Journal of Knot Theory and Its Ramifications
Vol. 22, No. 7 (2013) 1350036 (11 pages)

DOI: 10.1142/S0218216513500363

AN INFINITE FAMILY OF KNOTS WHOSE MOSAIC NUMBER IS REALIZED IN NON-REDUCED PROJECTIONS

LEWIS D. LUDWIG*, ERICA L. EVANS† and JOSEPH S. PAAT‡

Department of Mathematics and Computer Science, Denison University, Granville OH 43020, USA
**ludwigl@denison.edu*
†evans_el@denison.edu
‡paat_j@denison.edu

Received 23 June 2011
Accepted 7 May 2013
Published 2 July 2013

ABSTRACT

Lomonaco and Kauffman [Quantum knots and mosaics, *Quantum Inf. Process.* **7**(2–3) (2008) 85–115] introduced the notion of knot mosaics in their work on quantum knots. It is conjectured that knot mosaic type is a complete invariant of tame knots. In this paper, we answer a question of C. Adams by constructing an infinite family of knots whose mosaic number can be realized only when the crossing number is not. That is, there is an infinite family of non-minimal knots whose mosaic numbers are known.

Keywords: Mosaic number; crossing number; knot mosaics.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

This work answers a question posed by Colin Adams at a knot conference in 2009. First, we provide the background to the question. In 2008, Lomonaco and Kauffman [5] introduced knot mosaics in their work on quantum knots. Using the notation of Lomonaco and Kauffman, we now develop the definition of knot mosaic. Let $\mathbb{T}^{(u)} = \{T_i : 0 \leq i \leq 10\}$ be the set of 11 symbols called (unoriented) *tiles* depicted in Fig. 1. Let $n \in \mathbb{N}$, then we define an (unoriented) n-mosaic to be an $n \times n$ matrix $M = (M_{ij}) = (T_{k(i,j)})$ of (unoriented) *tiles* with rows and columns indexed from 1 to n. We denote the set of n-*mosaics* by $\mathbb{M}^{(n)}$. If we call the midpoint of the edge of a tile which is also the endpoint of a curve drawn on the tile a *connection point*, notice that T_0 has zero connection points, T_1–T_6 have two connection points each, and T_7–T_{10} each has four connection points. A tile in a mosaic is said to be *suitably*

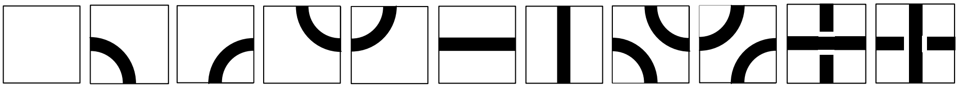
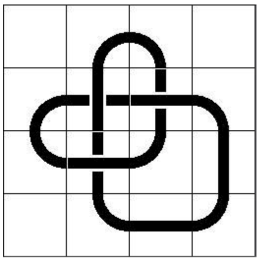


Fig. 1. Tiles T_0–T_{10} respectively.



$$= \begin{pmatrix} T_0 & T_2 & T_1 & T_0 \\ T_2 & T_{10} & T_9 & T_1 \\ T_3 & T_9 & T_4 & T_6 \\ T_0 & T_3 & T_5 & T_4 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 10 & 9 & 1 \\ 3 & 9 & 4 & 6 \\ 0 & 3 & 5 & 4 \end{pmatrix} \in \mathbb{K}^4$$

Fig. 2. The trefoil knot as a 4-mosaic.

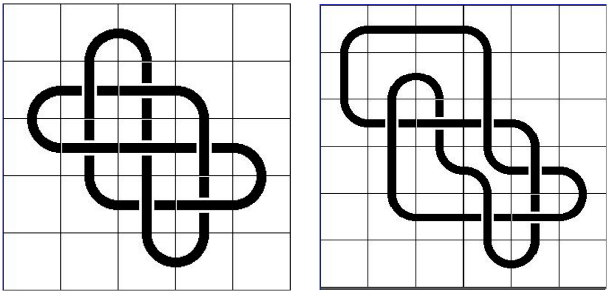
connected if all its connection points touch the connection points of contiguous tiles. An (unoriented) *knot n-mosaic* is a mosaic in which all tiles are suitably connected. Let $\mathbb{K}^{(n)}$ denote the subset of $\mathbb{M}^{(n)}$ of all knot n-mosaics. See Fig. 2 for an example of a trefoil as a 4-mosaic with its corresponding matrix.

