INVERTIBLE SHEAVES

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1. Defining Sheaves

Definition 1. *Given a variety V, the* ***ring of regular functions*** *(or coordinate ring) is $R(V) = k[x_1, \ldots, x_n]/I(V)$. The functions in $R(V)$ are called* ***regular functions****. Note that $R(V)$ is a finitely generated k-algebra.*

Definition 2. *Given a commutative ring R define the* ***Spectrum*** *as $\operatorname{Spec}(R) = \{P \subset R \mid P$ is a proper, prime ideal$\}$.*

This can be augmented with the Zariski Topology in the following way: The closed sets in $|\operatorname{Spec}(R)|$ are those V such that there is some $I \subset R$ s.t. V consists of all prime ideals in R containing I. Formally, $V(I) = \{x \in |\operatorname{Spec} R| \ : \ f(x) = 0 \ \forall \ f \in I\} = \{[P] \in |\operatorname{Spec} R| \ : \ P \supset I\}$. A point in $|\operatorname{Spec}(R)|$ is a prime ideal P strictly contained in R. We will denote this $[P] \in \operatorname{Spec}(R)$. The only closed points are maximal ideals in R.

Define $\Gamma(D_f, \mathcal{O}_X) = R_f$ to be the localization of R (this is also a sheaf). If $P \in \operatorname{Spec}(R)$ then the stalk at P is a locally ringed space which is a localization of R at P. Note that with this topology, $|\operatorname{Spec}(R)|$ is compact and has as basis $\{D_f\}_{f \in \mathbb{R}}$ where $D_f = \{$prime ideals P such that $f \notin P\}$. Also, Spec is a contravariant functor from the category of commutative rings to the category of topological spaces and all ring homomorphisms $f : S \to R$ induce continuous maps: $\operatorname{Spec}(f) : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$. As a side note, $|\operatorname{Spec}(R)|$ is a structure sheaf, has a sheaf defined on it, and is a locally ringed space. We'll talk more about this in a moment.

Examples:

(1) If $R = \mathbb{Z}$ then $|\operatorname{Spec}(R)|$ consists of primes union (0) (this is a special point because it's dense).
(2) If $R = \mathbb{C}$ then $|\operatorname{Spec}(R)|$ consists of the complex line union (0). All points on the complex line are closed (because $(x - a)$ is a maximal ideal) but (0) is dense in R.
(3) $\operatorname{Spec}(\mathbb{C}[x]_{(x)})$ has only two points. It has the ideal (x) which is closed and it has the point (0) which is dense.
(4) If $R = \mathbb{Z}_p$ then $|\operatorname{Spec}(R)|$ is just the point $\{0\}$
(5) $\operatorname{Spec}(\mathbb{Z}[x])$ has four types of points: $(0), (p)$, irreducible polynomials f, and (f, p) where the polynomials are irreducible mod p.

For every f in R there is some related map on $|\operatorname{Spec}(R)|$. Also, if $[P] \in |\operatorname{Spec}(R)|$ then we have $R \to R/P$, an integral domain. This maps to the Residue Field at x, $K(x)$ which is just the quotient field of R/P. We define the value of f at $[P]$ as $f(x) \in K(x)$

Examples:

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(1) If $R = \mathbb{C}$, $a \in R$, and $P = (x-a)$ then we have $\mathbb{C}[x] \to \mathbb{C}[x]/(x-a) \cong \mathbb{C}$ so the value of $f(x)$ is $f(a) \in \mathbb{C}$. If $P = (0)$ then we have $\mathbb{C}[x] \to \mathbb{C}[x]/(0) \to \mathbb{C}[x]$ and $f \mapsto f + (0) \mapsto f \in \mathbb{C}[x]$

