

JUVITOP ON BOUSFIELD LOCALIZATION

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1. MOTIVATION AND HISTORY

This section is mostly taken from Margolis [7]

Homotopy theory studies topological spaces up to homotopy, so it must study the functor $\pi_* : Top \rightarrow \Pi$ -algebras. Passage to the homotopy category is declaring π_* -isomorphisms to be isomorphisms. Because this functor is difficult to compute, one way to do homotopy theory is to study simpler functors which also map to graded abelian categories, e.g. $\pi_*^{\mathbb{Q}}$, $H_*(-; \mathbb{Z})$, K_* , or more generally E_* for E some generalized homology theory. The study of these functors reveals information about spaces as seen through the eyes of these various homology theories. It also reveals information about the spectra representing these homology theories, namely how much of Top they can “see.” To study this we must work in a category where the isomorphisms are exactly the E_* -isomorphisms, i.e. the maps f such that E_* is an isomorphism. These are the maps seen as isomorphisms by E . Is there always such a category? What about if we’re working in Spectra instead of Top?

In full generality, when you have a category \mathcal{C} and a class of maps \mathcal{W} to invert, you’re asking for a **functor $\mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$ which is universal with respect to the property that it takes \mathcal{W} into isomorphisms.** The difficulty comes in proving $\mathcal{C}[\mathcal{W}^{-1}]$ exists. One way to do this is to show that \mathcal{C} has a model structure with weak equivalences \mathcal{W} . There are other ways as well which we’ll discuss below, but we note here that in the end it is possible to do this in a model category framework and to get a model structure with \mathcal{W} as the weak equivalences. Let’s fix some notation:

An object B is **\mathcal{W} -local** if for all $v : X \rightarrow Y$ in \mathcal{W} , we have a bijection $\mathcal{C}(Y, B) \rightarrow \mathcal{C}(X, B)$. This means B can’t see the difference between \mathcal{W} and isomorphisms. A **\mathcal{W} -localization** of an object A is a map $w : A \rightarrow B$ where $w \in \mathcal{W}$ and B is \mathcal{W} -local. It is easy to see that w is universal, so B is unique up to \mathcal{W} -isomorphism. In particularly nice cases, \mathcal{W} is the class of maps f such that $L(f)$ is an isomorphism for a nice endofunctor L . In particular, call $L : \mathcal{C} \rightarrow \mathcal{C}$ a **Bousfield localization functor** if there is a natural transformation $\eta : 1 \rightarrow L$ such that $\eta_{LX} = L\eta_X : LX \xrightarrow{\cong} L^2X$. When the class \mathcal{W} is the E_* -isomorphisms for a homology theory E , we can make these definitions nicer. Namely, C is **E -acyclic** if $E_*(C) = 0$ and X is **E -local** if $[C, X] = 0$ for all E -acyclic C . So E -localization depends on E -acyclics only (i.e. same acyclics implies $L_E = L_F$).

Now, it’s easy to see that the objects of $\mathcal{C}[\mathcal{W}^{-1}]$ should be the same objects as in \mathcal{C} . To construct the morphisms you need to know that \mathcal{W} satisfies a **calculus of fractions** (see [4]). Formally, this means you need \mathcal{W} to contain id_X for all X , to satisfy the Ore condition (if $X \rightarrow Y \leftarrow Z$ and $\leftarrow \in \mathcal{W}$ then there is some $X \leftarrow K \rightarrow Z$ making the diagram commute and $\leftarrow \in \mathcal{W}$), and to satisfy cancellability (if $X \xrightarrow{\rightarrow} Y \rightarrow^w Z$ coequalizes the two arrows then there’s $K \rightarrow^w X$ which equalizes them). Informally, these rules are exactly what you need to define the morphisms by equivalence classes of spans $A \leftarrow \bullet \rightarrow B$ where the backwards arrow is in \mathcal{W} . The calculus of fractions tells

you this is well-defined, has a composition rule, is associative, and contains identity maps. The problem is that you **might have a class worth of maps between two objects**.