In their article, Lomonaco and Kauffman defined Reidemeister-like moves for knot mosaics and conjectured that knot mosaic type is a complete invariant of tame knots. A proof of this conjecture has been submitted by T. Kuriya and O. Shehab [4]. With this in mind, knot mosaic gives yet another way to study knots. In particular, one important invariant with regard to knot mosaics is the *mosaic number*. Define the mosaic number of a knot K to be the smallest integer n for which K can be represented as a knot n-mosaic, denoted $m(K)$. Lomonaco and Kauffman asked whether mosaic number is related to crossing number.

At first examination, one may assume that the mosaic number is realized when the crossing number is realized. This seems intuitively reasonable since the fewest number of crossing tiles (T_9 or T_{10}) is used when the crossing number is realized and this may in turn minimize the overall number of tiles needed to represent a given knot. However, a similar notion was disproved when Bernhard [2] produced an infinite family of knots whose unknotting number was realized only when additional crossings were added beyond the crossing number. Again, a very counterintuitive notion. This led Colin Adams to pose the following question:

> Does there exist an infinite family of knots such that for each knot, the mosaic number is realized only when its crossing number is not?

We answer this question in the affirmative. For the remainder of the work, the reader is referred to Adams [1] for undefined terms or concepts.



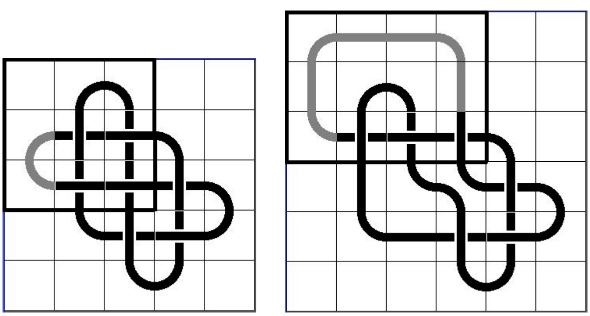
(a) (b)

Fig. 3. The knot 6_1 as a 5-mosaic and as a 6-mosaic.

2. Prelude to the Construction of the Infinite Family

To develop the construction of our infinite family of interest, we first consider the special case of the knot 6_1. By Fig. 3(a), we see the knot 6_1 represented as a 5-mosaic, with seven crossings. However 6_1 cannot be placed on a 4×4 mosaic, because such a mosaic only has at most four possible crossing tiles, as crossing tiles cannot be placed on an outer row or column of a mosaic and still be suitably connected. So the mosaic number of 6_1 is 5 (i.e. $m(6_1) = 5$). However, using an exhaustive search, Jacob Shapiro of Purdue University showed that no projection of 6_1 could be placed on a 5×5 mosaic with only six crossings. Therefore, the projection of 6_1 in Fig. 3(a) is our first example of a knot whose mosaic number is realized when its crossing number is not. Notice that we can project 6_1 on a 6×6 mosaic board so that its crossing number of six is realized, as shown in Fig. 3(b).

The key difference between the mosaics in Fig. 3 is evident in the upper-left 3×3 submosaic of 6_1 as a 5-mosaic and the upper-left 3×4 submosaic of 6_1 as a 6-mosaic depicted in Fig. 4. Roughly speaking, the grey strand on the mosaic in Fig. 4(a) is stretched to the grey strand on the mosaic in Fig. 4(b) via an expansion of the



(a) (b)

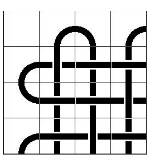
Fig. 4. Changing the knot 6_1 from seven to six crossings.

board and a series of Reidemeister moves. All the other tiles remained fixed. This will be our strategy in constructing our infinite family. Start with a square board of odd size, greater than or equal to 7. Take the upper-left 3×3 submosaic of 6_1 and place it on the board. Fill the remaining inner tiles with alternating crossings and note that the remaining outside tiles will be forced to produce a unique suitably connected mosaic, Fig. 6(c).

