ABSTRACT MATHEMATICS AS A UNIVERSAL MEDIUM FOR SCIENTIFIC INQUIRY

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Thanks for the opportunity to speak. As Melanie said, I wear two hats here at Denison: I teach and have research in both math and cs. In fact, I teach a bit of everything: in my intro CS class we cover applications from economics, sociology, political science, education and comm. In my stats and math classes I've covered applications to physics, biology, chemistry, geoscience, neuroscience, and psych. I'm a product of the liberal arts, so I try to figure out what is common to all these fields and how they relate to my own view of the world. That's what I'll share today.

When I gave my job talk I spoke about a very concrete problem where I analyzed a navigation algorithm which an autonomous agent (e.g. a robot) could use while exploring an unfamiliar terrain. In the end, finding the answer (in this case: determining whether the algorithm was a good idea or not) involved computing a certain probability and taking a limit as the mesh on the terrain gets infinitely fine. When most people think of math, this is what they think it's good for: computing things, making statements with probabilities attached, and taking limits. When I do work in computer science (e.g. my recent work with Jessica Tang), this is often the type of work I do.

In the sciences, most of the prominent methods for incorporating mathematics involve setting up stochastic processes, dynamical systems, or statistical models that capture the relevant processes in the scientific subject. At bottom, these techniques all involve interplays of numbers. However, a biologist's sophisticated understanding of the non-mechanistic nature of lifeheredity, reproduction, hierarchical nesting, symbiosis, metabolism, etc. remains trapped in the realm of ideas. Such ideas can often be reduced to numerical models.

Today I want to talk about another use of mathematics, which is as a unifying perspective. Here we try to extract the essence of an idea, blurring out the trees to see the forest (i.e. removing dependence on the real world problem that motivated the idea). Abstract math is a way of encoding the ideas not just numerical information. Much of what I'm saying comes from a paper I wrote over the summer for the Journal of Humanistic Mathematics, as part of a new project called User's Guides designed to make math papers more human readable.

Upshot: the idea can now be used anywhere that looks formally "the same" as where it originated. This means we've built a bridge between two different fields, so that ideas from one can be used in the other.

Example: optimization has been around for thousands of years. The Greeks tried to understand the shortest (best) path on a globe, the Babylonians tried to understand the fewest number of steps required to complete various tasks, architects tried to understand how to make the strongest shapes with the fewest materials. With the advent of calculus we gained a powerful way to state optimization problems as well as a tool for solving them. In the computer age it became extremely important to be able to come up with the best possible solution to a complex problem. Separately, evolution is about how species improve themselves over time. Bad traits are weeded out while good traits are encouraged. At time goes on the species gets better and better. Can we use the

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ideas of evolution as a way to do optimization? Yes! That's **evolutionary computing**, and it allows a population of candidate solutions to mutate and reproduce over hundreds of thousands of generations. A fitness function weeds out the weak and the individuals produced often have high fitness, hence can be good solutions to the original problem.

It is, however, a two way street with biology. Computer science has the advantage that they can see the evolution unfolding. You can vary the parameters to see how evolution speeds up when constraints are tight. Computer scientists observe that sometimes "junk DNA" is kept around because what looks like the optimal solution might turn out to only be a local maximum or might not be a maximum at all in a changing landscape. This observation in turn feeds biology so that they can see if the same observation is there in their studies and what might be driving it. Another example is evolution speeding up around changes (e.g. the move to the ice-age). You can see how mass extinctions lead to a speed up in evolution, giving evidence for a biological hypothesis.

The idea of optimization is fundamental for the human mind, so it's not surprising it came up in many different fields. By realizing that all these people were thinking of the same fundamental thing you can translate the ideas from one field into another, and put their minds together into a single power. There are many ideas, like evolution, that are fundamental and can therefore appear in different guises across the sciences. Building bridges between these different fields over these ideas allow you to leverage the ideas of the great minds in all fields and bring them to bear on the problem you are interested in.

When we use this philosophy on math itself we arrive at a formalism we can use to encode ideas in this way. At its core, mathematics is the study of objects (e.g. sets, vector spaces, curves (polynomial equations), shapes, networks, groups, rings, topological spaces, dynamical systems, probability spaces) and the relationships between them, e.g. functions, linear transformations, when one curve can be continuously mapped to another, when one shape can be deformed to another, when one network occurs inside another or has a similar relationship structure, etc. In math, the word "morphism has become standard to describe these types of maps, which preserve the structure of the objects in question.