To get around this set theoretic issue, you should prove an equivalence of categories $\mathcal{C}[\mathcal{W}^{-1}] \cong \mathcal{C}_{\mathcal{W}}$ the full subcategory of \mathcal{C} consisting of \mathcal{W} -local objects. Because it's a subcategory of \mathcal{C} , the morphism problem is resolved. It's easy to see that all this works in the case where \mathcal{W} is the L -isomorphisms for some Bousfield localization L ; this functor L is both directions of the equivalence. In general, however, to prove this equivalence of categories you must prove every object $A \in \mathcal{C}$ admits a \mathcal{W} -localization. This construction is where the hard work is done. Once you know this you define a functor L which takes A to its \mathcal{W} -localization and you must prove this is a functor unique up to \mathcal{W} -equivalence. It is because of the universal property of \mathcal{W} -localization.

Specializing the previous paragraph, one could take \mathcal{C} to be the homotopy category and \mathcal{W} to be the E_* -isomorphisms. However, one could also take \mathcal{C} to be the Quillen model structure on spaces and \mathcal{W} to be the E_* -isomorphisms. This gives the calculus of fractions (on a subcategory of fibrant objects), gives tools to make the proof that $\mathcal{C}[\mathcal{W}^{-1}] \cong \mathcal{C}_{\mathcal{W}}$ go easier, and gives a better understanding of the morphisms in $\mathcal{C}[\mathcal{W}^{-1}]$. It also gives a stronger conclusion, namely that we get a model structure whose weak equivalences are generated in a nice way by the E_* -isomorphisms and whose homotopy category is $HoTop[\mathcal{W}^{-1}]$. We'll discuss how model categories enter the picture at the end of the talk, but record here that even Bousfield's original proof for spectra required him to **take a homotopy colimit, i.e. to work in the (model) category of spectra** rather than its homotopy category. Furthermore, Bousfield's proof that every object admits E -localization is strongly reminiscent of the **small object argument** which appears in the study of model categories.

In 1975, Bousfield [2] proved that there is a homotopy category of spaces whose isomorphisms are the E_* -equivalences (in this case, \mathcal{W} -localization is also known as E -localization L_E). Around this same time, Ravenel was studying localization of spectra, and in particular things like $v_n^{-1}BP$. These localizations were the spectra analogue of obvious localizations in *GrRing* after applying π_* . They could easily be seen to exist via general realization principles, and Ravenel had also done a nice job getting his hands on them concretely enough to use them for computations in the non-local world. By 1979, Bousfield [3] had extended his proof about spaces to work also for homology theories of spectra (interestingly, it fails for cohomology theories, as we'll discuss next section). Furthermore, Bousfield proved an **analogue of Whitehead's Theorem**, which lets you get back from local data on spaces to global data. This says that an **E -equivalence between E -local spaces lifts to a weak equivalence**. In practice, most people use the universal property of E -localization rather than digging into Bousfield's proof. However, we'll sketch it next section anyway just to show that it's not insanely mysterious.

2. BOUSFIELD'S PROOF

This section is mostly from Bousfield's 1979 paper [3] and Margolis's book [7], but also from Nerses Aramian's lecture notes [1] for a talk he gave at UIUC.

2.1. Making it work for a small subcategory. Adams and Baird (unpublished, early 70s) noticed that for a small category \mathcal{C} one does not have the set theoretic issues mentioned above. They noticed that everything worked when you restrict from *Top* or *Sp* to the subcategory of objects less than some huge cardinal κ (formally, this is the subcategory where $|\pi_*(X)| < \kappa$ for all X). In particular, one can construct the E -localization of any object by iteratively taking wedges of E -acyclic objects (attaching E -acyclics does not change E -homology) and passing to cokernels

and colimits to make sure the map from X to $L_E(X)$ is an E -equivalence. The construction goes basically as follows:

- Base Case: $X_0 = X$
- Successor Case: Given X_λ , define $X_{\lambda+1}$ as the cofiber of $\bigvee_{f:A \rightarrow X_\lambda} A \rightarrow X_\lambda$, where A runs through all E -acyclic objects of our small category
- Limit Case: Define $X_\lambda = \operatorname{colim}_{\gamma < \lambda} X_\gamma$

