HANDOUT FOR GSS ON DIMENSION OF RING SPECTRA

1. Some algebra

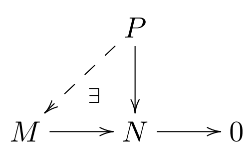
Morally, Algebra $\subseteq$ Homological Algebra $\subseteq$ Stable Homotopy Theory.

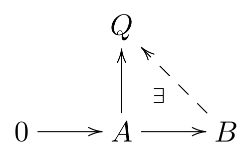
We have dimension for ring theory; what does it give us in stable homotopy theory?

Krull dimension of R is $\sup\{P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n \mid$ each P_i is a prime ideal of $R\}$.

The simplest rings are fields, which have Krull dimension zero. A field has all modules free. The next simplest modules after free modules are projective (they are direct summands of free modules). So the next simplest rings should have all modules projective. Such a ring is called semisimple. Turns out $R \cong R_1 \times \cdots \times R_n$ for $R_i = M_r(D)$ and D a division algebra.

We say module P is projective if: A module Q is injective if:





i.e. maps out of P lift along epimorphisms and maps into Q extend along monomorphisms

Given a module M a projective resolution of M is an infinite exact sequence of modules $\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$, with all the P_i's projective.
Similar for injective resolution but $0 \to M \to I_0 \to I_1 \to \cdots$.

The projective dimension of M (pd(M)) is the minimal length of a projective resolution of M.

Ex: If P is projective, $\text{pd}(P) = 0$ since $\cdots \to 0 \to 0 \to P \to P \to 0$ is a projective resolution.

Ex: For $R = \mathbb{Z}$, $\text{pd}(\mathbb{Z}/n) = 1$ since $\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ is minimal projective resolution, where the first map is mult by n and the second is quotient.

The right global dimension of R is $\sup\{pd(M) \mid M \in R - mod\}$

Ex: $\text{r.gl.dim}(k[x_1, \ldots, x_n]) = n$ because of the module $(x_1, \ldots, x_n)$

Ex: $\text{r.gl.dim}(k[x]/(x^2)) = \infty$ because k is an R-module and the minimal projective resolution is an infinite chain $\cdots \to k[x]/(x^2) \to k[x]/(x^2) \to k \to 0$, where each map takes $x \to 0$ and $1 \to x$

Fact: $\text{r.gl.dim}(R) = 1 \Rightarrow$ submodules of projective modules are projective. Next simplest after semisimple. Ex: all PIDs. NOTE: $\forall\ R$, R is a projective R-module. Not so for injective.

Note: A ring with injective dimension zero is called quasi-Frobenius and every injective (projective) left R-module is projective (injective). Also, Krull dimension is zero. Example: $\mathbb{Z}/p$.

$N \in R - mod$ is flat if whenever $0 \to A \to B \to C \to 0$ is exact then
$\Rightarrow 0 \to A \otimes N \to B \otimes N \to C \otimes N \to 0$ is exact.

Date: February 24, 2011.

Examples: $\mathbb{Q} \oplus \mathbb{Z}$ is flat but not injective or projective. $\mathbb{Q}/\mathbb{Z}$ is injective but not projective or flat. $\mathbb{Z}$ is projective but not injective. Injective $\mathbb{Z}$-modules are exactly divisible groups.

The flat dimension of M is the minimal length of a flat resolution. The right weak dimension of R is $\sup\{fd(M) \mid M \in R - mod\} = \max\{n \mid Tor_n^R(M, N) \neq 0$, some $M, N \in R - mod\}$

R is Von Neumann Regular if w.dim$(R) = 0$. This implies all modules over R are flat. Rings of weak dimension 1 have submodules of flat modules being flat.

NOTE: Projective $\Rightarrow$ Flat, so w.dim$(R) \leq$ r.gl.dim(R).

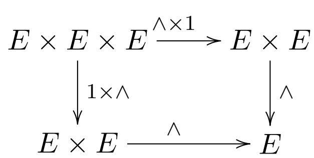
Serre's Theorem: If R is commutative and has finite global dimension then R is regular, so Krull dim = r.gl. dim. This allows us to apply homological algebra to commutative algebra.

