## THE SPECTRAL CATEGORY AND VON-NEUMANN REGULAR RINGS

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ABSTRACT. All rings considered are associative with identity and all occurring modules are unital right modules. We denote by Mod R the category of all *R*-modules.

The spectrum of a ring R is known to be the "set" of isomorphism classes of indecomposable injective right R-modules. When R is right-Noetherian the spectrum describes all injective Rmodules, since each injective module is a direct sum of indecomposable submodules. We want to briefly show that for any R (or even for every Grothendieck category), this spectrum can be replaced by the so-called spectral category; one obtains this spectral category by formally inverting all essential monomorphisms. Approaches to such considerations are provided by the work of JOHNSON[6] and UTUMI[8].

As an application one obtains for each module invariant, that the invariant coincides with that which FUCHS[3] introduced under strong assumptions.

## 1. The Spectral Category of a Grothendieck Category

1.1. Let  $\mathfrak{A}$  be a Grothendieck category<sup>1</sup>, i.e. an abelian category with exact direct limits and a generator. Exactness of direct limits is equivalent to the following statement:

(\*) For each family  $(A_{\lambda})_{\lambda \in \Lambda}$  of objects of  $\mathfrak{A}$ 

$$B \subset \bigoplus_{\lambda \in \Lambda} A_{\lambda} \text{ is } B = \sup_{\Gamma} \left( B \cap \bigoplus_{\lambda \in \Gamma} A_{\lambda} \right)$$

where all finite subsets of  $\Lambda$  factor through  $\Gamma$ .

(Often you see (\*) as a special case of AB5)[5]. Suppose conversely that  $(A_{\lambda})_{\lambda \in \Lambda}$  is an increasing filtered family of subobjects of an object A and A' is another subobject of A. When

$$p: \bigoplus_{\lambda \in \Lambda} A_{\lambda} \to A$$

 $<sup>^{1}</sup>$ Unlike FREYD[2] we require not only that the objects form a set (i.e. the category is small), but also the existence of generators

is the canonical morphism, then:

$$A' \cap \sup_{\lambda \in \Lambda} A_{\lambda} = p(p^{-1}A') = p\left(\sup_{\Gamma} \left(p^{-1}A' \cap \bigoplus_{\lambda \in \Gamma} A_{\lambda}\right)\right) = \sup_{\Gamma} p\left(p^{-1}A' \cap \bigoplus_{\lambda \in \Gamma} A_{\lambda}\right)$$
$$= \sup_{\Gamma} \left(A' \cap p\left(\bigoplus_{\lambda \in \Gamma} A_{\lambda}\right)\right) = \sup_{\Gamma} \left(A' \cap \sup_{\lambda \in \Gamma} A_{\lambda}\right) = \sup_{\lambda \in \Lambda} \left(A' \cap A_{\lambda}\right)$$

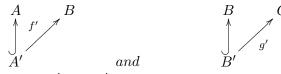
where all finite subsets of  $\Lambda$  factor through  $\Gamma$  and the last equation holds because  $(A_{\lambda})_{\lambda \in \Lambda}$  is an increasing filtration.

1.2. A monomorphism  $i : A \to B$  is called essential if the condition  $i(A) \cap B' = 0$  for  $B' \subset B$  implies B' = 0.

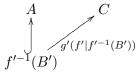
We define the spectral category Spec  $\mathfrak{A}$  of  $\mathfrak{A}$ : the spectral category has the same objects as  $\mathfrak{A}$ . For A and B objects of Spec  $\mathfrak{A}$ ,

$$(\operatorname{Spec}\mathfrak{A})(A,B) = \lim_{\longrightarrow}\mathfrak{A}(A',B)^2$$

where the direct limit is taken over all essential subobjects  $A' \subset A$ . Elements  $f \in \text{Spec } \mathfrak{A}(A, B)$ and  $g \in \text{Spec } \mathfrak{A}(B, C)$  are now determined by the diagrams:



where A' and B' are essential in A and B. Then  $f^{-1}(B')$  is essential in A and gf is defined via the diagram:



The categories  $\mathfrak{A}$  and Spec  $\mathfrak{A}$  become connected by the canonical functor  $P : \mathfrak{A} \to \operatorname{Spec} \mathfrak{A}$  which is the identity on objects and which has  $\mathfrak{A}(A, B)$  as the natural image in

$$\lim_{\overrightarrow{A'}} \mathfrak{A}(A', B)$$

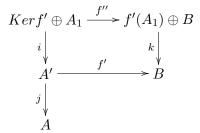
#### 1.3.