3. Construction of the Infinite Family

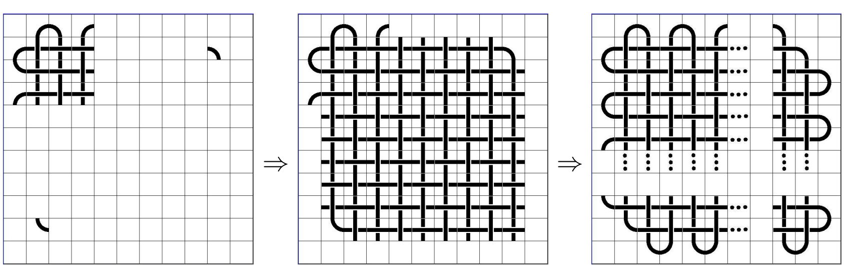
We now construct an infinite family of knot mosaics, $\mathcal{L} = \{L_{2n+1} : n \in \mathbb{N}\}$, such that for each $n \in \mathbb{N}$, the mosaic number of L_{2n+1} can only be realized when its crossing number is not. We construct each $L_{2n+1} \in \mathcal{L}$ in the following way:

(1) For $n = 2$, please see Fig. 3(a).
(2) For $n > 2$, construct L_{2n+1} on the $(2n+1) \times (2n+1)$ mosaic board B_{2n+1} in the following manner:
   (a) Start with the submosaic S_4 (see Fig. 5) in the upper-left corner of B_{2n+1}, then place tile T_1 in position $(2, n-1)$ and tile T_3 in position $(n-1, 2)$ of B_{2n+1}, see Fig. 6(a).
   (b) Place alternating crossing tiles T_9 and T_{10} in the remaining interior positions, see Fig. 6(b).
   (c) Notice that the remaining outer perimeter tiles are now forced in order for L_{2n+1} to be suitably connected to arrive at L_{2n+1}, Fig. 6(c).



$$= \begin{pmatrix} T_0 & T_2 & T_1 & T_2 \\ T_2 & T_{10} & T_9 & T_9 \\ T_3 & T_9 & T_9 & T_{10} \\ T_2 & T_9 & T_{10} & T_9 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 10 & 9 & 9 \\ 3 & 9 & 9 & 10 \\ 2 & 9 & 10 & 9 \end{pmatrix} \in \mathbb{K}^4$$

Fig. 5. Submosaic S_4.



(a) (b) (c)

Fig. 6. Constructing L_{2n+1} on B_{2n+1} from S_4.

Now to show that $\mathcal{L}$ is the infinite family we seek, we must:

(1) Compute the crossing number, $cr(L_{2n+1})$ for each $n \geq 2$.
(2) Compute the mosaic number, $m(L_{2n+1})$ for each $n \geq 2$.
(3) Show that when the mosaic number of L_{2n+1} is realized, the crossing number is not.

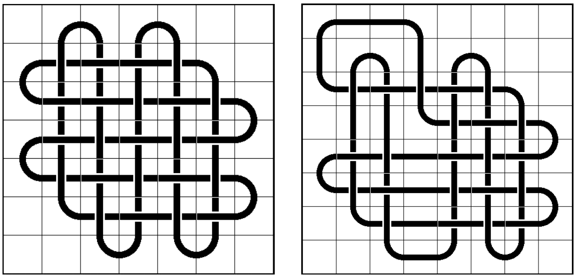
For ease of exposition, unless stated otherwise, we will assume $n \geq 2$ when considering L_{2n+1}. We begin our checklist with the following proposition.

Proposition 3.1. *The crossing number of* L_{2n+1} *is* $4n^2 - 4n - 2$, *that is* $cr(L_{2n+1}) = 4n^2 - 4n - 2$.