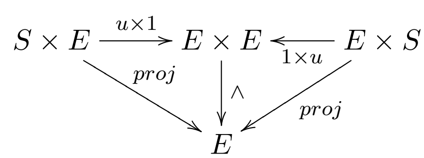
To GSS readers: Section 1 is a great analogy for Section 2, but today we're going to focus solely on Section 2. Many examples of S-algebra dimension will be given

2. Some topology

A spectrum X is a sequence (X_i) of topological spaces with maps $\Sigma X_i \to X_{i+1}$ where Σ is reduced suspension. Example: $S = (S^n)$ the sphere spectrum.

An S-algebra E is a spectrum which is also a generalized cohomology theory with a "nice" cup product. E comes with $\wedge : E \times E \to E$ and $u : S \to E$ satisfying:





NOTE: We've erased dimension, but now we have no points because of the grading. Thus, Krull Dimension for these E fails. Note also: these diagrams make E into an S-module.

To define dim(E) we have two tools. First, the homotopy $\pi_*(E)$ of E is a graded ring. Second, we can talk about E-modules, i.e. S-algebras M with an action $E \wedge M \to M$. Such M satisfy $\pi_*(M)$ is a $\pi_*(E)$-module. To study E-modules correctly we need $\mathcal{D}(E)$ = derived category of E: objects are E-modules, Morphisms$(M_1, M_2) = \mathcal{D}(E)(M_1, M_2) = \{S-$algebra morphisms:$M_1 \to M_2\}/\sim$ where $f \sim g$ if $f = g \circ s^{-1}$ for s a quasi-isomorphism (i.e. $\pi_*(s)$ is an isomorphism).

Note: $X \in \mathcal{D}(E)$ is projective iff $\pi_*(X)$ is a projective $\pi_*(E)$-module. Define $pd(X) = 0$. Say $pd(Z) \leq n$ if there exists $Y, P \in \mathcal{D}(E)$ with $pd(P) = 0, pd(Y) \leq n - 1$ s.t. $Y \to P \to \widetilde{Z} \to \Sigma Y$ where $\widetilde{Z}$ is a retract of Z. Flat dimension is similar.

Define r.gl.dim$(E) = \sup\{pd(Y) \mid Y \in \mathcal{D}(E)\}$ and say E is semisimple if r.gl.dim$(E) = 0$.

Fact: Semisimple E has $\pi_*(E) \cong R_1 \times \cdots \times R_n$ for R_i = graded field k or $R_i = k[x]/(x^2)$

A map $f : M_1 \to M_2$ is ghost if $\pi_* f = 0$. This means $\pi_*(E)$ can't see f. With this,

ghost dim$(E) = \min\{n \mid$ every composite of $n+1$ ghosts in $\mathcal{D}(E)$ is zero$\}$.

E is called Von Neumann Regular if gh.dim$(E) = 0$. Because gh.dim$(E) = \sup\{pd(X) \mid X$ is compact in $\mathcal{D}(E)\}$, we get gh.dim$(E) \leq$ r.gl.dim(E).

3. Motivation for Spectra 2

We want to compute homotopy groups, because **homotopy is a strong invariant** of the space. For example, Whitehead's Theorem says if X, Y are connected spaces with the homotopy type of a CW-complex then $f : X \to Y$ is a homotopy equivalence iff $\pi_i(f) : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for all $i > 0$. So homotopy groups allow us to study spaces up to homotopy equivalence.

Before spectra, the homotopy of spaces doesn't form a generalized homology theory because it doesn't satisfy excision. Blakers-Massey Excision Theorem says $\pi_n(X/A, B/A) \cong \pi_n(X, B)$ when $\pi_i(X, A) = 0$ for all $i < a$, $\pi_i(X, B) = 0$ for all $i < b$, and $n < a + b - 2$. So excision only holds when our spaces are highly connected.

One way to make it easier is to use the Freudenthal Suspension Theorem to say there exists a direct limit of $[S^{n+1}, \Sigma X] \to [S^{n+2}, \Sigma^2 X] \to [S^{n+3}, \Sigma^3 X] \to \cdots$. Call this limit $\pi_n^s(X)$, the stable homotopy group. The graded group $\pi_*^s(S^0)$ is the stable homotopy of spheres.

We want some category where the Hom-sets consist of stable homotopy classes of maps. Turns out $Ho\mathcal{S}$ does the job for $\mathcal{S}$ = Spectra. Also turns out $Ho\mathcal{S}$ is S-alg.

