QUAL TALK NOTES

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1. Motivation for Spectra 1

It is VERY hard to compute homotopy groups. We want to put as much algebraic structure as possible in order to make computation easier. You can't add maps in $HoTop$ but you can in Spectra (i.e. Spectra is an Ab-category). The motivation to go to $\mathbf{S} - \mathbf{Alg} = \mathbf{Ho}\mathcal{S}$ then becomes that you want an abelian category

We want to study the ring-like objects that arise in this category. "Ring-like" means ring-object, i.e. using the lens of category theory. They have no points, so you can't do traditional algebra. To measure complexity of these we'll use dimension.

2. Definitions

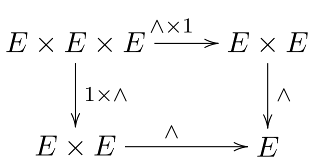
Definition 1 (Spectrum)**.** *A spectrum X is a sequence (X_i) of topological spaces (path conn. CW-complexes) with maps from $\Sigma X_i \to X_{i+1}$ where Σ is reduced suspension.*

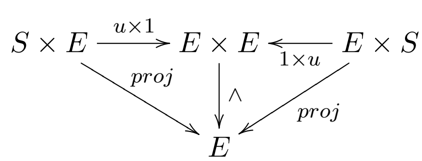
Example: $S = (S^n)$ the sphere spectrum. NOTE: We've erased dimension, so now maps that are morally the same, e.g. a map $S^5 \to S^4$ and a map $S^6 \to S^5$ are now actually the same. But the penalty is, we have no points. Define $X_* = \pi_k(X) = [\Sigma^k S, X]$. It's a graded ring, with smash product given by $(f : S^n \to X) \wedge (g : S^m \to X) : S^{n+m} \to X$. We get stable homotopy as $\pi_k(X) = \operatorname{colim}_n \pi_{k+n}(X_n)$ where the directed system is $\pi_{k+n}(X_n) \to \pi_{k+n+1}(S^1 \wedge X_n) \to \pi_{k+n+1}(X_{n+1})$

Example: Any cohomology theory, e.g. $H\mathbb{Q}$, KO

Definition 2 (S-algebra)**.** *An **S-algebra** is a generalized cohomology theory with a cup product that is associative up to infinitely coherent homotopy.*

An S-algebra E comes with $\wedge : E \times E \to E$ and $u : S \to E$





Mention: to make $\wedge$ work you need to have an <u>action of Σ_n</u> on X_n. Analogy with $Ch(R)$ shows that $A \otimes B \to B \otimes A$ takes $a \otimes b \mapsto (-1)^{|a|\cdot|b|} b \otimes a$. In our case we need the Σ_n to pick up the sign that contains the cost of moving b past a. Note: this works best if X_n is a simplicial set, but it all pushes through for any X_n with an action of Σ_n.

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An S-algebra E is an <u>S-module</u> because we have $S \wedge E \to E$. In particular, $S^i \wedge (S^j \wedge E) \cong (S^i \wedge S^j) \wedge E \cong S^{i+j} \wedge E$.

An E-module X has $E \wedge X \to X$ satisfying the usual action rule.

EXAMPLE: HR = Eilenberg-Maclane spectrum. This puts $K(R_n, n)$ in dimension n so the homotopy is $R = (R_n)$. This is a way for rings to sit in the category of S-algebras.

EXAMPLE: Chain complexes are $H\mathbb{Z}$-modules with $H\mathbb{Z} \wedge Y \to Y$ simply multiplication as an $H\mathbb{Z}$-module map. There's a spectra map going the other way, but not an $H\mathbb{Z}$-module map. Indeed, we have a Quillen equivalence: $\mathcal{D}(H\mathbb{Z}) \to \mathcal{D}(\mathbb{Z})$ =ChainComplexes. This is because cofibrant objects are built of $\Sigma^i H\mathbb{Z}$'s and these are simply n-cells mod $(n-1)$-cells. Done because $C_n M = \pi_n(M^n, M^{n-1})$.

