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LINKING IN STRAIGHT-EDGE EMBEDDINGS OF K_7

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ABSTRACT

In 1983 Conway and Gordon and Sachs proved that every embedding of the complete graph on six vertices, K_6, is intrinsically linked. In 2004 it was shown that all straight-edge embeddings of K_6 have either one or three linked triangle pairs. We expand this work to characterize the straight-edge embeddings of K_7 and determine the number and types of links in every embedding which forms a convex polyhedron of seven vertices.

Keywords: Complete graph; convex polyhedra; straight-edge embedding; linking.

Mathematics Subject Classification 2010: 57M15, 57M25

1. Introduction

In 1983, Conway and Gordon [3] and Sachs [11] proved that the complete graph on six vertices, K_6 is *intrinsically linked*. A graph is intrinsically linked if every embedding of the graph in $\mathbb{R}^3$ contains a homologically non-trivial link of two or more components. This result has spawned a significant amount of work, including the complete classification of minor minimal examples for intrinsically linked graphs by Robertson, Seymour and Thomas [10]. After the completion of this classification, work has turned to finding graphs in which every embedding has a more complex structure [2, 4–6].

Continuing this work, we consider straight line embeddings of the complete graph. Although every embedding of K_6 has 10 disjoint triangle pairs that may be linked, in 2004 Hughes [7] and in 2007 Huh and Jeon [8] proved that every *straight-edge* embedding of K_6 has only one or three pairs of linked (triangle) components. There are four such embeddings of K_6: one with four external vertices, one with five external vertices and two with six external vertices. In the above work [7], it was showed that three of the four graphs are topologically equivalent, so they needed to only consider two embeddings denoted K_6^1 and K_6^2 (see Figs. 1 and 2).

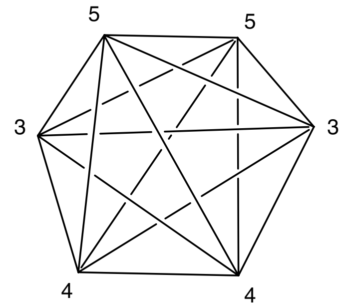


Fig. 1. K_6^1: one (3–3) link; external degree of each vertex indicated.

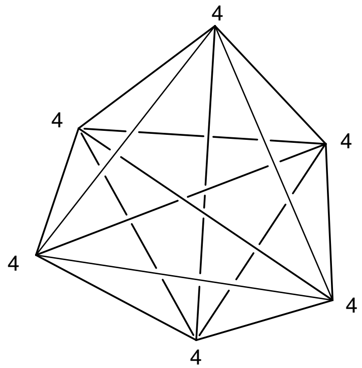


Fig. 2. K_6^2: three (3–3) links link; external degree of each vertex indicated.

We expand this work by classifying and enumerating all 2-component links contained in straight-edge embeddings of K_7 that form convex polyhedra with seven vertices (it is well known that there are five such graphs up to isomorphism, see, for example, [12]; see Figs. 4–8.) This work may be of interest to molecular chemists who are trying to synthesize topologically complex molecules. One could imagine that the vertices of these graphs represent atoms and the edges are the bonds of a molecule.

Theorem 1.1. *The minimum number of linked components in any straight-edge embedding of* K_7 *which forms a convex polyhedron of seven vertices is twenty-one, and the maximum number of linked components in* K_7 *is forty-eight. Specifically, we have the following*:

First, a brief explanation of the table in Fig. 3. Consider the column headed with K_7^1. According to the table, *any* straight-edge embedding of K_7 which has an external degree set denoted by K_7^1 will always have 7 (3–3) links and 14 (3–4) links. Now consider the column headed K_7^4. Due to the arrangement of the vertices on the

	K_7^1	K_7^2		K_7^3		K_7^4				K_7^5		
(3-3)	7	7	9	7	9	9	11	13	15	13	15	17
(3-4)	14	14	18	14	18	18	22	23,26	27,30	23,26	27,30	31

Fig. 3. Number of (3–3) and (3–4) links in straight line embeddings of K_7.

hull, as well as the positioning of the internal edges, this is a much more complicated case. A straight-edge embedding of K_7 which has an external degree set denoted by K_7^4 can have 4 possible amounts of (3–3) links. Moreover, there exists embeddings of K_7^4 with 13 (3–3) links and 23 (3–4) links as well as embeddings with 13 (3–3) links and 26 (3–4) links.

In order to prove the main result, we systematically consider each of the five families of straight-edge embeddings of K_7 and remove a vertex and its adjoining edges to obtain either K_6^1 or K_6^2. We begin by making some elementary observations about the hull of convex polyhedra formed by K_7 when a vertex is removed. These results are used to analyze the possible linked pairs in each of the K_7^i $(i = 1, \ldots, 5)$.