(2) If $R = \mathbb{Z}$, $f = 15$, and $x = [P] \in |\operatorname{Spec}(R)|$ then if $P = (0)$ we have $\mathbb{Z} \to \mathbb{Z}/(0) \to \mathbb{Q}$ and $15 \mapsto 15 \mapsto 15$. If $P = (p)$ for some prime p then we have $\mathbb{Z} \to \mathbb{Z}/(p) \to \mathbb{Z}_p$ and $15 \mapsto 15 + (p) \mapsto 15 \pmod{p}$

We now define a sheaf of functions on $X = |\operatorname{Spec}(R)|$, but note that shaves can be defined MUCH more generally on the open sets of any topology:

Definition 3. *A **sheaf** is an assignment which assigns to every open $U \subset X$ a set $\mathcal{O}_X(U)$ of regular functions on U (more generally, a set of "sections" on U) s.t.*

(1) $V \subset U \Rightarrow \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ via a restriction of U to V

(2) We obtain a category $\underline{X}$ with open sets as objects and inclusions as arrows such that $\mathcal{O}_X : \underline{X}^{op} \to \underline{Ring}$.

(3) Given an open cover $U = \bigcup_\alpha U_\alpha$, if $f \in \mathcal{O}_X(U)$ and $f|_{U_\alpha} = 0$ for all α then $f = 0$. Also, if $f_\alpha \in \mathcal{O}_X(U_\alpha)$ and $f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$ then there is some $f \in \mathcal{O}_X(U)$ such that $f|_{U_\alpha} = f_\alpha$ for all α. This is referred to as the gluing property. If this property is omitted we get a pre-sheaf.

One immediate fact is that $\mathcal{O}_X(X) = R$. Also, given $f \in R$ we can define R_f to be the localization where we invert all powers of f. We can then define $X_f = |\operatorname{Spec} R| \setminus V(f) = \{[P] \mid f \notin P\} \cong |\operatorname{Spec} R_f|$ and note that $\mathcal{O}_X(X_f) = R_f$. These two facts determine $\mathcal{O}_X$. Note that if $(X, \mathcal{O}_X)$ is a ringed space then the sheaf $\mathcal{O}_X$ is called the **structure sheaf** of X.

This is not a talk about sheaves in general, but I feel I must mention that a pre-sheaf can be turned into a sheaf by the functor of "sheafification" which is adjoint to the forgetful functor going from sheaves to pre-sheaves. I also feel I must define a stalk since you will hear people talking about them.

Definition 4. *The stalk $\mathcal{F}_x$ of a sheaf $\mathcal{F}$ captures the properties of a sheaf "around" a point $x \in X$ as we look at smaller and smaller neighborhood of x. Formally, $\mathcal{F}_x := i^{-1}\mathcal{F}(\{x\})$, where i is the inclusion map: $\{x\} \to X$.*

The natural morphism $F(U) \to F_x$ takes a section s in $F(U)$ to what is called its germ. This generalizes the usual definition of a germ.

2. Schemes and Coherent Sheaves

Definition 5. *An Affine Scheme is a pair $(|X|, \mathcal{O}_X)$ such that there exists a cover $|X| = \bigcup_\alpha U_\alpha$ and $(U_\alpha, \mathcal{O}_X|_{U_\alpha}) \cong (|\operatorname{Spec} R_\alpha|, \mathcal{O}_{\operatorname{Spec} R_\alpha})$. Here $|X|$ is a topological space and $\mathcal{O}_X$ is a sheaf of rings in $|X|$.*

Perhaps an easier way to think of this is as a locally ringed space isomorphic to $\operatorname{Spec}(A)$ *for some commutative ring A. A Scheme is a pair $(|X|, \mathcal{O}_X)$ which is locally an affine scheme (i.e. a locally ringed space which is locally isomorphic to the spectrum of a ring). Schemes form a category if we take as morphisms the morphisms of locally ringed spaces.*