3. Algebraic Motivation: why we care about dimension

Moral: Algebra $\subseteq$ Homological Algebra $\subseteq$ Stable Homotopy Theory

Krull Dim of $R = \sup\{P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n \mid \text{ each } P_i \text{ is a prime ideal of } R\}$. Note: this definition fails for spectra because spectra have no points, hence no good definition of prime ideal.

The simplest rings are fields, which clearly have Krull dim zero because no ideals. Dimension is telling us about complexity.

Dimension gives amazing theorems in algebra:

R is **semisimple** iff all modules over R are projective iff R is a direct sum of simple submodules. My favorite way to define such a ring as one with **global dimension zero**

Semisimple implies Artinian and Noetherian.

Theorem 1 (Artin-Wedderburn Theorem). *R is semisimple iff $R = R_1 \times \cdots \times R_n$ where $R_i = M_n(D)$ for D a division algebra*

Maschke's Theorem says k[G] is semisimple, so it sufficies to study irreducible representations

A ring R is right hereditary if every right ideal is projective as a right R-module. True iff r.gl.dim$(R) \leq 1$

Theorem 2 (Serre's Theorem). *If commutative R has finite global dimension then R is regular, i.e. for all prime $\mathcal{P}$, the min number of generators for $\mathcal{M} \subset R_{\mathcal{P}}$ is Krull dim($R_{\mathcal{P}}$).*

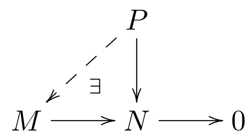
Note: Converse holds if R is semilocal, i.e. $R/\operatorname{rad}(R)$ is artinian, hence semisimple.

A point x on a algebraic variety X is nonsingular if and only if the local ring $\mathcal{O}_{X,x}$ of germs at x is regular

Lam 5.84 (Serre): R is commutative Noetherian local ring then gl.dim$(R) < \infty$ iff R is a regular local ring. In this case gl.dim = Krull dim.

4. Global Dimension

We say module P is **projective** if:



A module M is **flat** if the functor $-\otimes_R M$ is exact.

A **projective resolution** of M is $\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$, with all the P_i's projective.

Definition 3. ***Projective dimension*** $= pd(M) =$ *min. length of a projective resolution.*

Ex: If P is projective, $\text{pd}(P) = 0$ since $\cdots \to 0 \to 0 \to P \to P \to 0$ is a projective resolution.

Ex: For $R = \mathbb{Z}$, $\text{pd}(\mathbb{Z}/n) = 1$ since $\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ is minimal projective resolution, where the first map is mult by n and the second is quotient.

Examples: $\mathbb{Q} \oplus \mathbb{Z}$ is flat but not injective or projective. $\mathbb{Q}/\mathbb{Z}$ is injective but not projective or flat. $\mathbb{Z}$ is projective but not injective. Injective $\mathbb{Z}$-modules are exactly divisible groups.

Definition 4 (Right Global Dimension)**.** *r.gl.dim(R)* $= \sup\{pd(M) \mid M \in R - mod\}$

Ex: $\text{r.gl.dim}(k[x_1, \ldots, x_n]) = n$ because of the module $(x_1, \ldots, x_n)$

Ex: $\text{r.gl.dim}(k[x]/(x^2)) = \infty$ because k is an R-module and the minimal projective resolution is an infinite chain $\cdots \to k[x]/(x^2) \to k[x]/(x^2) \to k \to 0$, where each map takes $x \to 0$ and $1 \to x$. You can't get a smaller projective resolution because if you take this one and Hom into k then times t becomes zero and you get infinitely many non-zero Ext terms.

Fact: $\text{r.gl.dim}(R) = 1 \Rightarrow$ submodules of projective modules are projective. This is the next simplest ring after a semisimple ring. Ex: all PIDs.

Definition 5 (Weak Dimension)**.** $r.w.dim(R) = \sup\{fd(M) \mid M \in R - mod\}$

Projective $\Rightarrow$ Flat, so $\text{r.w.dim}(R) \leq \text{r.gl.dim}(R)$. If R is Noetherian then $\text{w.dim}(R) = \text{r.gl.dim}(R)$ because $fd(M) = pd(M)$ for all M.

