mathbb{Z}$$

$\square$