Example: The smallest non-affine scheme is $|X| = \{p, q_1, q_2\}$ with open sets $\varnothing, \{p\}, X_1 = \{p, q_1\}, X_2 = \{p, q_2\}, X$ and sheaf $\mathcal{O}_X(\varnothing) = 0, \mathcal{O}_X(\{p\}) = \mathbb{C}[x], \mathcal{O}_X(X_1) = \mathcal{O}_X(X_2) = \mathcal{O}_X(X) = \mathbb{C}[x]_{(x)}$

Affine schemes are closely related to commutative rings and it's perfectly fine to think of all affine schemes as having $X = \operatorname{Spec}\mathbb{C}[x]_{(x)}$ for this talk. Basically, a scheme is a topological space together with commutative rings for all open sets which arise via gluing together spectra of commutative rings. Formally, it is a locally ringed space X admitting a covering by open sets U_i s.t. the restriction to the structure sheaf $\mathcal{O}_X|_{U_i}$ is an affine scheme. Every scheme comes equipped with a unique morphism to $\operatorname{Spec}(\mathbb{Z})$.

Sheaves contain locally defined data attached to the open sets of a topology. We can get the open sets to be arbitrarily small but we can also build up information from the local information in a coherent way (thanks to the gluing). There are also maps (or morphisms) from one sheaf to another; sheaves (of a specific type, such as sheaves of Abelian groups) with their morphisms on a fixed topological space form a category. On the other hand, to each continuous map there is associated both a direct image functor, taking sheaves and their morphisms on the domain to sheaves and morphisms on the codomain, and an inverse image functor operating in the opposite direction. These functors, and certain variants of theirs, are essential parts of sheaf theory.

Suppose $(X, \mathcal{O}_X)$ is a ringed space. Let $\mathcal{O}_X(U)$ be the ring of regular functions on U and let $\mathcal{F}(U)$ be an $\mathcal{O}_X(U)$ module. Note that the category of such modules is abelian.

Definition 6. *A **quasi-coherent sheaf** is a sheaf of $\mathcal{O}_X$-modules locally isomorphic to the cokernel of a map between free $\mathcal{O}_X$-modules.*

*A **coherent sheaf** F is a quasi-coherent sheaf which is locally of finite type and for all U open in X satisfies $\ker(\phi)$ is of finite type for all morphisms $\phi : \mathcal{O}_U$-module$\rightarrow F_U$ of finite rank.*

A sheaf of rings is coherent if it is coherent considered as a sheaf over itself. For affine varieties V with affine coordinate ring there is a covariant equivalence of categories between that of quasi-coherent sheaves and sheaf-morphisms on the one hand and that of R-modules and R-module homomorphisms on the other. If R is Noetherian then coherent sheaves correspond to finitely generated modules.

3. Invertible Sheaves

Definition 7. *An **invertible sheaf** is a coherent sheaf S on X (still a ringed space) for which there is an inverse T with respect to tensor product of $\mathcal{O}_X$-modules. That is, we have $S \otimes T$ isomorphic to $\mathcal{O}_X$, which acts as identity element for the tensor product.*

Invertible sheaves are locally free sheaves of rank 1. It is my goal to relate these objects to line bundles, divisors, and the Picard Group. The Picard Group is the group (under tensor product) of isomorphism classes of invertible sheaves on X. Note that Pic is a functor from the category of invertible sheaves to the category of abelian groups.

Definition 8. *Given a variety X over $\mathbb{C}$, attach a 1-dimensional vector space (i.e. a complex line) at each point to get the **Line Bundle** $\mathcal{L}$ over X.*

There is an open cover $\{U_i\}$ of X such that $\mathcal{L}|_{U_\alpha} \cong U_\alpha \times \mathbb{C}$. Consider $U_\alpha \cap U_\beta$. We may patch this as $U_\alpha \times \mathbb{C}|_{U_\alpha \cap U_\beta} \rightarrow U_\beta \times \mathbb{C}|_{U_\alpha \cap U_\beta}$ (where this map is called $g_{\alpha,\beta}$ and is not identically zero). More generally, we may let g be a linear transformation into the vector space of invertible matrices.