2. Necessary Preliminaries

To distinguish the five convex polyhedra, we label each by its *external degree* set, $[\overline{\delta}_1, \overline{\delta}_2, \overline{\delta}_3, \overline{\delta}_4, \overline{\delta}_5, \overline{\delta}_6, \overline{\delta}_7]$. Here, the external degree $\overline{\delta}_i$ of a vertex v_i is the number of edges on the hull of the convex polyhedron incident to that vertex. A variation of Steinitz's Theorem [13] guarantees that every 3-connected planar graph has essentially only one planar embedding, see, for example, [14, Exercise 8.2.46]. So the placement of the vertices on the convex hulls of the embeddings in Figs. 4–8 are unique up to isomorphism.

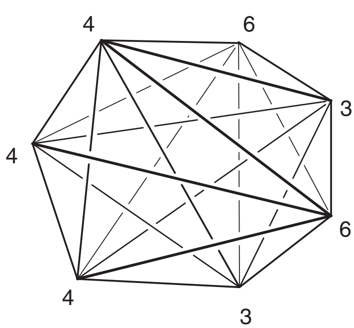


Fig. 4. K_7^1 $[3, 3, 4, 4, 4, 6, 6]$.

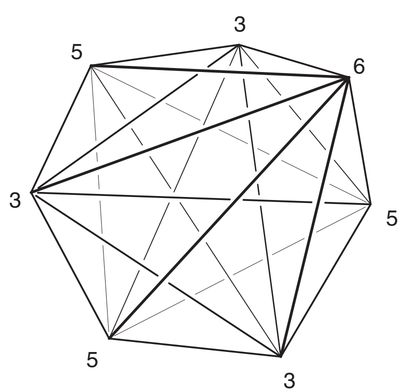


Fig. 5. K_7^2 $[3, 3, 3, 5, 5, 5, 6]$.

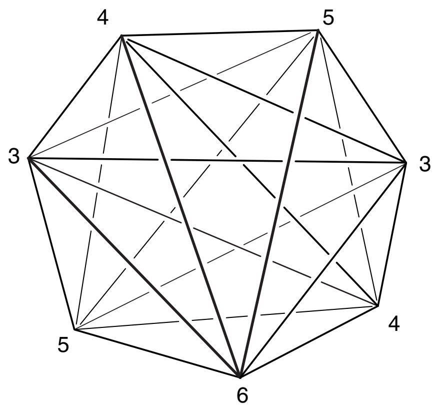


Fig. 6. K_7^3 [3, 3, 4, 4, 5, 5, 6].

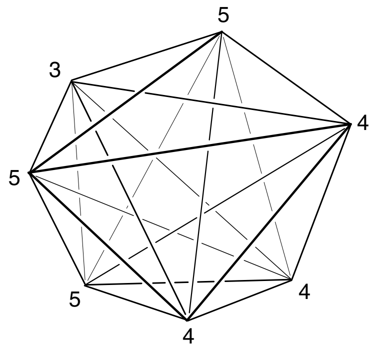


Fig. 7. K_7^4 [3, 4, 4, 4, 5, 5, 5].

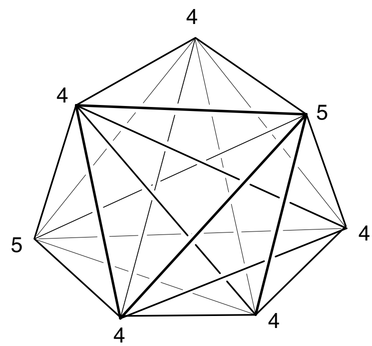
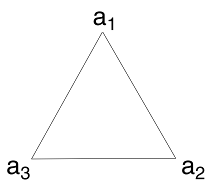
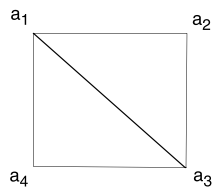
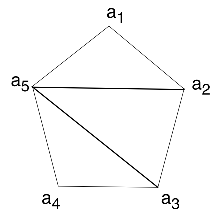


Fig. 8. K_7^5 [4, 4, 4, 4, 4, 5, 5].

Theorem 2.1 (Steinitz's Theorem). *A graph G is the edge graph of a polyhedron if and only if G is a simple planar graph which is 3-connected.*

If we refer to the degree of a vertex, we actually mean the *external degree*, unless otherwise stated. Also, for ease of notation, when we refer to an *embedding of K_7* we actually mean a straight-edge embedding of K_7 that forms a convex polyhedron with 7 vertices. As K_7 has seven vertices, it can have two types of links: a link composed of two 3-cycles or a link composed of a 3-cycle and a 4-cycle. We refer to these as (3–3) *links* or (3–4) *links*.