Proof. By construction, we see that L_{2n+1} can be placed on a $(2n+1)$-mosaic board with $(2n-1)^2 - 2$ crossings. As Fig. 7(a) demonstrates, the construction of L_{2n+1} on a $(2n+1)$-mosaic is not alternating. Specifically, the crossing tiles in positions $(3, 2)$ and $(3, 3)$ are non-alternating. However, by expanding the board and performing a series of Reidemeister moves, we can create a reduced alternating projection of L_{2n+1} on a $(2n+2)$-mosaic board that has one less crossing as depicted in Fig. 7(b). In 1987, Kauffman [3], Murasagi [8] and Thistlethwaite [9] proved the following conjecture by Tait: the crossing number of an alternating knot occurs in a reduced alternating projection. Since the new projection is a reduced alternating knot, we know its crossing number is realized. That is, $cr(L_{2n+1}) = (2n-1)^2 - 3 = 4n^2 - 4n - 2$. □

Proposition 3.2. *The mosaic number of* L_{2n+1} *is* $2n+1$. *That is* $m(L_{2n+1}) = 2n+1$.

Proof. By construction, we see that L_{2n+1} fits on a $(2n+1)$-mosaic. Moreover, by Proposition 3.1, L_{2n+1} needs $4n^2 - 4n - 2$ crossing tiles. However, a $2n$-mosaic only has $4n^2 - 8n + 4$ possible positions for crossing tiles and $4n^2 - 8n + 4 < 4n^2 - 4n - 2$ for $n \geq 2$, so $m(L_{2n+1}) = 4n^2 - 4n - 2$. □



(a) (b)

Fig. 7. L_{2n+1} as a $(2n+1)$-mosaic with $(2n-1)^2 - 2$ crossings and as a $(2n+2)$-mosaic with $(2n-1)^2 - 3$ crossings.

We have proven two of three components for the main result. We now develop a series of propositions that will be used to prove our main result: when the mosaic number of L_{2n+1} is realized, the crossing number is not. To begin, we first recall the famous Tait Flyping Conjecture:

> Given any two reduced alternating diagrams D_1 and D_2 of an oriented, prime alternating knot, D_1 may be transformed to D_2 by a sequence of flypes.

A flype consists of twisting a part of knot, a tangle T, by 180 degrees. Figure 8 demonstrates a flype. The Tait Flyping Conjecture was proven by Menasco and Thistlethwaite using a blend of geometric techniques and polynomials [7]. While Tait's original statement dealt with knots on the surface of a sphere, we can think of the sphere as being significantly large so that our drawings appear on the plane.

Notice that by our construction, any reduced alternating projection of L_{2n+1} has only one possible flype, for example see Fig. 9. Hence there are only two projections of L_{2n+1} that we have to consider which we refer to as D^1_{2n+1} and D^2_{2n+1}. For example, for $n = 3$, D^1_7 and D^2_7 are depicted in Fig. 9. By considering these

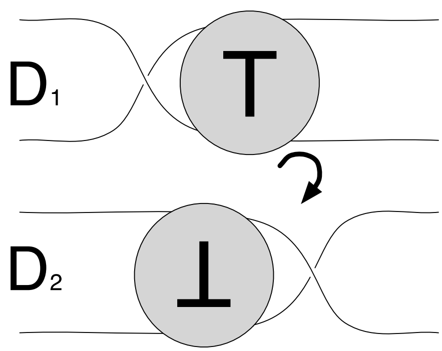
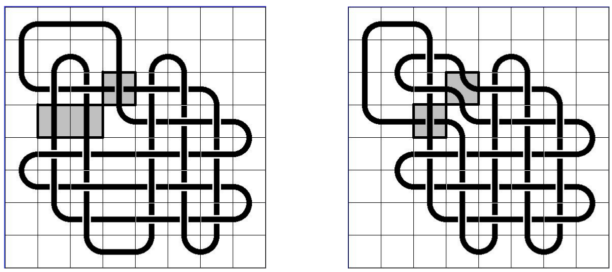


Fig. 8. A flype.