**Theorem.** Suppose  $\mathfrak{A}$  is a Grothendieck category. Then Spec  $\mathfrak{A}$  is a Grothendieck category in which each morphism decomposes (i.e. Ker f and Im f are direct summands in the source and target of f).

*Proof.* It is clear that Spec  $\mathfrak{A}$  is an additive category and P is an additive functor. Let  $f \in (\operatorname{Spec} \mathfrak{A})(A, B)$  be a morphism as considered in 1.2. Let  $A_1$  be the complement of Ker f' in A' (i.e.  $A_1 \cap \operatorname{Ker} f' = 0$  and  $A_1$  is maximal with respect to this property) and  $B_1$  the complement

<sup>&</sup>lt;sup>2</sup>If A and B are objects in the category  $\mathfrak{A}$  we write  $\mathfrak{A}(A, B)$  instead of Hom<sub> $\mathfrak{A}</sub>(A, B)$ </sub>

of  $f'(A_1)$  in B. This results in a commutative diagram



where f'' is induced by f' and decomposes via the essential inclusions i, j, k. Since Pf'' is isomorphic to f and decomposes, f also decomposes. In particular, Spec  $\mathfrak{A}$  is an abelian category and P is left exact.

Furthermore Spec  $\mathfrak{A}$  has infinite direct sums and P commutes with direct sums, since a direct sum of essential monomorphisms is an essential monomorphism in the Grothendieck category  $\mathfrak{A}$ .

In general, P commutes with intersections (because of left-exactness) and with a supremum of a filtered system of subobjects (analogous to the proof for direct sums). Therefore (\*) from 1.1 holds in Spec  $\mathfrak{A}$ .

Finally, PU is a generator in Spec  $\mathfrak{A}$  if U is a generator in  $\mathfrak{A}$ .

1.4. The canonical functor P is an isomorphism iff every morphism in  $\mathfrak{A}$  is decomposable. A Grothendieck category in which each morphism decomposes is therefore called a spectral category.

1.5.

**Theorem.** For objects  $A, B \in \mathfrak{A}$  the following are equivalent

- i) PA is isomorphic to PB
- ii) There is an object  $C \in \mathfrak{A}$  and essential monomorphisms  $i: C \to A$  and  $j: C \to B$
- iii) A and B are isomorphic to their injective hulls

The proof is clear

Therefore there is a 1-1 correspondence between the isomorphism classes of injective objects of  $\mathfrak{A}$  and the isomorphism classes of all objects of Spec  $\mathfrak{A}$ . An object  $A \in \mathfrak{A}$  is co-irreducible iff PA is simple. At the same time, an object A is coirreducible if it's nonzero and each different subobject is essential in A.

2. Characterizing the Spectral Category with the help of regular rings

2.1.

**Theorem.** Let  $\mathfrak{S}$  be a spectral category with generator U and  $R = \mathfrak{S}(U,U)$ . Then R is regular<sup>3</sup> and is an injective R-module. The functor

$$S \xrightarrow{F} \mathfrak{S}(U, S)$$

 $<sup>{}^{3}</sup>R$  regular means each principal ideal is a direct summand in R

is an equivalence of  $\mathfrak{S}$  onto the full subcategory of Mod R consisting of direct summands of powers of R.

*Proof.* That the functor F is an equivalence onto the full subcategory follows from [7]. From [7] it also follows that infinite direct products are in  $\mathfrak{S}$  because F is an equivalence to the induced quotient category of Mod R (the existence of inverse limits follows from the existence of direct limits, and is also true under much more general assumptions (see e.g. [1]).) Since each morphism in  $\mathfrak{S}$  decomposes and U is a generator, it is also an injective cogenerator. To each  $S \in \mathfrak{S}$  there is thus a monomorphism i from S into the product

$$\prod_{\lambda} U_{\lambda}$$

for  $U_{\lambda} \cong U$ . Since *i* decomposes, *Fi* also decomposes, i.e. *FS* is a direct summand of

$$F(\prod_{\lambda} U_{\lambda}) \cong \prod_{\lambda} F(U_{\lambda})$$

however,  $F(U_{\lambda})$  is isomorphic to R.