The table in Fig. 9 addresses the consequences of removing a vertex of degree 3 to 6 from a straight line embedding of K_7. Let v be a vertex on the hull of the polyhedron formed by K_7. This vertex will have $i = 3, \ldots, 6$ neighboring vertices on the hull that form an i-gon. When v is removed, the resulting polyhedron hull is formed by K_6, so the faces of the polyhedron are triangles. Therefore, the resulting

$\deg(v) = 3$ Type 3	$\deg(v) = 4$ Type 4	$\deg(v) = 5$ Type 5
		
$\Delta\deg(a_1, a_2, a_3) = -1$	$\Delta\deg(a_1, a_3) = 0$ $\Delta\deg(a_2, a_4) = -1$	$\Delta\deg(a_1, a_4) = -1$ $\Delta\deg(a_2, a_4) = 0$ $\Delta\deg(a_5) = +1$

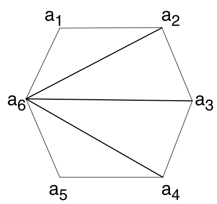
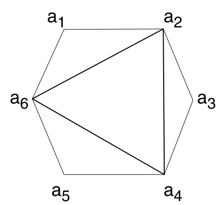
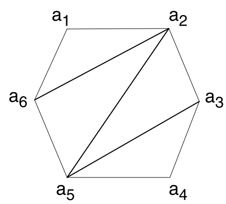
$\deg(v) = 6$ Type 6.1	$\deg(v) = 6$ Type 6.2	$\deg(v) = 6$ Type 6.3
		
$\Delta\deg(a_1, a_5) = -1$ $\Delta\deg(a_6) = +2$ $\Delta\deg(a_2, a_3, a_4) = 0$	$\Delta\deg(a_1, a_3, a_5) = -1$ $\Delta\deg(a_2, a_4, a_6) = +1$	$\Delta\deg(a_1, a_4) = -1$ $\Delta\deg(a_2, a_5) = +1$ $\Delta\deg(a_3, a_6) = 0$

Fig. 9. The resulting degree changes when a vertex v of given degree is removed from K_7.

i-gon will be triangulated by the new edges exposed after the removal of v and its adjacent edges, as denoted in the table. Those familiar with graph theory will recognize Type 6.1–3 as the maximal outerplanar graphs on six vertices. Notice that Type 6.1 is a special case. A Type 6.1 removal always results in K_6^1, as a degree 3 vertex will increase to a degree 5.

3. Proof of the Main Result: (3–3) Links

We now classify the number of (3–3) links and (3–4) links found in the five families of convex embeddings of K_7 determined by their external degree sets. We will see that the number of links in four of these five embeddings varies depending on the conformation considered.

Again, our method is to consider an embedding of K_7 with particular degree set and remove a vertex. We then examine which version(s) of K_6 results (result) and count the number of (3–3) links accordingly.

Proposition 3.1. *Every embedding of* K_7^1 *has* 7 (3–3) *links. Every embedding of* K_7^2 *and* K_7^3 *have either* 7 *or* 9 (3–3) *links.*

Proof. Fix a vertex v in K_7^i $(i = 1, 2, 3)$. If there exists a different vertex u with degree 6, then the removal of v will cause the degree of u to decrease by 1, resulting in K_6^1. As K_6^1 only has 1 (3–3) link, K_7^1 has 7 (3–3) links, one for each removal of a given vertex.

This shows that K_7^2 and K_7^3 will have at least 6 (3–3) links, one for each vertex removed that is not of degree 6. Now consider the vertex, s, of degree six in K_7^2 and K_7^3. For K_7^2, a straightforward counting argument shows that a Type 6.3 polygon from the table in Fig. 9 cannot occur. If a Type 6.1 polygon occurs, K_6^1 results. If a Type 6.2 polygon occurs, K_6^2 results. Similarly, for K_7^3, a Type 6.2 polygon cannot occur. If a Type 6.1 polygon occurs, K_6^1 results. If a Type 6.3 polygon occurs, *either* K_6^1 or K_6^2 occurs. We conclude that K_7^2 and K_7^3 have either 7 or 9 (3–3) links. □

Now consider the (3–3) links in K_7^4. By Steinitz's Theorem, Fig. 10 is the only planar representation of the edge graph of the polyhedron formed in case K_7^4. Notice that the degree 3 vertex is only incident to the degree 5 vertices. Similarly, Fig. 11 is the only planar representation of the edge graph of the polyhedron formed by K_7^5. Here it should be noted the degree 5 vertices are not adjacent on the external hull.

Proposition 3.2. *Every straight-edge embedding of* K_7^4 *has either* $9, 11, 13,$ *or* 15 (3–3) *links.*

Proof. By Fig. 10 and the table in Fig. 9, if the degree 3 vertex is removed from K_7^4, K_6^2 results producing 3 (3–3) links. Since the degree 3 vertex is not adjacent to any degree 4 vertex, the removal of a degree 4 vertex results in K_6^1 producing 1 (3–3) link each.

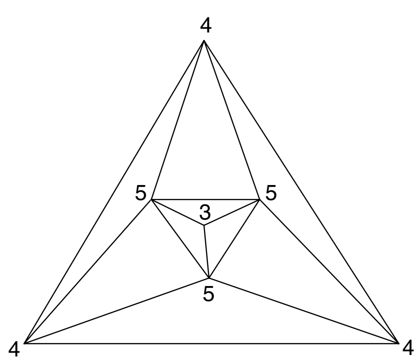


Fig. 10. Planar K_7^4.

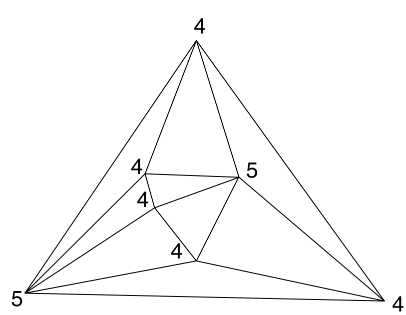


Fig. 11. Planar K_7^5.