4. Proof of 1.7

Useful Theorem from Lam (4.23): Let $0 \to K \to F \to P$ be exact in $\mathcal{M}_R$ where F is free with basis $\{e_i\}$. Then P is flat iff $\forall\, c \in K\ \exists\, \theta \in \mathrm{Hom}_R(F, K)$ with $\theta(c) = c$.

Proof: $(\Leftarrow)$: Get $K \cap FI \subset KI$ for $I \subset R$ any left ideal. Write $c \in K \cap FI$ as $\sum e_i r_i$ and take θ with $\theta(c) = c$. Then $c = \theta(e_i) r_i + \cdots + \theta(e_m) r_m \in KI$

$(\Rightarrow)$: Write $c = \sum e_i r_i$ and let $I = \sum R_i r_i$. So $c = \sum c_\alpha s_\alpha$ for $c_\alpha \in K$ and $s_\alpha \in I$. So $c = \sum_j (\sum_\alpha c_\alpha t_{\alpha j}) r_j$ lets us define θ to send e_{i_j} to $\sum_\alpha (c_\alpha t_{\alpha j}) \in K$ and the other e's to zero. Then $\theta(c) = c$

weak dim $R \leq$ gh.dim R...

Spse RHS $= n < \infty$. Let X be an E-module. Then we know there's a free resolution of X_* by$P_0, \ldots, P_n$. Create the SES's $0 \to M_{k+1} \to P_k \to M_k \to 0$ where $M_k = \ker(d_{k-1})$. Exactness is because it's inclusion followed by d_k with range restricted so it becomes onto. The P_i are realized in $\mathcal{D}(E)$ by Q_i. Use Lam's theorem above by getting $P_n \to M_{n+1}$ sending c to itself (do so via perfect complexes). This proves the n-th element in downstairs chain is flat, so $fd(M) \leq n$.

r.gl.dim$(E) \leq$ r.gl.dim(E_*)...

Spse RHS $= n < \infty$. We'll show $pd(X) \leq pd(X_*)$ for all X, following Christiensen 8.3. Let $pd(X_*) = k$. Let $X^0 = X$ and construct $P^0 \to X^0$ s.t. $P^0_n \to X^0_n$ and $(P^0_n)_* \to (X^0_n)_*$ are epi and P^0 is projective. Let $X^1 = \Sigma \ker(P^0 \to X^0)$ be a choice of cofiber in the exact triangle upstairs. Continue in this way to get $0 \to A \to Q_{k-2} \to \cdots \to Q_1 \to X_* \to 0$ with each Q_i projective. A must be projective because $pd(X_*) = k$. So we can realize all the Q_i and A upstairs and we see that X^{k-1} is projective, i.e. ghosts out are zero. This tells us X^{k-2} has length at most 2 (i.e. composite of two ghosts out is null). Continuing we see $X = X^0$ has length k, so a composite of k ghosts out is null. This proves $pd(X) \leq k$.

EASIER WAY: Look at the universal coefficient spectral sequence. If $pd(X) \leq k$ then there's nothing above the k-line in E_2 and this means there can't be anything above that line in E_∞. But this means any composite of k ghosts is null, since composing moves you up in filtration by at least one each time. This is Hovey's Prop 1.5

gh.dim(E) $\leq$ sup{con.flat.dim(X) with X arbitrary} $\leq$ w.dim(E_*)...

The first inequality follows because gh.dim(X) is equal to sup{con.flat.dim(X) with X compact} = sup{flat dim(X) with X compact} = sup{flat dim(X) with X arbitrary}. The key here is that compact and flat implies projective.

The second inequality is because con.flat.dim(X) $\leq$ flat dim(X_*) for all X. This is because given a resolution $0 \to F \to P_{n-1} \stackrel{d_{n-1}}{\to} \cdots \to P_0 \to X_* \to 0$ where F is flat over E_*, we have exact $0 \to K_{i+1} \to P_i \to K_i \to 0$ for $K_i = \ker(d_{i-1}), K_0 = X_*$, and $K_n = F$. Because the P_i are projective this is uniquely realizable by triangles $X_{i+1} \to Q_i \to X_i \to \Sigma X_{i+1}$ where $(X_i)_* = K_i$ and $(Q_i)_* = P_i$. Because P_i is a retract of a direct sum of copies of E_*, Q_i is a retract of a coproduct of copies of E. This gives $\Sigma^{i-1} X_i \to Y_i \to X \to \Sigma^i X_i$ for all i. This gives exact $\Sigma^{i-1} Q_i \to Y_i \to Y_{i+1} \to \Sigma^i Q_i$ via the 3x3 lemma on $X \to \Sigma^i X_i$ with $X \to \Sigma^{i+1} X_{i+1}$ under it.