(a) (b)

Fig. 9. Example of the flype in L_7.

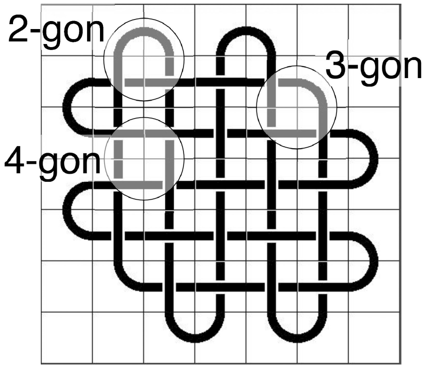


Fig. 10. Example of a 2-gon, 3-gon, and 4-gon.

reduced alternating projections, we also have that L_{2n+1} is prime, so Tait's conjecture applies. L_{2n+1} is prime due to a result by Menasco [6] that states if a knot is composite, then this is immediately apparent from any alternating projection of the knot in which trivial crossings have been eliminated.

Now that we only need to consider two projections of L_{2n+1}, we next consider the m-gons formed in these projections. Please see Fig. 10 for an example of a 2-gon, 3-gon and 4-gon. We see that the types of m-gons in any embedding of L_{2n+1} is completely determined as established by our next proposition.

Proposition 3.3. *Every reduced alternating projection of* L_{2n+1}, $n \geq 3$, *is solely composed of at least one* 2*-gon*, 3*-gon*, 4*-gon and* 5*-gon and only one* $(8n-11)$*-gon.*

Proof. As L_{2n+1} has only two knot projections, we need only to consider projections D^1_{2n+1} and D^2_{2n+1} of L_{2n+1} as in Fig. 9. Notice with these two projections, for any $n \geq 3$, the upper-left 5×5 submosaic remains unchanged for each of these projections. These 5×5 submosaics contain at least one 2-gon, 3-gon, 4-gon and 5-gon. Moreover, the region outside the 5×5 submosaic is comprised of crossing tiles (T_9 or T_{10}), quarter circle tiles (T_1–T_4) or T_5. By construction, these tiles will only produce 2-gons, 3-gons, or 4-gons.

The last m-gon to consider is the m-gon that creates the perimeter D^1_{2n+1} and D^2_{2n+1} as depicted in Fig. 9. We proceed by induction starting with $n = 2$. In this case, we simply count the perimeter m-gon of the projection of L_5 on B_6, see Fig. 3(b), and note it is a 5-gon. Assume that the perimeter of D^1_{2n+1} (or D^2_{2n+1}) is an m-gon on B_{2n+2}, where $m = 8n-11$. To create a reduced alternating projection of L_{2n+3} on B_{2n+4}, we add two rows to the bottom and two columns to the right side of our $(2n+1) \times (2n+1)$ mosaic board. L_{2n+3} will then have two additional rows and columns of crossings, which will increase the number of edges appearing in perimeter m-gon of L_{2n+3} by a total of 8 — two new edges for each of the four

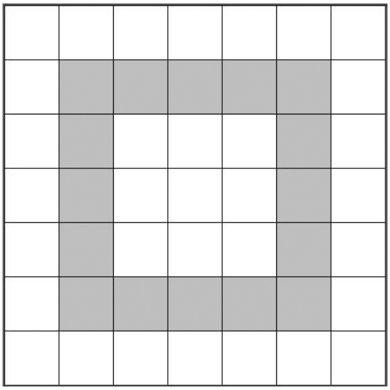


Fig. 11. The inner boundary of a 7×7 mosaic board.

"sides" of the m-gon. Hence, $m = 8n - 11 + 8 = 8(n+1) - 11$ for the perimeter m-gon of a reduced alternating projection of L_{2n+3} on B_{2n+4}. □

Before our next proposition, we provide a definition for ease of exposition. Consider a $(2n+1) \times (2n+1)$ mosaic board, by *inner boundary*, we are referring to the tile positions $\{(2,m) : 2 \leq m \leq 2n\} \cup \{(2n,m) : 2 \leq m \leq 2n\} \cup \{(m,2) : 2 \leq m \leq 2n\} \cup \{(m,2n) : 2 \leq m \leq 2n\}$, see Fig. 11.