Let $L_E(X) = X_\kappa$ for the huge κ above. For any A we show $[A, L_E(X)] = [QA, L_E(X)] = 0$ where QA is the cofibrant replacement of A in Sp . Note that QA necessarily contains less than κ many cells, so that $QA \rightarrow L_E(X)$ factors through some X_λ and so $QA \rightarrow X_\lambda \rightarrow X_{\lambda+1}$ is null by definition of the cokernel. This proves $L_E(X)$ is E -local. Furthermore, $X \rightarrow L_E(X)$ is an E -equivalence. To see this, note that it holds for all $X_\lambda \rightarrow X_{\lambda+1}$ by the cofiber sequence in E -homology, and if $X \rightarrow X_\gamma$ is an E -equivalence for all $\gamma < \lambda$ then $E_*(X_\lambda) = E_*(\operatorname{colim} X_\gamma) \cong \operatorname{colim} E_*(X_\gamma)$, so $X \rightarrow X_\lambda$ is an E -equivalence. This is why the proof **breaks down for cohomology theories: they do not commute with colimit or hocolimit**. So far the only known way to prove one can localize at cohomology theories (due to Casacuberta, Scevenels, and Smith) is to assume a large cardinal axiom known as **Vopenka's Principle** which makes this part of the argument unnecessary.

2.2. Getting from a small subcategory to all of Sp via hocolims. The problem with Adams and Baird's proof is that they make no mention of why one should be allowed to pass from Sp to $Sp_{<\kappa}$. How do we know that the E -equivalences remain unchanged? How do we know that X_κ matches X_ρ for some $\rho > \kappa$? Or that in the end these constructions will stabilize to the same $L_E(X)$? The trick is to replace the cokernel above by the cofiber on the model category level and to **replace the colimit by a homotopy colimit**. In Jacob Lurie's language, what is needed is to **show that there is a small subcategory \mathcal{C}_0 which generates Sp under homotopy colimits**. The method Bousfield originally used to do this came to be known as proving homology functors are cardinality continuous (cf Margolis). Morally, this works because we can take for **objects of \mathcal{C}_0** the set S of homotopy type representatives for spectra A **with** $|\pi_* A| < |\pi_* E|$. The proof above shows that $[A, X] = 0$ for all these A , and Bousfield's work shows this is enough to conclude $[B, X] = 0$ for all spectra B because **any B can be represented as a homotopy colimit of spectra in S** .

The key is to take any map $f : X \rightarrow A$ where $|\pi_*(X)| < |\pi_* E|$ and A is E -acyclic, and factor f through some E -acyclic $X^{(1)}$ with $|\pi_*(X^{(1)})| < |\pi_* E|$. Do this first for finite X , then for general Y by writing Y as a homotopy colimit of a directed system of finite X . Next get a chain $Y^{(n)}$ and define $Y^{(\infty)}$ as the homotopy colimit. Then **any E -acyclic A can be written as a hocolim of spectra X with $|\pi_*(X)| < |\pi_* E|$** by applying the above to get $A \simeq \operatorname{hocolim} X_\alpha \rightarrow \operatorname{hocolim} X_\alpha^{(\infty)} \rightarrow A$ and showing this is a π_* isomorphism.

Use this result to conclude that **if a functor $F^*(C)$ is trivial on the subset of X with $|\pi_*(X)| < |\pi_* E|$ then $F^*(C)$ is trivial for all C** (we'll apply this to $F = [-, L_E(X)]$). The proof writes C as a hocolim of N_α in the set and uses the Milnor Exact Sequence $\lim^1 F^{*-1}(N_\alpha) \rightarrow F^*(C) \rightarrow \lim F^*(N_\alpha)$ to conclude $F^*(C)$ is trivial. Applying all this to the Adams and Baird proof shows that it's enough to check $[A, L_E X] = 0$ on this subset of A , so completes the proof that $L_E(X)$ is an E -localization.

In the end, Bousfield constructed an E -localization $A \rightarrow A_E$ where A_E comes out of this complicated construction. Actually, one has a triangle in the homotopy category: $C_E A \rightarrow A \rightarrow A_E \rightarrow \Sigma(C_E A)$ where $C_E A$ is E -acyclic.

2.3. Margolis's proof. Margolis's Chapter 7 goes back to the Adams and Baird proof (i.e. the calculus of fractions version) and add an axiom on the functor we're trying to invert. In Proposition 3, he proves: if $P : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is exact, preserves coproducts, and has factorization for maps $h : P(X) \rightarrow P(Y)$ through $P(U)$ where $X \rightarrow U \leftarrow Y$ with the backwards arrow in \mathcal{W} , then there is an H -localization functor.