R is **Von Neumann Regular** iff $\text{w.dim}(R) = 0$ iff all modules over R are flat.

$\text{w.dim}(R) = 1$ implies submodules of flat modules are flat.

$\text{r.gl.dim}(R) = 1$ implies submodules of projective modules are projective.

5. Derived Category

The correct category to study modules over an S-algebra E is $\mathcal{D}(E)$. Objects are E-modules, maps from M_1 to M_2 are $\{S-\text{algebra morphisms:} M_1 \to M_2\}/\sim$ where $f \sim g$ if $f = g \circ s^{-1}$ and s_* is an isomorphism.

CORRECT CATEGORY because triangulated. There is no abelian category of E-modules, i.e. none where you can add maps and get kernels and cokernels. Triangulated is the best you can do. It's also compactly generated and has derived tensor products and derived Hom objects.

Definition 6. *A map $f : X \to Y$ in $\mathcal{D}(E)$ is* ***ghost*** *if $f_* = 0$*

Such maps CANNOT BE SEEN BY π_*

EXAMPLE: any map from $HR \to \Sigma^k HR$ is ghost if $k > 0$ because $\pi_n(HR) = R$ iff $n = 0$, so $\pi_n(\Sigma^k HR) = \pi_{n-k}(HR) = 0$

We have a categorical equivalence: $\mathcal{D}(HR) \cong \mathcal{D}(R)$. We also have $\mathcal{D}(S) \to \mathcal{D}(H\mathbb{Z})$ via $X \mapsto H\mathbb{Z} \wedge X$ i.e. extension of scalars. Going the other way send $Y \mapsto Y$ and it's restriction of scalars.

$X \in \mathcal{D}(E)$ is **projective** iff X_* is a projective E_*-module. Define $\mathrm{pd}(X) = 1$. Projective E_*-modules are realizable.

Proof of Realizable: Retracts of coproducts of free will realize the projective guys. M is a retract of $\oplus_n \Sigma^n E_*$ so X will be a retract of $\bigvee_n \Sigma^n E$.

Definition 7. $\mathrm{pd}(X) \leq n+1$ *iff $Y \to P \to \widetilde{X} \to \Sigma Y$ with P projective,* $\mathrm{pd}(Y) \leq n$*, and X a retract of $\widetilde{X}$.*

6. Dimensions of Ring Spectra

Definition 8. *$pd(X) \leq n+1$ iff $Y \to P \to \widetilde{X} \to \Sigma Y$ with P projective, $pd(Y) \leq n$, X a retract of $\widetilde{X}$*

Definition 9. *$r.gl.dim(E) = \sup\{pd(X) \mid X \in \mathcal{D}(E)\}$*

Definition 10 (Ghost Dimension)**.** *$gh.dim(E) = \sup\{pd(X) \mid X \in \mathcal{D}(E)$ is compact$\}$*

Fact: $\mathrm{gh.dim}(E) \leq \mathrm{r.gl.dim}(E)$

Fact: $\mathrm{r.gl.dim}(E) = \mathrm{r.gl.dim}(E_*)$. Also, $\mathrm{gh.dim}(E) = \mathrm{gh.dim}(E_*)$.

Fact: If $E = HR$ then $\mathrm{r.gl.dim}(E) = \mathrm{r.gl.dim}(R)$. Same for weak dim.

Proposition 1. *$X \in \mathcal{D}(E)$ is projective iff the natural map $\mathcal{D}(E)(X, Y) \to \mathrm{Hom}_{E_*}(X_*, Y_*)$ is iso for all Y*

We use this in practice all the time, especially to show when ghosts are null.