Poincaré quote.

Definition 1. *A **category** is a collection of objects and the maps between them. The maps have to satisfy two rules: every thing must be related to itself by simply being itself (so there is always an identity map $1_X : X \to X$), and if one thing is related to another and the second is related to a third, then the first is related to the third (so maps are composable).*

This definition matches our notion of the word category in everyday speech. It's a collection of things, all of which are related in some way.

An **isomorphism** is a morphism $f : X \to Y$ such that there is some $h : Y \to X$ with $f \circ h = 1_Y$ and $h \circ f = 1_X$.

The notion of category provides a fundamental and abstract way to **describe mathematical entities and their relationships**. **Virtually every branch of modern mathematics** can be described in terms of categories. Thus, if we can phrase a concept in terms of categories, it will have versions in most fields of mathematics. Because of the power of the results, proofs can be technical, but life can be made much easier by finding the right definitions and proofs classically first, as we did above.

Examples from math:

- Set, the category of sets and set functions. Isomorphisms are bijections. Math 210.

- Vect, the category of vector spaces and linear transformations. Two are isomorphic if they have the same dimension. Math 231.
- Grp, the category of groups and group homomorphisms. Isomorphisms. Math 332
- Graphs and graph homomorphisms $f : V(G) \to V(H)$ such that $f(v)f(u)$ is an edge of H whenever vu is an edge of G. Math 275. Also: these came up in Dave Smith's talk.
- Dynamical Systems and finitary maps. May's research.
- Operator algebras and maps between them. Matt Neal.
- My project with Mike Westmoreland to get a categorical foundation for quantum processes.

Examples from the sciences:

- $\mathbb{R}$: objects are real numbers and put a map $x \to y$ if $x \leq y$. In general you can do this for any partially ordered set.
- Classical mechanics can be viewed as studying the state of the world around us as time goes on. So it works just like the example above, except an object is the whole state of the universe at time t.
- States of the economy as time goes on.
- Crystallography: Objects are arrangements of atoms in a molecule, morphism is a symmetry.
- Databases: an object can be a table, a morphism can be a shared column (called a foreign key).
- Going a bit more meta, an experiment is like a category. Objects could be observables and a relationship could tell us if they're correlated.
- Even more meta, the collection of all experiments is a category. Objects are experiments and we say two are related if they got the same conclusion (perhaps just on one question of interest across all experiments).
- In material science, objects could be materials and we could draw $A \to B$ if A is an ingredient or part of B, so water $\to$ concrete. A different way to view it as a category would be to draw $A \to B$ if A is less electrically conductive than B, so concrete $\to$ water.
- Robert Rosen introduced in the 90s a category of morphogenetic networks to study morphogenetic problems. Objects are elements and their different states, morphisms come from neighborhoods.

Reusable methodologies can be formalized, and that doing so is inherently valuable. Category theory also provides a language for experimental design patterns, introducing formality while remaining flexible.

Example: Category theory can serve as a **mathematical model for mathematical modeling**. Our minds simultaneously keep several models of the world, often in conflict. The value of a model can therefore be measured by how well it fits with other models. What is true will be present across all models, so we should study the relationship between models.

Our interactions with the subject, which is ostensibly out there, are actually interactions with our own familiar models of it.

Minsky quote

The **yoga of category theory** is that one must study maps between objects to study the objects. Applying this to categories themselves leads you to functors $F : \mathcal{C} \to \mathcal{D}$, i.e. maps from objects to objects and morphisms to morphisms compatible with id_A and $f \circ g$ (i.e. they preserve structure). Formally, $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Examples:

- Forgetful functor: Vect $\to$ Set
- Free functor: Set $\to$ Vect.
- Abelianization functor: Grp $\to$ Ab by modding out by commutator subgroup.
- A dynamical system can be viewed as a functor from the time category to the set of possible states.
- Homology: Top $\to$ Graded Ab Groups. This relates our 332 to 322, hence algebraic topology.