This isolates the difficulty to forcing $\mathcal{C}[\mathcal{W}^{-1}](X, Y)$ to be small. Margolis adds an axiom to a functor $H : \mathcal{C} \rightarrow \text{Set}$ which makes this work. H is said to be **cardinality continuous** if for each cardinal c there is a cardinal $d = d(c)$ such that $|H(X)| \leq c$ implies there is some $Y \in \mathcal{C}_{<d}$ and $f : Y \rightarrow X$ where $H(f)$ is a bijection. More or less, this is assuming exactly what Bousfield needed in his proof. Margolis proves in Proposition 5 that homology functors have this property. Again, at a critical moment he must use the fact that homology functors commute with colimits.

The benefit of Margolis's approach is that it highlights what one would need to prove that cohomological localization exists:

Proposition 1 (Proposition 6). *Let H be a cohomology functor represented by a spectrum W and suppose there is a cardinal $e = e(W)$ such that for all $Y \in \text{Sp}$ and nonzero $f : Y \rightarrow W$ there is some $g : Z \rightarrow Y$ in $\text{Sp}_{<e}$ with $fg \neq 0$. This occurs if there are no f -phantom maps to W . Then H is cardinality continuous.*

2.4. Extra Facts (courtesy of Bousfield 1979).

- If X is a module spectrum over a ring spectrum E (i.e. $X \in E\text{-mod}$) then X is E -local.
- The **product** of a set of E -local spectra is E -local.
- A **retract** of an E -local spectrum is E -local.
- E -localization commutes with Σ , with Σ^{-1} and with finite wedge.
- L_E need not preserve smash products, but it does preserve monoids, commutative monoids, A_∞ -algebras, and E_∞ -algebras.

Recall that L_E is determined by the E -acyclics. There is an E -acyclic spectrum aE such that the smallest class containing aE closed under coproducts and the two-out-of-three property on triangles is the whole class of E -acyclics. This aE is incredible: its existence shows that **the class of E -acyclics is generated by a single object**. He proves it exists, but it's difficult to get your hands on. The construction is similar to the theorem above, i.e. goes by transfinite induction.

One can use the existence of E -localization to make an E_* -Adams Spectral Sequence $E_2^{s,t}(X, Y) = \text{Ext}_{E_*E}^{s,t}(E_*X, E_*Y) \Rightarrow [X, E \wedge Y]_*$, where convergence occurs iff $\lim_r^1 E_r^{s,t}(X, Y) = 0$ for all s, t .

3. EXAMPLES AND APPLICATIONS

This section is mostly from Bousfield [3] and Ravenel [9], [8]

3.1. Arithmetic Localization and Completion. From Tom Goodwillie at <http://mathoverflow.net/questions/8of-maps-between-p-local-and-q-local-spectra>

“Localizing a spectrum at a prime p (or at a set of primes) is the kind of Bousfield localization that is most like localization in algebra. In this case π_n of the localization of X is the algebraic localization of $\pi_n X$, i.e. the result of inverting all the primes other than p . The local spectra are the spectra X such that for every n the abelian group $\pi_n X$ is one on which every prime other than p acts invertibly, and the acyclic objects are the ones which become trivial upon localizing, i.e. the

spectra X such that for every n the group $\pi_n X$ is a torsion group without p -torsion. Localization of X is the same as smash product of X with the localization of the sphere spectrum. If you want to describe the acyclic objects as the E_* -acyclic objects for some E , you can of course let E be the localization of the sphere. Note that we are not starting with an E and using the general machine of Bousfield to make the localization functor, although of course we could.

Completing at a prime is again not too far from being a purely algebraic matter. Here we can choose E to be the mod p Moore spectrum, i.e. the homotopy cofiber of the map $p : S \rightarrow S$ from the sphere spectrum to itself. The acyclic objects are the spectra for which each homotopy group has p acting invertibly. The localization LX (which is called the p -completion) can be described as the holim, over natural numbers k , of the smash product of X with the mod p^k Moore spectrum. If each homotopy group of X is finitely generated, then $\pi_n LX$ can be described as the tensor product of $\pi_n X$ with the group \mathbb{Z}_p of p -adic integers, and also in this case LX is the smash product $X \wedge LS$. But neither of these statements is true for general X : for example, the p -completion of $H\mathbb{Q}$ is trivial while $\mathbb{Q} \otimes \mathbb{Z}_p$ is the field of p -adic numbers and $X \wedge LS = H(\mathbb{Q} \otimes \mathbb{Z}_p)$. And (therefore) $LH(\mathbb{Q}/\mathbb{Z}) = \Omega H\mathbb{Z}_p$."