Proposition 2. *$pd(X) \leq n$ iff every composite of $n+1$ ghosts $f_{n+1} \circ \cdots \circ f_1$ is null where $\mathrm{Dom}(f_1) = X$. This holds iff $E_2^{s,t} = \mathrm{Ext}_{E_*}^{s,t}(X_*, Y_*) \Rightarrow \mathcal{D}(E)(X, Y)_{t-s}$ has $E_\infty^{s,*} = 0 \ \forall\ s > n$*

Here we have algebra on the E_2 term converging to topology on the E_∞ term.

EXAMPLE: There are these really important, well-studied maps of cohomology theories called the Steenrod Squares. They're tricky to define properly, but one way is as an attempt to make the cup product be stable. Anyway, $\mathrm{Sq}^i : H^n(X; R) \to H^{n+i}(X; R)$. Given any spectrum X, let $a \in H^n(X)$. Then we have $X \xrightarrow{a} H\mathbb{F}_2 \xrightarrow{Sq^i} \Sigma^{n+i} H\mathbb{F}_2 \to \dots$. The composite of Sq^i and a is a ghost, even if a is not. For a specific example, let $X = \mathbb{R}P^k$ (sits in spectra as $\Sigma^\infty \mathbb{R}P^k$) and let $\langle a \rangle = H^1(\mathbb{R}P^k; \mathbb{F}_2) = \mathbb{F}_2$. Sq^1 is a ghost.

EXAMPLE: $\mathrm{gh.dim}(S) = \infty$. Suppose it was $n < \infty$. Consider $X = \mathbb{R}P^k$ for $k = 2^{n+1}$. Using a above:
$X \xrightarrow{a} \Sigma^1 H\mathbb{F}_2 \xrightarrow{Sq^1} \Sigma^2 H\mathbb{F}_2 \xrightarrow{Sq^2} \Sigma^4 H\mathbb{F}_2 \dots$. We get a composite of n ghosts which is non-null.
Hence, $\mathrm{pd}(X) \geq n+1$ so $\mathrm{gh.dim}(S) \geq n+1$, contradiction.
Note: $\mathrm{Sq}^i : \Sigma^n H\mathbb{F}_2 \to \Sigma^{n+i} H\mathbb{F}_2$. Applying π_n we see LHS$\neq 0$ but RHS$= 0$. So Sq^i is ghost.

7. Analogy to Ring Theory Holds

Recall: **depth**(R) = length of the longest regular sequence $((x_1, \dots, x_n)$ s.t. $\sum x_i R \neq R$ and x_i not a zero-divisor in $R/(x_1 R + \cdots + x_{i-1}R))$

Theorem 3. *If E is a commutative S-algebra then depth$(E_*) \leq$ gh.dim$(E) \leq \min\{$w.dim(E_*),r.gl.dim$(E) \leq$ r.gl.dim(E_*)*

Theorem 4. *If E is a commutative S-algebra and E_* is Noetherian with gl.dim$(E_*) < \infty$ then gh.dim(E) =r.gl.dim(E) = r.gl.dim(E_*)*

Proof: by Serre every $R_{\mathcal{P}}$ is regular local since R is commutative of finite global dim. Thus, R is Cohen-Macaulay by defn: all localizations at prime ideals are local Cohen-Macaulay (because regular local rings). So depth = gl.dim.

Cool fact: Commutative R is a regular local ring iff gl.dim$(R) < \infty$.

We need to have the Noetherian condition on E_* because without IDEALS we have no definition for E to be Noetherian.

Key fact: Regular local rings are Cohen-Macaulay. The proof uses Auslander-Buchsbaum: Let (R, P) be a local ring. If M is a finitely presented R-module of finite projective dimension then pd(M) =depth$(P, R)-$depth(P, M). If R is Noetherian, regular, and local then all modules are finitely generated, so that condition means nothing. Also, pd(M) is the length of every minimal free resolution of M.