Science Examples:

- If A be the set of amino acids and Str(A) the set of all strings formed from A. The process of translation gives a functor turning a list of RNA triplets into a polypeptide.
- Quantum field theory was categorified by Atiyah in the late 1980s, with much success (at least in producing interesting mathematics). In this domain, an object is a reasonable space, called a manifold, and a morphism is a manifold connecting two manifolds, like a cylinder connects two circles. Such connecting manifolds are called cobordisms. Topological quantum field theory is the study of functors Cob $\to$ Vect that assign a vector space to each manifold and a linear transformation of vector spaces to each cobordism.
- Recall Ashwin's talk last semester about different ways to order cars (that was the Tuesday lunch series version; the DSA was about ordering spaceships). If you fix your preferences then this ordering makes them a category. Consider the price function that tells you the cost of a car, and lands in $\mathbb{R}_{>0}$. In order for this to be a functor it must respect the ordering: is it true that better cars cost more and worse cars cost less? My girlfriend would say so, but I do love my 2001 Ford Focus.
- If you're running an experiment and in all cases so far have observed 4 traits. You've created a mental model for what's going on, but then you observe several cases where only the first 3 traits are true. You shift to a new mental model and that process is a functor.
- An experiment can be thought of as a functor from the category of pairs (Experimenter, Variables) to the category of measurements of the variables under observation. Viewing it this way makes it explicit that the experimenter can affect the outcome, something well-known in psychology and sociology.
- Pattern matching
- Transference in the liberal arts sense of the word. The ability to rigorously change our point of view.

A functor is like a conductor of mathematical truth. Certain structures and conceptual frameworks show up again and again in our understanding of reality, and functors let us apply them to other fields.

Gromov quote

Abstract thinking is a fundamentally human trait, and I'm proposing we apply it across the sciences. Our ability as a species to think abstractly has been linked with our intelligence on all measures. Certainly abstraction has become a necessary skill in math and computer science. In CS we teach it in every class; if students can use abstraction to get to the essence of the problem then their solution will hold very generally any time someone is trying to solve this problem. It saves them from having to come up with variations on the same idea for every instance of the problem, and allows them to use the same code in all solutions.

Draw picture of a plane and all the spurs coming off it. This is what I mean by medium. Too often people think they "just can't do math" because they don't like arithmetic, but I claim math is about ideas and everyone can do it!

When we formalize our ideas our understanding is clarified. This is why we write papers.
<u>Our ability to communicate</u> is also enhanced, and we begin to see connections and future work.

Core Ideas that come up in my work:

(1) Objects and the relationships between them.
(2) The use of functors to build bridges.
(3) Breaking an object up into simple pieces; understanding how to build complicated structure from these pieces.
(4) Localization: shifting view so that two objects you previously viewed as different are now viewed as the same.
(5) Replacing an object by one which is easier to work with but has the same fundamental properties you are trying to study.
(6) Mapping an object to a small bit of information about the object. Showing that two are different because they differ on this bit. Trying to find a complete set of invariants so you know precisely when two are the same.

There is also a **humanistic side**, where we help determine which ideas the human mind keeps coming back to over and over again from different backgrounds. Mathematics certainly has an aesthetic side (how do people decide what to work on? How do they decide what constitutes a "great paper" and which gets published?) and I find the abstract language is a great way to draw it out.

Show cloud photo and explain.

I won't have time to talk about all these, but let's try (3) above.

Example:

- Chemistry breaks down to the study of atoms and the molecules they make up.
- Physics breaks the world down even further, into strings.
- Molecular Biology studies the cell. Robert Rosen introduced a categorical presentation of (M,R)-systems, which model the activities of a cell. This is a category of automata (sequential machines).
- Geoscience breaks materials down into their simplest constituent pieces.
- Neuroscience and neurons.
- CS and data packets.

- Economics and game theory try to isolate a single cause and effect relationship by holding all other variables constant ("decision making on the margin")
- Political science and the action of individuals.
- Understanding how materials are built up of their constituent parts. For example, a tendon is made of collagen fibers that are assembled in series and then in parallel, in a specific way. Each collagen fibre is made of collagen fibrils that are again assembled in series and then in parallel, with slightly different specifications. We can continue down, perhaps indefinitely, though our resolution fails at some point. A collagen fibril is made up of tropocollagen collagen molecules, which are twisted ropes of collagen molecules, etc.