From Bousfield...

Let J be a set of primes and let $\mathbb{Z}_{(J)}$ be the subring of \mathbb{Q} where p is invertible iff $p \notin J$. Let $X_{(J)} = X \wedge M(\mathbb{Z}_{(J)})$ for the Moore spectrum. This is arithmetic localization of spectra. Let X_p^\wedge be the p -adic completion of X , i.e. the inverse limit $\lim_n (X \wedge M(\mathbb{Z}/p^n))$. Let $X_J^\wedge = \prod_{p \in J} X_p^\wedge$. This is completion of spectra. We now show that both arithmetic localization and completion are special cases of Bousfield localization and indeed get a number theoretic **characterization for $L_E(X)$ if both X and E are connective**.

Theorem 1. *Let E_* be a connective homology theory and X a connective spectrum. Let J be complementary to the set of primes where E_i is uniquely p -divisible for each i . Then $L_E X = X_J^\wedge$ if each element of E_* has finite order, and $L_E X = X_{(J)}$ otherwise.*

In the same vein, the times p^n map on some X has cofiber $X \wedge M(p^n)$. The homotopical version of completion is the inverse holim $_n (X \wedge M(p^n))$. This turns out to equal $L_{M(p)} X$. The fact that **completion is a special case of localization** is one of the few places where algebraic topology is simpler than homological algebra.

K -theory is unbounded and had been studied separately, with various computations being done there which show the situation can be quite a bit more complicated, e.g. $\pi_{-2} L_K S^0 \cong \mathbb{Q}/\mathbb{Z}$.

3.2. Chromatic Homotopy Theory. In [8] Ravenel noticed lots of periodicity in the E_2 -term of the Adams-Novikov spectral sequence, saw that it came from algebraic periodicity of the E_1 -term of the chromatic spectral sequence, and attempted to find geometric periodicity (e.g. related to $BP_* BP$ -comodules) to explain that.

Let L_n denote $L_{E(n)}$ for Morava E -theory. Our goal is to computationally relate $BP_*(X)$ and $BP_*(L_n X)$. Recall that $E(0) = H\mathbb{Q}$ and $E(1)$ is one of the $p - 1$ isomorphic summands of K localized at p . Finding geometric descriptions of $E(2)$ is the motivation for elliptic cohomology (namely TMF). For the higher $E(n)$ it's supposed to be TAF.

Side Note: The E -locals for L_0 are $\{X \mid [M(p), X] = 0\}$, and the E -locals for L_1 are $\{X \mid [M(p, v_1), X] = 0\}$, i.e. for generalized mod p Moore space. In each case you can use Bousfield's work to find aE generating these and thereby get a good handle on what localization is doing. Incidentally, you might think L_2 is given by $\{X \mid [M(p, v_1, v_2), X] = 0\}$ but actually that's the finite version L_2^f . The telescope conjecture was asking if $L_2 = L_2^f$. We'll discuss this a little bit more below.

We can get an equivalence relation on spectra given by $E \sim F$ if $L_E = L_F$. The equivalence classes are called **Bousfield classes** and all together they form the **Bousfield lattice**. Remark: If $\langle E \rangle \leq \langle F \rangle$ then $L_F \rightarrow L_E$.

Fact: $\langle L_n \rangle = \langle E(n) \rangle = \langle v_n^{-1}BP \rangle = \bigvee_0^n \langle K(i) \rangle$

Using the last description and the remark, we get natural transformations $L_n \rightarrow L_{n-1}$ for all n . This gives the chromatic tower for a p -local spectrum X :

$L_0X \leftarrow L_1X \leftarrow L_2X \leftarrow \cdots \leftarrow X$ and hence a chromatic filtration of π_*X given by $\ker(\pi_*X \rightarrow \pi_*(L_nX))$. The **Chromatic Convergence Theorem** states that for a p -local finite CW complex X , this tower converges, i.e. $X \cong \varprojlim L_nX$