Suppose conversely that p is an idempotent endomorphism of a power  $\mathbb{R}^{\aleph}$  of R and  $M = p(\mathbb{R}^{\aleph})$ . Then p is of the form Fq for  $q \in \mathfrak{S}(U^{\aleph}, U^{\aleph})$ . Therefore M = Ker(1-p) = F(Ker(1-q)). We now finally show that R is self injective: Let  $I \subset R$  be a right ideal. We want to show that the natural image of R in  $\text{Hom}_R(I, R)$  is surjective. There are the following identifications:

$$\operatorname{Hom}_{R}(I,R) = \operatorname{Hom}_{R}(\lim M,R) = \lim \operatorname{Hom}_{R}(M,R) = \lim \mathfrak{S}(V,U) = \mathfrak{S}(\lim V,U)$$

where M is a finitely generated submodule of I, hence a direct summand of R, hence it has the form  $FV, V \subset U$ . But

 $\lim V$ 

is a direct summand in U; therefore the natural image

$$\operatorname{Hom}(U,U) = R \to \mathfrak{S}(\lim V,U) = \operatorname{Hom}_R(I,R)$$

is surjective

2.2.

**Theorem.** Let R be a regular, self-injective ring. Then the full subcategory of Mod R of direct summands of powers of R is a spectral category.

*Proof.* Let  $\mathfrak{C}$  be the localizing subcategory [4, pg. 377] of all modules C of Mod R with  $\operatorname{Hom}_R(C, R) = 0$ .

A right ideal  $I \subset R$  is essential iff  $R/I \in \mathfrak{C}$ . Let  $R/I \in \mathfrak{C}$  and  $e \in R$  with  $I \cap eR = 0$ . Given any morphism  $f : eR \to R$ , form a monomorphism from eR to R/I by continuing f to R/I

hence  $\operatorname{Hom}_R(eR, R) = 0$  so eR = 0 as eR is a direct summand of R.

Conversely suppose I is essential in  $R, f: R/I \to R$  is a homomorphism, and g is the composition of f with projection from R to R/I. The image g(R) is cyclic, hence projective, therefore Ker g is a direct summand in R and is essential, hence Ker g = R and f = 0.

We now know that Mod  $R/\mathfrak{C}$  is a Grothendieck category [4, pg. 378] and that the canonical functor from Mod R to Mod  $R/\mathfrak{C}$  possesses a right adjoint functor S. This S induces an equivalence of Mod  $R/\mathfrak{C}$  to the full subcategory of all  $\mathfrak{C}$ -closed objects of Mod R. This means a module M is  $\mathfrak{C}$ -closed when 0 is the unique submodule of M in  $\mathfrak{C}$  and when  $R/I \in \mathfrak{C}$  implies every morphism  $f: I \to M$  extends to R. In particular, each injective module is  $\mathfrak{C}$ -closed when 0 is the unique submodule in  $\mathfrak{C}$ . The direct summands of powers of R are also  $\mathfrak{C}$ -closed.

Conversely let M be a  $\mathfrak{C}$ -closed module. Let I be any right ideal and let I' be the complement of I, so each homomorphism  $f: I \to M$  extends to  $I \oplus I'$  and also to R. Thus  $I \oplus I'$  is essential in R and therefore  $R/I \oplus I'$  is in  $\mathfrak{C}$ . This means that M is injective, hence in particular that the  $\mathfrak{C}$ -closed modules form a spectral category.

For  $f \in \operatorname{Hom}_R(M, R)$ , the  $R_f$  are also copies of R. The canonical image  $\phi(M)$  of M in

# $\prod_{f} R_{f}$

is injective because Ker  $\phi$  belongs to  $\mathfrak{C}$ . Hence M is a direct summand in  $\prod R_f$ . The  $\mathfrak{C}$ -closed modules are hence exactly direct summands of powers of R.

### 3. Applications

3.1.  $\mathfrak{S}$  is a spectral category so each object S of  $\mathfrak{S}$  is a direct sum of its base So S, i.e. a sum of simple subobjects of S, and its radical Ra S, i.e. the intersection of maximal subobjects of S. We denote by  $\mathfrak{S}_d$  (d for discrete) or  $\mathfrak{S}_k$  (k for continuous) the full subcategory of all objects whose radical or base is 0, so the functor  $S \to (So S, Ra S)$  is an equivalence of  $\mathfrak{S}$  with  $\mathfrak{S}_d \times \mathfrak{S}_k$ . If  $\mathfrak{E}$  is another representative system of isomorphism classes of simple objects in  $\mathfrak{S}$  then it's known that the functor  $S \to \mathfrak{S}(E, S)_{E \in \mathfrak{E}}$  is also an equivalence of the semi-simple category  $\mathfrak{S}_d$  and the product category

$$\prod_{E \in \mathfrak{E}} \operatorname{Mod} \mathfrak{S}(E, E)$$

the category of vector spaces over the skew field  $\mathfrak{S}(E, E)$ 

From 2.2 and the previous comments we obtain

**Theorem.** Each regular, self-injective ring R is the ring direct product of its base So S and the finite radical  $Ra_eR$  (this is the intersection of all maximal direct summands of R). This  $Ra_eR$  is a

regular, self-injective ring whose base vanishes.