The removal of a degree 5 vertex, v, has two cases. To establish them, consider the two degree 4 vertices adjacent to v and the degree three vertex. Orient the polyhedron such that the plane formed by these three vertices lies in the xy-plane of $\mathbb{R}^3$ and vertex v has a positive z-coordinate. With this orientation, at most two vertices could have a positive z coordinate:

Case I. v is the only vertex in this orientation with a positive z coordinate, Fig. 12.

Case II. There exists another degree 5 vertex, u, with positive z-coordinate, Fig. 13.

To see why there is only two cases, we consider the possibilities of the other two vertices, a degree five and a degree four, having a positive z coordinate. If all three degree 5 vertices had positive z-coordinates, the degree 3 vertex would be adjacent to two degree four vertices on the polyheral hull, contradicting Fig. 10. On the other hand, suppose the remaining degree 4 vertex had a positive z coordinate. This would cause the vertex itself to be internal to the convex hull or cause the vertex to be externally connected to the degree five vertex v, contradicting the hypothesis.

In Case I, removal of v will expose the two internal edges connecting the degree 4 vertices and the degree 3 vertex, resulting in a Type 5 removal from the table in Fig. 9. Hence Case I results in K_6^2.

In Case II, when the edge connecting vertex v and the degree 5 vertex below the plane is removed, one of the degree 4 vertices will have two internal edges exposed (the edges adjoining it to u and the degree 3 vertex). Again, a Type 5 removal results, but as the degree 4 vertex increases in degree by one, K_6^1 results in this case.

We now calculate the frequency of the (3–3) links. By the argument above, the number of (3–3) links in K_7^4 is only affected by the positioning of the degree 5 vertices. To see how the removal of a degree five vertex affects the (3–3) links, consider Figs. 15–18. These perspectives are created by placing the three degree 4 vertices in the xy-plane and giving the degree 3 vertex a positive z coordinate, then looking down the positive portion of the z-axis. These represent aerial views of the tetrahedron formed by the degree 3 vertex and the three degree 4 vertices with the degree 5 vertices, v, u, and x, placed accordingly. Without loss of generality, we

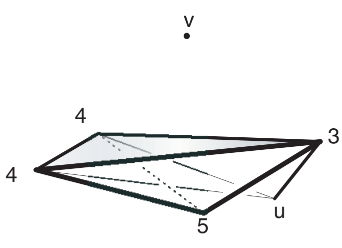


Fig. 12. Case I.

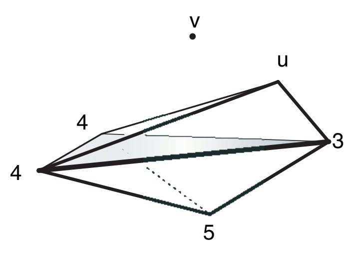


Fig. 13. Case II.

assume the vertices u, v, and x do not intersect the edges of the tetrahedron. The argument given earlier in the proof insures there are only four possible arrangements of the degree 5 vertices up to symmetry. In these four cases, each of the degree five vertices will either behave like Case I or Case II. This is summarized in the Fig. 14. □

We end this section by discussing the (3–3) links in K_7^5.

Proposition 3.3. *Every straight-edge embedding of* K_7^5 *has either* 13, 15, *or* 17 (3–3) *links.*

Proof. From Fig. 11 and the table in Fig. 9 we see that whenever a degree 5 vertex is removed, K_6^1 is formed. When a degree 4 vertex is removed, a Type 4 removal

	u	v	x	(3-3) links
Case 1	Case I	Case I	Case I	15
Case 2	Case I	Case II	Case I	13
Case 3	Case I	Case II	Case II	11
Case 4	Case II	Case II	Case II	9

Fig. 14. Summary of (3–3) links in K_7^4 due to removal of degree 5 vertex.

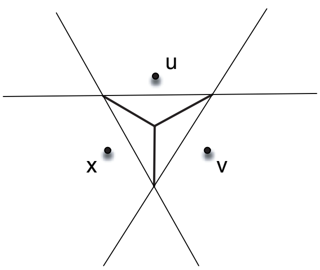


Fig. 15. Case 1.

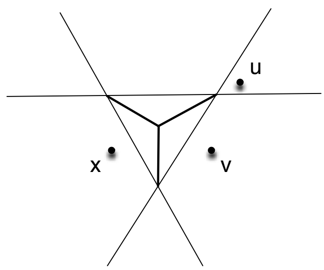


Fig. 16. Case 2.

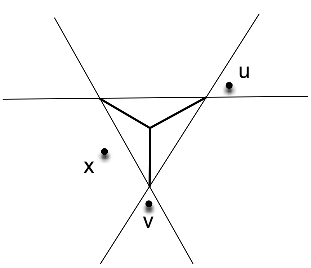


Fig. 17. Case 3.