Theorem 1. *The line bundle $\mathcal{L}$ is an invertible sheaf if $\mathcal{L}(U)$ is a rank 1 $\mathcal{O}_X$-module.*

For the above theorem, we need to state how to patch because sheaves need this property. But again we can use $g_{\alpha,\beta}$. We require it to be rank 1 because tensoring keeps rank 1 and this lets us form an abelian group.

Definition 9. *A Weil divisor is a finite formal linear combination of codimension 1 subvarieties*

A Cartier divisor is a collection of $\{U_i, f_i\}$ *such that* $\{U_i\}$ *is an open cover of* X *and* $f_i \in K(U_i)$ *the function field of rational functions.*

Given a (Weil) divisor we may define a relation $D < D'$ if $D' - D$ has non-negative coefficients. Then we get a vector space of functions $L(D) = \{f/g \mid (f/g) > -D\} = \{f/g \mid (f/g) + D > 0\}$. Finally, we get an equivalence relation $D \sim D'$ if there is some f/g with $D + (f/g) = D'$ under the above notion of equality. It is an easy proposition to see that $D \sim D' \Rightarrow L(D) \sim L(D')$ and the converse also holds.

To patch with divisors we need $f_i/f_j \neq 0$ anywhere on $U_i \cap U_j$. With this condition, we can get a line bundle from a divisor by simply taking the cross product of our open sets with $\mathbb{C}$. All we need is the patching function $g_{\alpha,\beta}$ which is non-zero on $U_i \cap U_j$. We can simply define $g_{\alpha,\beta} = f_\alpha/f_\beta$ and we have this property.

To get an invertible sheaf from $D = \{U_i, f_i\}$ define $\mathfrak{L}(D)(U_i) = \mathcal{O}_X(U_i)$-module generated by $1/f_i$. This means it's $\{f/f_i \mid f \in \mathcal{O}_X(U_i)\}$. Because $f_i \in \mathcal{O}_X(U_i)$ we know that 1 is in our module.

Recall that $D \sim D'$ iff $L(D) \cong L(D')$. This implies $\mathfrak{L}(D_1 + D_2) \simeq \mathfrak{L}(D_1) \otimes \mathfrak{L}(D_2)$. We can therefore define the Picard group as the group of line bundles mod this isomorphism, as the group of divisors mod this isomorphism, or as the group of invertible sheaves mod this isomorphism.

On a smooth curve C we get a canonical divisor K_C which captures tangency information and is associated to the cotangent bundle.

Now come some extra sections that I won't have time to talk about but which I find interesting.

4. Grothendieck Topology

Definition 10. *A Grothendieck topology is a structure on a category* C *which makes the objects of* C *act like the open sets of a topological space. A category together with a choice of Grothendieck topology is called a site.*

With this notion we can define sheaves on a category and get closer to derived categories and stacks.

The motivation for this concept is the Weil conjectures. Andr Weil proposed that certain properties of equations with integral coefficients should be understood as geometric properties of the algebraic variety that they defined. His conjectures postulated that there should be a cohomology theory of algebraic varieties which gave number-theoretic information about their defining equations. This cohomology theory was known as the "Weil cohomology", but using the tools he had available, Weil was unable to construct it.

A Grothendieck topology on C is a collection of sets (called coverings) for every object x which act as the categorical morphisms. This collection will be denoted $\mathrm{Cov}(x)$ and satisfies:

(1) For all objects x, $\{x \stackrel{id}{\to} x\} \in \mathrm{Cov}(x)$
(2) If $\{X_\alpha \to X\}_\alpha \in \mathrm{Cov}(X)$ and $Y \to X$ is any arrow then the fiber product $X_\alpha \times_X Y$ exists and is in $\mathrm{Cov}(Y)$ for all α
(3) If $\{X_\alpha \to X\} \in \mathrm{Cov}(X)$ and for all α, $\{X_{\alpha_\beta} \to X_\alpha\}_\beta \in \mathrm{Cov}(X_\alpha)$ then $\{X_{\alpha_\beta} \to X_\alpha \to X\} \in \mathrm{Cov}(X)$