All we need to prove is that if gl.dim$(R) < \infty$ then depth(R) =gl.dim(R). This is page 482 of Eisenbud. Let $k = R/\mathcal{M}$ and let $x_1, \dots, x_n$ be a minimal set of generators for $\mathcal{M}$. By the Principal Ideal Theorem (about codim$(P) \leq c$ where P is minimal prime containing $x_1, \dots, x_c$) we have Krull dim $(R) \leq n$. We must show depth$(R) \geq n$ since depth is $\leq$ Krull dim. Turns out the Koszul complex $K(x_1, \dots, x_n)$ has length n and is contained in the minimal free resolution of k. Thus, pd$(k) \geq n$ and this proves depth$(R) \geq n$.

Fact: E_* semisimple $\Rightarrow E$ semisimple. We'll see in a moment the converse is false.

EXAMPLE: $E_{n*} = W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u, u^{-1}]$ so gh.dim E_n = gl.dim E_n = gl.dim $E_{n*} = n$. First, $W\mathbb{F}_{p^n}$ is the Witt Ring of $\mathbb{F}_{p^n}$. It's the smallest DVR in characteristic zero with residue field $R/\mathcal{M} \cong \mathbb{F}_{p^n}$ and it's unique up to isomorphism. This is a local field and a finite extension of $\mathbb{Z}_p$ of degree p^n. The chain of extensions $\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \dots$ corresponds to $W(\mathbb{F}_{p^n}) \xrightarrow{p^n} \mathcal{P} \xrightarrow{p^n} \mathcal{P}^{\in} \dots$.

Next, $W\mathbb{F}_{p^n}$ has dimension 1 because it's a ring of integers in a field. Similarly, it's Noetherian. Adjoining $n-1$ variables gives dimension $1+n-1 = n$ and keeps it Noetherian. Because we adjoin both u and u^{-1} they are units and so it doesn't affect dimension.

8. Analogy to Ring Theory Almost Holds

Theorem 5. *A semisimple S-algebra E with E_* commutative has $E_* \cong R_1 \times \cdots \times R_n$ where each R_i is either a graded field k or an exterior algebra $k[x]/(x^2)$ over a graded field k*

Thus, E semisimple $\not\Rightarrow E_*$ semisimple

Corollary: r.gl.dim$(E) = 0 \Rightarrow E_*$ is quasi-Frobenius, hence 0-Gorenstein, i.e. R has injective dimension 0 as an R-module

Conjecture: $\text{r. gl. dim}(E) = n \Rightarrow E_*$ is n-Gorenstein.

Theorem 6. *Suppose $E \to F$ in S-alg gives F_* free over E_*. Then* $\text{gh. dim}(E) \leq \text{gh. dim}(F)$

Proof: Consider the functor $F \wedge_E (-) : \mathcal{D}(E) \to \mathcal{D}(F)$ and its right adjoint the restriction functor. Because F_* is flat over E_* we have $F_* \otimes_{E_*} X_* \to (F \wedge_E X)_*$ is an iso. Both sides are homology functors and it's an iso for $X = E$ so it's an iso for all X. Thus, $F \wedge_E (-)$ preserves ghosts since a ghost in E has $\pi_* f = 0$ so it's image is ghost in F. If g is n-ghosts with domain compact in $\mathcal{D}(E)$ then $F \wedge_E g$ is n-ghosts with domain compact in $\mathcal{D}(F)$. Because F_* is free over E_* we know $F \wedge_E X$ is a coproduct of copies of X as an E-module, so g is a restriction of $F \wedge_E g$. This means we can't have $F \wedge_E g = 0$ unless $g \neq 0$.

9. Analogy to Ring Theory Fails

KO is 2-local periodic real K-theory

$KO_* = \mathbb{Z}_{(2)}[\eta, w, v, v^{-1}]/(\eta^3, 2\eta, w\eta, w^2 - 4v)$ where $\langle \eta \rangle = \pi_1(KO), \langle w \rangle = \pi_4(KO), \langle v \rangle = \pi_8(KO)$. Infinite global dim.

$H\mathbb{F}_2^*(KO) = [KO, H\mathbb{F}_2]^*$ = maps of spectra. This is an $\mathcal{A}-$module but not a ring. It's zero.