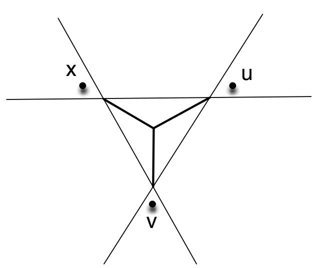


Fig. 18. Case 4.

occurs according to the table in Fig. 9. As the degree 5 vertices are not adjacent on the hull of K_7^5, the diagonal edge exposed with a Type 4 removal will either consist of two degree 4 vertices or two degree 5 vertices. If the two vertices associated with the diagonal edge are of degree 4, then K_6^2 will result due to this Type 4 removal. Similarly, if the two vertices are of degree 5, then K_6^1 will result.

We next consider the possible internal structure of K_7^5 that determines the number of (3–3) links. Recall the degree 5 vertices are not adjacent on the convex hull. This insures that each degree 5 vertex has an external edge to each degree 4 vertex. We now focus on the five-cycle $(v_1, v_2, v_3, v_4, v_5)$ consisting of the degree 4 vertices. Using the internal edges incident to the vertices in the five-cycle, we can divide the five-cylce into a number of different triangles, as depicted in Figs. 19–23.

Consider the three triangles depicted in Fig. 19 (respectively, 20–23). If we denote the degree 5 vertices as v_6 and v_7, we claim that the internal edge E

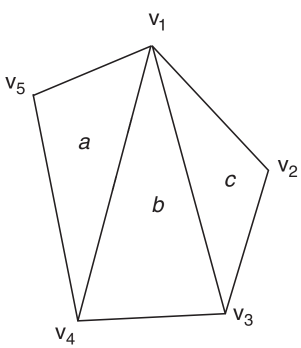


Fig. 19. Incident to V_1.

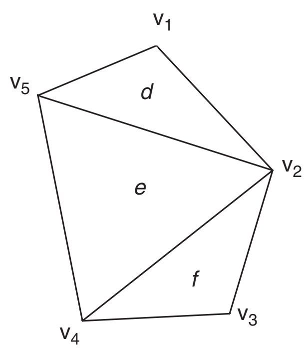


Fig. 20. Incident to V_2.

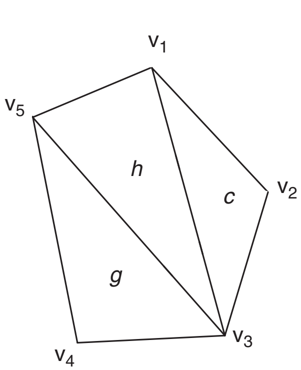


Fig. 21. Incident to V_3.

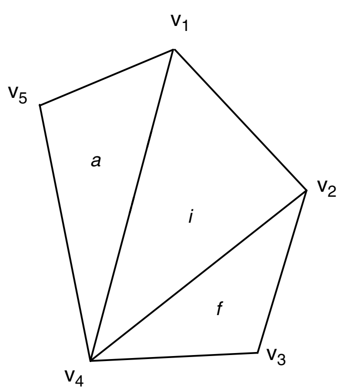


Fig. 22. Incident to V_4.

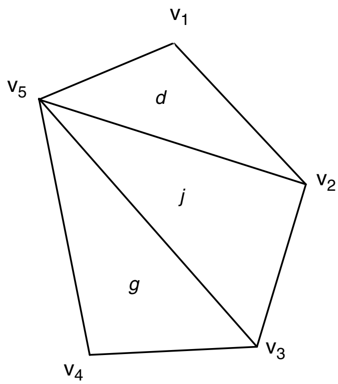


Fig. 23. Incident to V_5.

connecting v_6 and v_7 will pass through only one of the triangles $a - j$ depicted in each of Fig. 19 (respectively, 20–23). Since v_6 (v_7) connects to all of the degree 4 vertices by external edges, the 3 triangles from Fig. 19 (respectively, 20–23) and v_6 (v_7) form three tetrahedra, T_1, T_2, T_3, with T_i and T_{i+1} sharing a common face and T_1 and T_3 sharing only v_6 (v_7). Any ray emanating from v_6 can thus intersect the interior of at most one of the T_i. It follows that the edge E must intersect exactly one of the T_i, else it would not be an interior edge. Since E must puncture the face of T_i opposite v_6 (v_7), our claim thus follows.

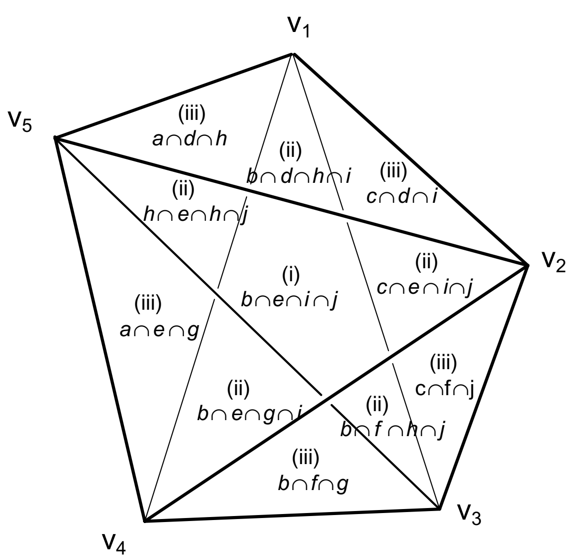


Fig. 24. K_5 formed by the degree 4 vertices.