5. Proof of 2.3

If E is a commutative S-algebra then depth(E_*) $\leq$ gh.dim(E) $\leq$ min{w.dim(E_*),r.gl.dim(E) $\leq$ r.gl.dim(E_*)

We already have everything except the first inequality, because of 1.7. Let $(x_1, \ldots, x_n)$ be a regular sequence in $R = E_*$. First, we know there is an E-module $E/(x_1, \ldots, x_n)$ realizing the R-module $R/(x_1, \ldots, x_n)$ by induction. R is a projective R-module so it is realizable (base case). We have an exact triangle $E/(x_1, \ldots, x_{k-1}) \stackrel{x_k}{\to} E/(x_1, \ldots, x_{k-1} \to E/(x_1, \ldots, x_k) \to \Sigma E/(x_1, \ldots, x_{i-1}$ by the usual quotient SES exactness. So we define $E/(x_1, \ldots, x_k)$ to be the thing filling the blank spot in the triangle.

Next, $\mathrm{Ext}^i_R(R/(x_1, \ldots, x_n), R) = 0$ iff $i \neq n$, again by induction. The base case is clear because R is a projective R-module. We know $0 \to R/(x_1, \ldots, x_{k-1}) \stackrel{x_k}{\to} R/(x_1, \ldots, x_{k-1} \to R/(x_1, \ldots, x_k) \to 0$ is exact. Applying Hom$(-, R)$ to this picture gets a long exact sequence where the Ext terms all vanish except at $i = n$.

Finally, the Universal Coefficient Spectral Sequence tells us:
$\mathrm{Ext}^{s,t}_R(R/(x_1, \ldots, x_n), R) \Rightarrow \mathcal{D}(E)(E/(x_1, \ldots, x_n), E)$.
So $E_2^{s,t} = 0$ whenever $s \neq n$. This means all differentials are zero so E_∞ must have an element of filtration n (i.e. $E_\infty^{s,t}$ has some non-zero part when $s = n$). So gh.dim $E \geq n$ by Prop 1.4.

6. Cohomology Theories

The EilenbergSteenrod axioms apply to a sequence of functors Hn from the category of pairs (X, A) of topological spaces to the category of abelian groups, together with a natural transformation $\partial : H_i(X, A) \to H_{i-1}(A)$ called the boundary map (here $H_{i-1}(A)$ is a shorthand for $H_{i-1}(A, \varnothing)$). The axioms are:

1. Homotopy: Homotopic maps induce the same map in homology. That is, if $g : (X, A) \to (Y, B)$ is homotopic to $h : (X, A) \to (Y, B)$, then their induced maps are the same.
2. Excision: If (X, A) is a pair and U is a subset of X such that the closure of U is contained in the interior of A, then the inclusion map $i : (X - U, A - U) \to (X, A)$ induces an isomorphism in homology.
3. Dimension: Let P be the one-point space; then $H_n(P) = 0$ for all $n \neq 0$.

(4) Additivity: If $X = \coprod_\alpha X_\alpha$, the disjoint union of a family of topological spaces X_α, then $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$.

(5) Exactness: Each pair (X, A) induces a long exact sequence in homology, via the inclusions $i : A \to X$ and $j : X \to (X, A)$:

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots$$

Brown Representability...for all cohomology theories $h^n(-)$ there is an object E_n such that $[X, E_n] \cong h^n(X) \cong [X, \Omega E_{n+1}] \cong [\Sigma X, E_{n+1}]$. This proves $E_n \cong \Omega E_{n+1} \cong \dots$