$(H\mathbb{F}_2)_*(KO) = \pi_*(H\mathbb{F}_2 \wedge KO) = [S^0, H\mathbb{F}_2 \wedge KO]$. It's zero.

ko is 2-local connective real K-theory

$KO = v^{-1}ko$, specifically it's the direct limit of $ko \overset{\cdot v}{\to} \Sigma^{-8} ko \overset{\cdot v}{\to} \dots$

$ko_* = \mathbb{Z}_{(2)}[\eta, w, v]/(\eta^3, 2\eta, w\eta, w^2 - 4v)$

$K_* = \mathbb{Z}[u, u^{-1}]$ with $|u| = 2$. $KO_* = \mathbb{Z}[\eta, w, v, v^{-1}]/(\eta^3, 2\eta, w\eta, w^2 - 4v)$. We work in $\mathbb{Z}_{(2)}$

$k_* = \mathbb{Z}[u]$ with $|u| = 2$. $ko_* = \mathbb{Z}[\eta, w, v]/(\eta^3, 2\eta, w\eta, w^2 - 4v)$

A max ideal in KO_* is $(\eta, w, 2)$. Certainly any max ideal must contain η because all prime ideals do. Kill off η and you have $\mathbb{Z}_{(2)}[w, v, v^{-1}]/(w^2 - 4v)$. If $2 \in M$ then so is w because v is a unit and we modded out by $w^2 - 4v$. But 2 must be in M to have $\mathbb{Z}_{(2)}/M$ be a field. If $2 \notin M$ then $2x + z = 1$ for some x, z. This means $\langle \{2\} \cup M \rangle = R$, a contradiction of maximality.

Theorem 7. $1 \leq \text{gl. dim}\, KO \leq 3$ *and* $4 \leq \text{gl. dim}\, ko \leq 5$

$C(\eta)$: $C(\eta^2)$

Lower bound: $\text{Sq}^2\text{Sq}^1\text{Sq}^2\text{Sq}^1 \neq 0$ for $ko-$module $ko \wedge A(1)$.

Upper bound: Follow Bousfield and view a $KO-$module as a CRT-module where CRT$= \{KO_*, K_*, KSC_*\}$

Thus, gl.dim $E_* = \infty$ means we cannot apply our theorems

More general "build from" statement does exist (here we built ku from $ko \wedge C(\eta)$):

Theorem 8. *If E is an S-algebra and X is a spectrum s.t. r.gl.dim($E \wedge X$) $= m$, pd(X) $= k$, and S can be built from X in ℓ steps then gl.dim(E) $\leq (k+1)(\ell+1)(m+1) - 1$*

If X is a finite type zero spectrum then S can be built in finitely many steps. These spectra have rational homology not equal to zero, so $H\mathbb{Q}$ detects them. Type n is the first time your n-th K-theory is nonzero. Same type implies can build one from another using coproducts, suspensions, and retracts. A is a retract of B if $A \to B \to A$. For KO example, ℓ could be huge, so the theorem just gives gl.dim(KO) $< \infty$. Finding the right X might make ℓ small, but $m = k = 1$ still.

Cor: $\mathrm{r.gl.dim}(E) = n \not\Rightarrow E_*$ is n-Gorenstein. A counterexample is KO because it's not n-Gorenstein for any n. We can see this because Gorenstein is a special case of Cohen-Macaulay, and KO_* is not Cohen-Macaulay: Krull dim $= 1$ (prime ideals are (η) and $(\eta, 2, w)$) but Depth $= 0$ (no non-zero divisors in the maximal ideal, so if x is any non-unit then x is a zero divisor, so no regular sequences at all).

From $KO \to K$ is complexification, i.e. $- \otimes \mathbb{C}$. From $K \to KO$ is forgetting. Doing c then forgetting gives two copies of the original vector bundle. Thus, if 2 is a unit we have K is a KO-summand.

10. Future Directions

(1) Find the exact global dim for KO and ko

(2) Use the theory of ideals to define Noetherian E and depth for E. Relate to E_* notions. Make the analogy to algebra stronger

(3) Other notions of dimension I didn't mention: find examples, see which are equivalent, relate to E_*