Proposition 3.4. *If the outside perimeter of a projection of L_{2n+1} is determined by an m-gon, there are at most m crossing tiles on the inner boundary of L_{2n+1}.*

Proof. Assume there is an m-gon defining the outer perimeter for a projection of L_{2n+1} and that there are $k > m$ crossing tiles on the inside boundary of L_{2n+1}. When a crossing tile is placed on the inner boundary, it will contribute two concurrent edges to the m-gon. If we sum all these edges, we note each is counted twice, so there are exactly k such edges along perimeter of L_{2n+1}, a contradiction. □

We can now make a useful observation. By Propositions 3.3 and 3.4, if a reduced alternating projection of L_{2n+1} can be placed on a $(2n+1)$-mosaic board, it must be the case that the $(8n-11)$-gon forms the outer perimeter of the projection and the inner boundary of this mosaic board must have $(8n-11)$ crossing tiles.

Before we prove our main result, we need two other observations. First, no reduced, alternating projection of L_{2n+1} can have an entire row or column of crossing tiles along the inner boundary. If it did, it would not be reduced. Secondly, a simple counting argument shows that any $r \times s$ submosaic S of a knot mosaic K has an even number of connection points to $K \backslash S$. For referencing purposes, we now state these observations as lemmas.

Lemma 3.5. *No reduced, alternating projection of L_{2n+1} can have an entire row or column of crossing tiles on the inner boundary.*

Lemma 3.6. *Any $r \times s$ submosaic S of a knot mosaic K has an even number of connection points to $K \backslash S$.*

We now state and prove the main result.

Theorem 3.7. *For all $L_{2n+1} \in L$, $m(L_{2n+1})$ can only be realized when $cr(L_{2n+1})$ is not.*

Proof. The case of L_5 has already been established, so consider $n \geq 3$. In this case by Propositions 3.1 and 3.2, $cr(L_{2n+1}) = 4n^2 - 4n - 2$ and $m(L_{2n+1}) = 2n + 1$. So it is sufficient to show that L_{2n+1} cannot be realized on a $(2n+1) \times (2n+1)$ mosaic board with $4n^2 - 4n - 2$ crossings. We now construct a reduced, alternating projection of L_{2n+1} and argue that the crossing number cannot be attained.

Notice that for a $(2n+1) \times (2n+1)$ mosaic board the inner boundary has $8n - 8$ tiles. By our observation following Proposition 3.4, $8n - 11$ of these tiles must be crossing tiles (T_9 or T_{10}). So if we are going to construct L_{2n+1} with exactly $4n^2 - 4n - 2$ crossing tiles, we must have three non-crossing tiles on the inner boundary. We show that this cannot be the case.

To satisfy Lemma 3.5 and insure the projection is reduced, exactly one of these non-crossing tiles must be placed in a corner of the inner boundary. Without loss of generality, we place the non-crossing tile T_2 in position $(2, 2)$, or location one in Fig. 12. Figure 12 labels the regions where the other two non-crossing tiles can be placed. There are seven regions to place two non-crossing tiles, so we will consider the 21 combinations of placing these two non-crossing tiles in the next six cases.

Case 1: $\{\mathbf{a}, \mathbf{d}\}$, $\{\mathbf{a}, \mathbf{3}\}$, $\{\mathbf{a}, \mathbf{b}\}$, $\{\mathbf{a}, \mathbf{2}\}$, $\{\mathbf{a}, \mathbf{c}\}$, $\{\mathbf{b}, \mathbf{2}\}$, $\{\mathbf{c}, \mathbf{3}\}$, $\{\mathbf{d}, \mathbf{2}\}$, $\{\mathbf{d}, \mathbf{b}\}$, $\{\mathbf{d}, \mathbf{c}\}$, $\{\mathbf{d}, \mathbf{3}\}$. If the other two non-crossing tiles are placed in any of these pairs of locations, there will be a row or column of crossing tiles on the inner boundary, contradicting Lemma 3.5.