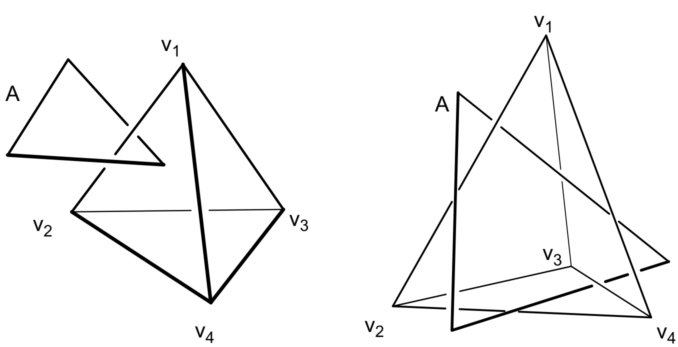
We now consider the K_5 subgraph determined by vertices $v_1 - v_5$, see Fig. 24. Using the triangles depicted in Figs. 19–23, we see that the K_5 has three types of triangular regions, labelled (i), (ii), or (iii). Since E is an internal edge, it will have to pass through one of the three types of regions. If E passes through region (i) in K_5, then a removal of any degree 4 vertex will not expose E and K_6^2 will result. If E passes through region (ii), then the removal of the degree 4 vertex associated with region (ii) will expose E, resulting in K_6^1. Removal of any other degree 4 vertex in this situation will not expose E and result in K_6^2. Finally, if E passes through region (iii), the removal of either degree 4 vertex associated with region (iii) will expose E. In this situation, removal of each degree 4 vertex will result in two copies of K_6^1 and three copies of K_6^2. $\square$

4. Proof of the Main Result: (3–4) Links

We now focus on the (3–4) links formed in straight-edge embeddings of K_7. Consider a triangle of K_7 that belongs to a (3–3) link. The four remaining vertices form K_4, so the triangle is actually linked to two or four triangles (see Figs. 25(a) and 25(b), note that the triangular faces B_1 through B_4 refer to both figures). This is a consequence of Lemma 4.2 and the fact that in straight-edge embeddings, the linking number of two triangles is either $-1, +1$, or 0. This idea is crucial to the arguments for the (3–4) links, so we state it as a proposition and will refer to these cases as 2-link tetrahedron and 4-link tetrahedron.

Proposition 4.1. *A triangle in a* (3–3) *link in* K_7 *is in two or four distinct* (3–3) *links.*

As the next two propositions show, the number of (3–4) links in an embedding of K_7 is determined by the number of 2-link tetrahedrons and 4-link tetrahedrons it contains. We will see that 4-link tetrahedrons are less common and a given



(a) 2-link tetrahedron (b) 4-link tetrahedron

Fig. 25. $B_1 = \{v_1, v_2, v_4\}, B_2 = \{v_1, v_2, v_3\}, B_3 = \{v_1, v_3, v_4\}, B_4 = \{v_2, v_3, v_4\}$.

embedding can have at most one. To begin, we need the following lemma which is a modification of Observations 1 and 2 in Flapan *et al.* [4].

Lemma 4.2. *Let A, B, and C be triangles in a straight-edge embedding of K_7 with A disjoint from B and C, and B and C sharing one edge. Define $B + C$ as $[E(B) \cup E(C)] - [E(B) \cap E(C)]$. That is, $B + C$ is the 4-cycle formed by the union of the edges of B and C with the intersecting edge between B and C removed. If $\mathrm{lk}(X, Y)$ is the mod 2 linking number of any two cycles X and Y, then $\mathrm{lk}(A, B) + \mathrm{lk}(A, C) = \mathrm{lk}(A, B + C)$.*

Proposition 4.3. *Every triangle that is part of a 2-link tetrahedron is linked to exactly two distinct quadrilaterals.*

Proof. Consider an embedding of K_7 with a 2-link tetrahedron consisting of a triangle A and two triangles of the tetrahedron, B_1 and B_2. Then $\mathrm{lk}(A, B_1) = 1$, $\mathrm{lk}(A, B_2) = 1$, $\mathrm{lk}(A, B_3) = 0$, and $\mathrm{lk}(A, B_4) = 0$. The three distinct quadrilaterals formed in the tetrahedron are $B_1 + B_2, B_1 + B_3$, and $B_1 + B_4$. From Lemma 4.2, we know $\mathrm{lk}(A, B_1 + B_2) = 0, \mathrm{lk}(A, B_1 + B_3) = 1$, and $\mathrm{lk}(A, B_1 + B_4) = 1$. This shows that A is linked to $B_1 + B_3$ and $B_1 + B_4$. Clearly if the edge shared between B_1 and B_2 is removed, the components A and $B_1 + B_2$ are splittable. □

While a 4-link tetrahedron has more (3–3) links than a 2-link tetrahedron, it only has 1 (3–4) link.