One form of the **telescope conjecture** states that for a type n complex X with a v_n -map f (this gives X an intrinsic periodicity), the telescope \widehat{X} of the tower $X \xrightarrow{f} \Sigma^{-d}X \xrightarrow{f} \Sigma^{-2d}X \xrightarrow{f} \cdots$ satisfies the property that $\widehat{X} \rightarrow L_nX$ is an equivalence. This is false for $n = 2$ (though true for the finite version of L_n), so we won't go that route. However, one can still do some useful computations, e.g. if $BP_*(X) = BP_*/I_n$ then $BP_*(L_nX) = v_n^{-1}BP_*/I_n$ and the E_2 term is known. In general, to compute $BP_*(L_nX)$ in terms of $BP_*(X)$, Ravenel proves and uses the **Localization Theorem**:

$BP \wedge L_nX = X \wedge L_nBP$ for any X and if $v_{n-1}^{-1}BP_*(X) = 0$ then $BP \wedge L_nX = X \wedge v_n^{-1}BP$, which implies $BP_*(L_nX) = v_n^{-1}BP_*(X)$. See section 6 of [8] for computations of $L_n(BP)$

Every localization satisfies the property that for all X , $E \wedge X$ is E -local. However, it need not be the case that $E \wedge X = L_EX$. A Bousfield localization is said to be smashing if $L_EX = X \wedge L_E(S^0)$. This occurs iff L preserves homotopy colimits and iff L commutes with all coproducts. Bousfield has spectral sequence conditions which help determine when this happens. Ravenel's **Smash Product Theorem** states that $L_nX \cong X \wedge L_nS^0$. This is very useful for computations and will probably come up again in the pre-Talbot seminar.

4. MODEL CATEGORIES

Model categories give a **general place you can do homotopy theory**, and this transforms algebraic topology from the study of topological spaces into a general tool useful in many areas of mathematics. The localization $\mathcal{M} \rightarrow \mathcal{M}[\mathcal{W}^{-1}]$ always lands in a homotopy category and always takes exactly the zig-zags of weak equivalences to isomorphisms. Let T be a set of maps in \mathcal{M} . Because the homotopy category is nice, we can do:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad \cdots \quad} & \text{?????} \\ \downarrow & & \downarrow \\ \text{Ho}(\mathcal{M}) & \longrightarrow & \text{Ho}(\mathcal{M})[T^{-1}] \end{array}$$

We'd like a model category $L_T\mathcal{M}$ which actually sits above $\text{Ho}(\mathcal{M})[T^{-1}]$. Because all three categories above have the same objects, its objects are determined. It's morphisms will be the same as those in \mathcal{M} , but we want the maps in T to become isomorphisms in $\text{Ho}(\mathcal{M})[T^{-1}]$ so we need them to be weak equivalences in $L_T\mathcal{M}$. So this category must have a **different model category structure**, where $\mathcal{W}' = \langle T \cup \mathcal{W} \rangle$ and clearly $\mathcal{W} \subset \mathcal{W}'$. You can't change only \mathcal{W} because it'll screw up the axioms. We want to keep the cofibrations fixed so we can build things out of them and have the two model structures related, so we have to shrink the fibrations: $\mathcal{F} \supset \mathcal{F}'$. If you assume \mathcal{M} is cellular and left proper then you can generalize Bousfield's Theorem (see Hirschhorn's book [5])

and get the model structure desired, but you have to be careful with how you generate \mathcal{W}' from T , i.e. you have to use simplicial mapping spaces.

The functor $\mathcal{M} \rightarrow L_T\mathcal{M}$ preserves a lot of properties. For many years everyone assumed it preserved monoids and commutative monoids (now you need a monoidal structure on \mathcal{M}) because it does for Spectra. Mike Hill (2011) showed that for the model category of G -equivariant spectra **Bousfield localization does NOT preserve commutative monoids**. My thesis finds hypotheses on \mathcal{M} and on the maps T so that Bousfield localization preserves strict commutative monoids, where strict means the diagrams commute on the nose rather than up to higher homotopy. The question for E_∞ is much easier. On the way I find conditions so that Bousfield localization preserves the pushout product axiom, the monoid axiom, and the Σ_n -equivariant monoid axiom that's needed on a model category \mathcal{C} in order to conclude commutative monoids in \mathcal{C} inherit a model structure.

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