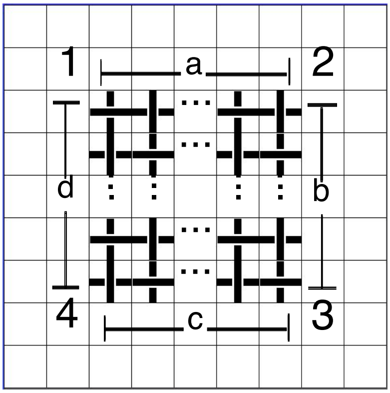


Fig. 12. Possible locations for the non-crossing tiles.

Case 2: $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$. This arrangement can be suitably connected along the outside perimeter in a unique way so as to create a reduced, alternating knot. However, such a construction does not produce a 5-gon, contradicting Proposition 3.3.

Case 3: $\{b, c\}$. If we try to place the non-crossing tiles in region b or c, both of the two available positions in these regions will be surrounded on three sides by crossing tiles, contradicting Lemma 3.6.

Case 4: $\{3, b\}$, $\{2, c\}$. This case is similar to Case 3.

Case 5: $\{4, b\}$, $\{4, c\}$. If the two non-crossing tiles are placed in either of these pairs of regions, there is only one way to suitably connect the projection so it is reduced, alternating. However, this projection does not contain a 5-gon, contradicting Proposition 3.3.

Case 6: $\{4, a\}$, $\{4, d\}$. This case is similar to Case 5.

Therefore there does not exist a valid way of placing three non-crossing tiles along the inside perimeter in an attempt to create L_{2n+1}. Therefore $c(L_{2n+1})$ cannot be realized on a $(2n + 1) \times (2n + 1)$ mosaic and our result is established. □

4. Questions

We close with several questions that are natural extensions of this work.

(1) Are there other infinite class of knots for which we can compute the mosaic number? $(2, q)$-torus knots? Pretzel knots?
(2) Can the mosaic number for all knots of 10 or fewer crossings be determined?
(3) Can the crossing number be used as a bound for determining mosaic number?
(4) Does there exist a knot whose mosaic number is n, but whose crossing number is only realized on a mosaic board of size $n + 2$?

Acknowledgments

Support for this research was provided by Denison University and the Anderson Science Endowment. The authors would like to thank Jacob Shapiro of Purdue University and Colin Adams of Williams College for their suggestions and encouragement. We would also like to thank the reviewer for his/her encouraging words and excellent suggestions.

References

[1] C. C. Adams, *The Knot Book* (American Mathematical Society, Providence, RI, 2004).
[2] J. A. Bernhard, Unknotting numbers and minimal knot diagrams, *J. Knot Theory Ramifications* **3**(1) (1994), DOI: 10.1142/S0218216594000022.

[3] L. Kauffman, New invariants in the theory of knots, *Amer. Math. Monthly* **95**(3) (1988) 195–242.

[4] T. Kuriya and O. Shebab, The Lomonaco–Kauffman conjecture, preprint (2008), arXiv.math:0811.0710.

[5] S. J. Lomonaco and L. Kauffman, Quantum knots and mosaics, *Quantum Inf. Process.* **7**(2–3) (2008) 85–115.

[6] W. Menasco, Closed incompressible surfaces in alternating knot and link complements, *Topology* **23**(1) (1984) 37–44.

[7] W. W. Menasco and M. B. Thistlethwaite, The Tait flyping conjecture, *Bull. Amer. Math. Soc.* **25**(2) (1991) 403–412.

[8] K. Murasugi, Jones polynomials and classical conjectures in knot theory, *Topology* **26** (1987) 187–194.

[9] M. B. Thistlethwaite, A spanning tree expansion of the Jones polynomial, *Topology* **26**(3) (1987) 297–309.