Proposition 4.4. *Every triangle that is part of a 4-link tetrahedron is linked to exactly one quadrilateral.*

Proof. Straightforward calculations of the linking numbers of Fig. 25(b) show $\mathrm{lk}(A, B_1 + B_2) = \mathrm{lk}(A, B_1 + B_4) = 0$ and $\mathrm{lk}(A, B_1 + B_3) = \pm 2$. As A and $B_1 + B_2$ (similarly A and $B_1 + B_4$) have a linking number of zero, but only 7 sticks, they cannot form the Whitehead link which has a stick number of 8 [1], so these links are splittable. The non-splittable link formed by A and $B_1 + B_3$ is a stick representation of what is widely known as King Solomon's "Knot". □

Now that we know 2-link and 4-link tetrahedrons can occur in straight-edge embeddings of K_7 and how they contribute to the (3–4) linking, the next obvious question is in what embeddings can they occur? Let us consider the 4-link tetrahedron case. To have a triangle, T, linked to all four faces of the tetrahedron, the triangle must have at least two internal edges with respect to the polyhedron formed by K_7. This type of linking causes all the faces of the tetrahedron to be internal. Since triangle T links all four triangles, T must be punctured by exactly one edge from each of the triangles. The only way for this to happen is for the triangle T to be punctured by exactly 2 non-adjacent edges of the tetrahedron. This cannot occur with K_7^1, K_7^2 or K_7^3 because given any triangle with at least two internal edges, there are not two additional internal edges that are non-adjacent. To have this number of internal edges would require seven vertices with at least one internal edge each. But these cases all have at least one degree 6 vertex, which is not incident to an internal edge. This argument provides our next proposition.

Proposition 4.5. *K_7^1, K_7^2, and K_7^3 only contain 2-link tetrahedrons.*

With Proposition 4.5 in mind, by a simple counting argument, we have the following.

Proposition 4.6. *If a straight-edge embedding of K_7 contains n (3–3) links in which every triangle in a (3–3) link is part of a 2-link tetrahedron, there are n distinct triangles forming these links.*

We are now ready to count the number of (3–4) links in K_7, that is, links formed by a triangle and quadrilateral. Combining Propositions 4.3, 4.5, 4.6, we have the number of (3–4) links in K_7^1, K_7^2, and K_7^3 as stated in Theorem 1.1. The cases K_7^4 and K_7^5 will take more work.

Proposition 4.7. *An embedding of K_7^4 with 9 or 11 (3–3) links contains only 2-link tetrahedrons.*

Proof. Consider an embedding of K_7^4 with 9 or 11 (3–3) links and suppose it contains a 4-linked tetrahedron. The triangle, T, in this 4-link tetrahedron should have two internal edges and the face of the triangle should be punctured by two other internal edges. Therefore, the degree set for this triangle must be [3–5] and the internal puncturing edges are each incident to a degree 5 vertex. Suppose we remove one of these degree five vertices incident to a puncturing edge. From the table in Fig. 9, we have a Type 5 triangulated pentagon.

From Fig. 10, we know the two degree 5 vertices are non-adjacent within the pentagon and neither of the degree 4 vertices is adjacent to the degree 3 vertex. Under these conditions, a straightforward placement argument gives two possible vertex degree arrangements depicted in Fig. 26, denoted Cases 1 and 2.

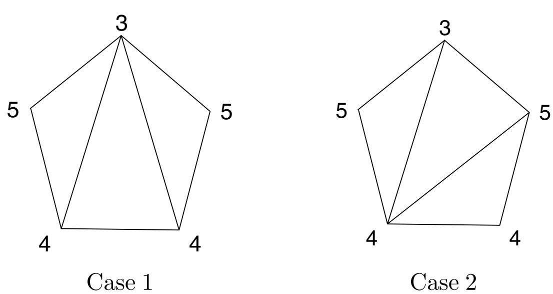


Fig. 26. The two possible triangulated pentagons in K_7^4.

With the triangulation in Case 1, K_6^2 is formed resulting in 3 (3–3) links. Similarly, for the other degree five vertex 3 (3–3) links are formed. The removal of the degree 3 results in 3 (3–3) links and the removal of the degree 4 vertices each result in 1 (3–3) link. Considering the removal of the original degree 5 vertex in the linking triangle, we now have at least 13 (3–3) links, a contradiction.

The triangulation in Case 2 is also not possible. Recall, for a 4-link tetrahedron to occur, there must be two internal edges each incident to a degree 4 and degree 5 vertex that puncture the triangle T, which itself is comprised of a degree 3, degree 4, and degree 5 vertex. The arrangement in Case 2 will result in T being punctured by at most one 4, 5 internal edge. □

Given Propositions 4.3, 4.6 and 4.7, we see that an embedding of K_7^4 with 9 or 11 (3–3) links will have 18 or 22 (3–4) links respectively.

Proposition 4.8. *An embedding of K_7^4 with* 13 *or* 15 (3–3) *links will have at most one* 4*-link tetrahedron.*

Proof. In order to have a 4-link tetrahedron, it is necessary to have a triangle disjoint from the tetrahedron with two internal edges that is punctured by two disjoint internal edges of the tetrahedron. These can occur in certain embeddings of K_7^4. Now we show they can only occur at most once.