Example: for homotopy theory $E_n = K(G, n)$ Eilenberg Maclane space. For K-theory, $E_n = BU$ so reduced homotopy of K-theory of X is $[X, BU_k]$ and non-reduced homotopy is $[X, BU_k \times \mathbb{Z}]$. Bott Periodicity says $K^{n+2}(X) \cong K^n X$. Also, $\pi_i(BU \times \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}, 0, \dots$ and $\pi_i(BO \times \mathbb{Z}) = \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0$

7. How to do computation

Ext: Given a projective resolution of A with $f_i : P_i \to P_{i-1}$, apply $\mathrm{Hom}(-, B)$ and define

$$\mathrm{Ext}^k_R(A, B) = \frac{\ker f^*_{k+1}}{\Im f^*_k} \qquad \mathrm{Tor}^R_n(A, B) = \frac{\ker f_n \otimes id}{\Im f_{n+1}}$$

Alternately, apply $- \otimes_R B$ and define Tor. Some examples of Ext are cohomology computations.

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0, k = n \ \ odd \\ \mathbb{Z}/2 & k < n \ \ odd \\ 0 & \end{cases} \qquad H^k(\mathbb{R}P^n) = H_k(\mathbb{R}P^n) \qquad H^k(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2\ \forall k$$

$$H_k(\mathbb{R}P^\infty) = \mathbb{Z}/2 \text{ if } k \text{ is odd} \qquad H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[x] \text{ as a ring}$$

Universal Coefficient Theorem: $H^n(X; G) \cong \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}(X), G)$ and $H_n(X; A) \cong (H_n(X, R) \otimes_R A) \oplus \mathrm{Tor}^R(H_{n-1}(X, R), A)$

From now on, work mod 2.

A cohomology operation $(\pi, n; G, m)$ is a family $\theta_X : H^n(X; \pi) \to H^m(X; G)$ for all X such that $f^*\theta_Y = \theta_X f^*$ for all f

Theorem: $[X, K(\pi, n)] \leftrightarrow H^n(X; \pi)$ by $f \leftrightarrow f^*(\iota_n)$. Hence, $\mathcal{O}(\pi, n; G, m) \leftrightarrow H^m(K(\pi, n); G)$

$\alpha \in \Omega X$ defines $c_\alpha : \Omega X \to \Omega X$ by $\beta \to \overline{\alpha} * \beta * \alpha$. Thus, we have an action of $\pi_1(X)$ on $\pi_n(X)$ given by $[f] \mapsto [c_\alpha \circ f]$

Define $\smile$: $H^n(X, \pi) \times H^m(X, \pi) \to H^{n+m}(X, \pi)$ by taking (f, g) to the function ϕ which does f on front n-face of simplicial complex and g on back m-face, i.e.

$$\phi = \begin{cases} f & \text{on } \sigma|_{0,\dots,n} \\ g & \text{on } \sigma|_{n,\dots,n+m} \end{cases}$$

Next, $u \smile v = (-1)^{pq} v \smile u$ for $p = \deg u, q = \deg v$. So cup product cannot commute with Σ, hence CUP PRODUCT IS NOT STABLE. We define $\smile_1$ to tell us how far $\smile = \smile_0$ is from being stable.

ϕ is the equivariant chain map arising from the carrier $\mathcal{C} : d_i \otimes \sigma \to C(\sigma \times \sigma) = C(X) \otimes C(X)$ which sends $w \otimes k \to k \otimes k$. Then $\smile_i$: $C^p(K) \otimes C^q(K) \to C^{p+q-i}(K)$ via $(u \smile_i v)(c) = (u \otimes v)\phi(d_i \otimes c)$.

Equivariant carriers are really hard!! Another way to define the squares is axiomatically, via $Sq^0 = 1, |u| = q \Rightarrow Sq^q u = u \smile u, q > |u| \Rightarrow Sq^q u = 0$, and Cartan for excisive pairs.

$Sq^i(u) = u \smile_i u$ is stable.

Given $g : X \to Y, g^* : H^*(Y) \to H^*(X)$ preserves $+, \times, \smile$, and Sq because cohomology is a functor: Top$\to$Ring

Here's $A(1) = \langle Sq^1, Sq^2 \rangle$. The left column points from bottom to top are 1, Sq^1, Sq^2, $Sq^1 Sq^2 = Sq^3$. The right column points from bottom to top are $Sq^2 Sq^1, Sq^3 Sq^1, Sq^2 Sq^3, Sq^5 Sq^1$