  Another example: spider silk.

Putting things back together again is often extremely difficult. For example, consider evolution again. It's still unclear to us precisely how the extremely complex structures we see come about. Or in neuroscience, how the firing of neurons relates to the actual thoughts in our minds.

How does category theory approach this problem? Well, in several of our examples we've seen categories where objects can be subobjects of others. Similarly, there is a notion of a *colimit*, which is an object built up from simpler pieces.

Example: The current state of any evolutive system is a colimit of previous states.

An evolutive system is a subcategory of time, for each t a category K_t (the state at time t), for each period [t,t] a functor $K_t \to K_t$. These are just functors Time $\to$ Cat

Example: Emergent Phenomena like the behavior of an ant colony, or of people clapping, etc.

Example: modeling the biological tendency toward homeostasis

Example: Suppose you have different temperature reading devices measuring a terrain, perhaps with some overlapping areas. You can patch them together to get a maximally accurate reading by taking the colimit. This is simply a categorification of some kind of weighted averaging operation (weighted by knowledge of the devices).

Example: Consider outer space. Different astronomers record observations using telescopes. We can patch together different observations of space as a colimit. Objects here using pixels and the set of wavelengths in the visible light spectrum (written in nanometers).

Example: The set of laws of the land; are there inconsistencies? Do they assemble properly?

Example: The individuals making up society, and realizing society as the sum of its parts, i.e. at the object built up from all these individuals. When something happens and individuals are effected, the net effects on the colimit can be studied this way.

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(4) My own research:

Declaring two different things to be functionally the same.

Adding more isomorphisms.

Example: viewing two different driving routes as the same if they take the same time.

Example: viewing two problems in the book or on the exam as equivalent if they are the same difficulty and test the same concept.

Example: A commonly considered abstraction is the **phoneme**, which abstracts speech sounds in such a way as to neglect details that cannot serve to differentiate meaning. Other analogous kinds of abstractions (sometimes called "emic units") considered by linguists include morphemes, graphemes, and lexemes.

SHAPES = Top, the category of topological spaces and continuous maps. Point-set topology deals with spaces up to homeomorphism (the notion of isomophism here), whereas homotopy theory is less discriminating and views spaces X and Y as the same if they are homotopy equivalent (i.e. if X can be gradually deformed to match Y). Math 322.

HoTop, the category of topological spaces and homotopy classes of maps. Isomorphisms are now homotopy equivalences. Formally: view two spaces as isomorphic if there are maps $f : X \to Y$ and $g : Y \to X$ with $fg \sim id_Y$ and $gf \sim id_X$. For example, a circle and a cylinder (or donut) are now the same but they are not homeomorphic because the circle has cut vertices. Or the letters P and O.

Talk about model categories and applications in topology, homological algebra, stable module category, graph theory, algebraic geometry, and computer science.

Can talk to all my colleagues and it's great!

In this line of work, an "application" is when we discover a new field of abstract math is related to one which is better understood. So that's not an application in the sense of making airplanes fly faster, but it's still important for the development of all of mathematics moving forward, both to prove better results (which could lead to real applications down the line, especially in mathematical physics) and as a general organizing principle so that we see what core idea we are really pursuing.

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(6) What is the least amount of information required to describe something?

Example: Macroeconomics. Trying to predict behavior at time t based on behavior at time t' just based on the macro environments at those times. It'd be great if you knew which indicators really mattered so you could make predictions like that.

Example: Biological classification is a method for dividing the set of organisms into distinct classes, called taxa. In fact, it turns out that such a classification, say a **phylogenetic tree**, can be understood as a partial order C on the set of taxa. The typical ranking of these taxa, including kingdom, phylum, etc., can be understood as morphism of orders.

Example: Trying to predict when two hearts are going to behave the same. You know they're different if they have wildly different EKGs, but just having the same EKG does not mean they'll respond the same way in all situations.

Punchline:

It's worthwhile to think abstractly.
Many people may have thought about similar problems
It would be good to have bridges between those fields

If you pose your questions within this framework then it will become easier for folks in other disciplines to use your ideas and for you to use theirs.