Note that the fiber product is the limit of the following diagram:

$$\begin{tikzcd} {} \arrow[r, "?"] \arrow[d, "?"] & Y \arrow[d] \\ X_\alpha \arrow[r] & X \end{tikzcd}$$

Example: If P and Q are properties of morphisms of schemes and Y is a fixed scheme then the $P-Q$ site on Y is a category called the full subcategory of schemes over Y. It's objects are P morphisms and its arrows are commutative diagrams (where $f_1, f_2 \in P$)

$$\begin{tikzcd} X_1 \arrow[rr] \arrow[dr, "f_1"] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

This example leads to

(1) Big/small site of a topological space (Q is the property of being a homeomorphism onto an open subset)

(2) Big/small Zariski site

(3) Etale Site (Q is etale maps)

(4) Big faithfully flat finite presentation site (Q is flat and finitely presented)

(5) Lisse-Etale site (P is smooth, Q is etale)

A topos is a category equivalent to the category of sheaves on a site.

5. Etale

If F is a sheaf over X, then the tale space of F is a topological space E together with a local homeomorphism $\pi : E \to X$; the sheaf of sections of π is F. E is usually a very strange space, and even if the sheaf F arises from a natural topological situation, E may not have any clear topological interpretation. For example, if F is the sheaf of sections of a continuous function $f : Y \to X$, then $E = Y$ if and only if f is a covering map.

The tale space E is constructed from the stalks of F over X. As a set, it is their disjoint union and π is the obvious map which takes the value x on the stalk of F over $x \in X$. The topology of E is defined as follows. For each element s of $F(U)$ and each $x \in U$, we get a germ of s at x (i.e. an equivalence class of functions). These germs determine points of E. For any U and $s \in F(U)$, the union of these points (for all $x \in U$) is declared to be open in E. Notice that each stalk has the discrete topology. Two morphisms between sheaves determine a continuous map of the corresponding tale spaces which is compatible with the projection maps (in the sense that every germ is mapped to a germ over the same point). This makes the construction into a functor.

This gives an example of an tale space over X. An tale space is a topological space E together with a continuous map $\pi : E \to X$ which is a local homeomorphism such that each fiber of π has the discrete topology. The construction above determines an equivalence of categories between the category of sheaves of sets on X and the category of tal spaces over X. The construction of an tale space can also be applied to a presheaf, in which case the sheaf of sections of the tale space recovers the sheaf associated to the given presheaf.

The map π is an example of what is sometimes called an tale map. "tale" here means the same thing as "local homeomorphism". However, the terminology "tale map" is more common in contexts where the right analogue of a local homeomorphism of manifolds is not characterized by the property of being a local homeomorphism. This is the case in algebraic geometry.

With this notion we can talk about stacks. A stack generalizes a scheme. A stack is a category X over the tale site satisfying the following three properties:

(1) We can define restrictions of objects over a scheme S to objects in open coverings of S: The category X is fibered in groupoids over the tale site.

(2) We can patch isomorphisms: Isomorphisms are a sheaf for X.

(3) We can patch objects: Every descent datum is effective.

Note that the tale site is the name for the usual category of schemes considered together with the tale Grothendieck topology.

Example: The moduli space of algebraic curves (Deligne-Mumford stack) defined as a universal family of curves of given genus g does not exist as an algebraic variety because in particular there are elliptic curves admitting nontrivial automorphisms. For elliptic curves over the complex numbers the corresponding stack is a geometrical factor of the upper half-plane by the action of the modular group.

PERHAPS LOOK NOW AT THE AMS ARTICLE “WHAT IS...A STACK?”