Suppose an embedding of K_7^4 has one 4-link tetrahedron with punctured triangle labeled $(3, 4_a, 5_a)$ (i.e. it has a 3, 4, and 5 degree vertex). As this triangle is part of a 4-link tetrahedron, it is punctured by the other degree 4, 5 internal edges, say edge 4_b5_b and 4_c5_c. Now consider one of the other triangles with two internal edges, say $(3, 4_b, 5_b)$. In order for $(3, 4_b, 5_b)$ to belong to a 4-link tetrahedron, it must be punctured by edge 4_a5_a and 4_c5_c. Notice that the planes determined by the triangles (3, 4a, 5a) and (3, 4b, 5b) intersect in a line, say $\mathcal{L}$. Observe that $\mathcal{L}$ must contain the degree 3 vertex and it must intersect the line segments (4b, 5b) and (4a, 5a). There is a positive distance from the point where $\mathcal{L}$ and 4_b5_b intersect and the point where $\mathcal{L}$ and 4_a5_a intersect. Thus, the edge 4_a5_a cannot puncture triangle $(3, 4_b, 5_b)$. Hence, $(3, 4_b, 5_b)$ cannot be contained in a 4-link tetrahedron. □

We see from Propositions 4.3, 4.6–4.8 that the number of (3–4) links in K_7^4 is as stated in Theorem 1.1. Next, we consider the (3–4) links of K_7^5.

Proposition 4.9. *An embedding of K_7^5 has at most one* 4*-link tetrahedron. Moreover, an embedding of K_7^5 with* 13 *or* 15 (3–3) *links may or may not contain a* 4*-link tetrahedron, but an embedding with* 17 (3–3) *links always contains a* 4*-link tetrahedron.*

Proof. Recall, a 4-link tetrahedron requires one triangle with two internal edges and the face of the triangle must be punctured by two non-adjacent internal edges of the tetrahedron. In K_7^5, there are only five triangles with two internal edges and they are formed by edges incident to the 5 degree 4 vertices. Notice that up to symmetry, there are only two ways to arrange these edges [9, Theorem 3.1]: the image depicted in Fig. 24 and its mirror image. Now we consider the two non-adjacent internal edges of the tetrahedron that puncture the internal triangle(s). By Fig. 24, we see that there is only one of the internal triangles punctured by an internal edge formed by the two degree 4 vertices (in our example, triangle $v_1v_3v_4$ is punctured by edge v_2v_5). The only other internal edge that can puncture the triangle is the edge incident to the two degree 5 vertices, denote it e. Therefore, a 4-link tetrahedron occurs in K_7^5 if e punctures this internal triangle, otherwise a 2-link tetrahedron occurs. □

Next, we summarize the number of (3–4) links that can occur in K_7^5. Considering Propositions 4.4 and 4.9 and Fig. 24, if the internal edge adjoining the two degree 5 vertices, e, punctures the triangle in

(1) region (i), 17 (3–3) links gives 31 (3–4) links,
(2) region (ii), 15 (3–3) links gives 27 (3–4) links,
(3) region (iii), 13 (3–3) links gives 23 (3–4) links.

However, if e does not puncture the internal triangle, we have a 2-link tetrahedron instead. So by Proposition 4.3, if e punctures

(1) region (ii), 15 (3–3) links gives 30 (3–4) links,
(2) region (iii), 13 (3–3) links gives 26 (3–4) links.

5. Questions

There are obvious directions in which this work could continue. For this work, each embedding of K_7 formed a convex polyhedron with seven vertices. What about an embedding of K_7 which forms a convex polyhedron with 4 vertices? That is, four of the vertices form the hull of the polyhedron and the other three vertices are internal. It seems reasonable that such embeddings are isomorphic to one of the five cases with seven external vertices, but this is not obvious.

Question 5.1. Given a straight-edge embedding G of $K_n, n \geq 7$, with $k \geq 4$ external vertices and $m = n - k$ internal vertices, is G always isomorphic to an embedding of K_n with n external vertices?

Another direction of study is to consider $K_n, n \geq 7$. While K_6 has only 10 disjoint triangle pairs to consider, K_7 has 70, and K_8 has 280. Moreover, with K_6 there were only (3–3) links. K_7 introduced (3–4) links and for K_8, one would have to consider (3–3), (3–4), (3–5), and (4–4) links. Whereas there were only 5 distinct convex polyhedral embeddings of K_7, it is well known there are 14 for K_8 (see, for example, [12]).

Question 5.2. Given a straightedge embedding of K_n, how many (k, m) links does it contain, where $3 \leq k \leq n-3$ and $3 \leq m \leq n-k$?

Clearly this is an ambitious question. Possibly a more attainable question is the following.

Question 5.3. Given a straight-edge embedding of K_n, what is an upper or lower bound for the number of (k, m) links it contains, $3 \leq k \leq n-3$ and $3 \leq m \leq n-k$?

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