3.2. For each  $E \in \mathfrak{E}$  and  $S \in \mathfrak{S}$  let  $r_E(S)$  be the well-defined dimension of  $\mathfrak{S}(E, S) = \mathfrak{S}(E, \operatorname{So} S)$ over  $\mathfrak{S}(E, E)$ . Often this is characterized as the magnitude  $r_E(S)$ ,  $E \in \mathfrak{E}$  of the basis of S up to isomorphism. The function  $r_E$  has the following properties:

- i) If S and T are isomorphic, then  $r_E(S) = r_E(T)$
- ii) If  $S = \bigoplus_{\lambda} S_{\lambda}$  then  $r_E(S) = \sum_{\lambda} r_E(S_{\lambda})$
- iii) If F is simple then

$$r_E(F) = \begin{cases} 1 \text{ for } F \cong E\\ 0 \text{ otherwise} \end{cases}$$

iv) If So S = 0 then  $r_E(S) = 0$ 

It is clear that  $r_E$  is the unique function from the objects of  $\mathfrak{S}$  to the cardinal numbers with properties i) through iv).

3.3. Suppose now  $\mathfrak{A}$  is any Grothendieck category with spectral category Spec  $\mathfrak{A}$ ,  $P : \mathfrak{A} \to \operatorname{Spec} \mathfrak{A}$  the canonical functor, and  $\mathfrak{F}$  a representative system of isomorphism classes of indecomposable injective objects in  $\mathfrak{A}$ . Then  $P\mathfrak{F}(=\mathfrak{F})$  is a representative system of isomorphism classes of simple objects of Spec  $\mathfrak{A}$ .

Carrying over the results of 3.1 and 3.2 for Spec  $\mathfrak{A}$  and  $\mathfrak{A}$  one obtains the following result:

**Theorem.** Suppose  $(I_{\gamma})_{\gamma \in \Gamma}$  and  $(J_{\lambda})_{\lambda \in \Lambda}$  are direct families of indecomposable injective objects, A is an object, and  $i : \bigoplus_{\gamma} I_{\gamma} \to A$  and  $j : \bigoplus_{\lambda} J_{\lambda} \to A$  are essential monomorphisms in  $\mathfrak{A}$ . Then there is a bijection  $b : \Gamma \to \Lambda$  with  $J_{b(\gamma)} \cong I_{\lambda}$  for all  $\gamma \in \Gamma$ .

Furthermore, for all  $I \in \mathfrak{F}$  and  $A \in \mathfrak{A}$ ,  $r_I(A) := r_{PI}(PA) = \text{dimension of } (\text{Spec }\mathfrak{A})(PI, PA)$  over  $(\text{Spec }\mathfrak{A})(PI, PI)$ . NOTE: original paper had " $I \in J$ ", a typo The function  $r_I$  has the following properties:

- i) If  $f: A \to B$  is an essential monomorphism then  $r_I(A) = r_I(B)$
- ii) If  $A = \bigoplus_{\lambda} A_{\lambda}$  then  $r_I(A) = \sum_{\lambda} r_I(A_{\lambda})$
- iii) If A is coirreducible then

 $r_I(A) = \begin{cases} 1 \text{ if I is isomorphic to the injective hull of } A \\ 0 \text{ otherwise} \end{cases}$ 

iv) If A contains no coirreducible subobject then  $r_I(A) = 0$ 

$$\sup_{\lambda} A_{\lambda} = \bigoplus_{\lambda} A_{\lambda}$$

(Zorn's Lemma). Then  $r_I(A)$  is the cardinality of the set of all  $A_{\lambda}$  whose injective hulls are isomorphic to I. Specifically, if U is a generator of  $\mathfrak{A}$  then A is U-cyclic, i.e. a choice of epimorphic image of U.

As a corollary one obtains e.g: suppose  $(A_{\gamma})_{\gamma \in \Gamma}$  and  $(A_{\lambda})_{\lambda \in \Lambda}$  are families of copies of an object A in  $\mathfrak{A}$ . Then

$$\bigoplus_{\gamma} A_{\gamma} \cong \bigoplus_{\lambda} A_{\lambda}$$

and there is an  $I \in \mathfrak{F}$  with  $0 < R_I(A) < \aleph_0^{\gamma}$  so  $\Gamma$  and  $\Lambda$  have the same cardinality.

Using the preceding on modules over a ring, the result that basis length of free modules is unique follows. Furthermore, this follows without any of the assumptions of Fuchs[3] on the ring.

### 4. References

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