Floating Point

CS-281: Introduction to Computer Systems

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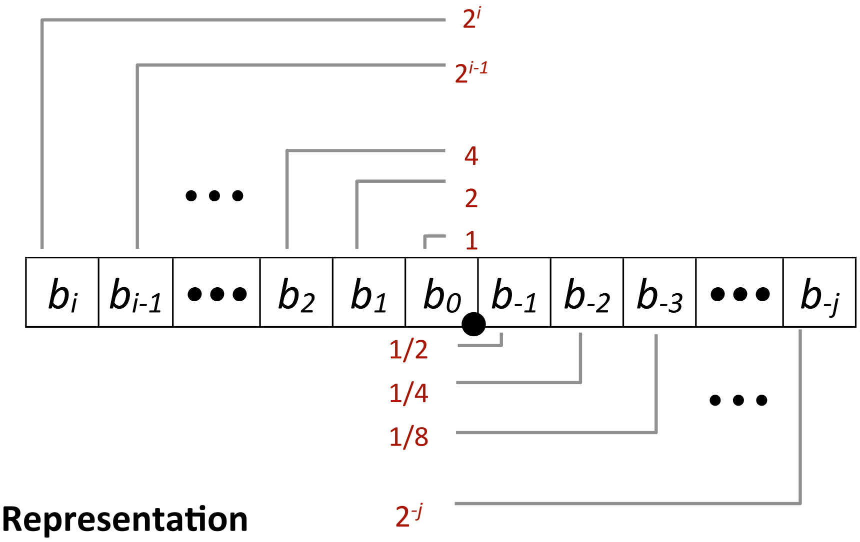
Fractional binary numbers

- **What is 1011.101_2?**

Fractional Binary Numbers



- **Representation**
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \times 2^k$

Fractional Binary Numbers: Examples

Value	Representation
5 3/4	101.11_2
2 7/8	10.111_2
63/64	0.111111_2

- **Observations**
- Divide by 2 by shifting right
- Multiply by 2 by shifting left
- Numbers of form $0.111111\ldots_2$ are just below 1.0
  - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
  - Use notation $1.0 - \varepsilon$

Representable Numbers

- **Limitation**
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

- **Value** **Representation**
  - 1/3 `0.0101010101[01]…`$_2$
  - 1/5 `0.001100110011[0011]…`$_2$
  - 1/10 `0.0001100110011[0011]…`$_2$

IEEE Floating Point

- **IEEE Standard 754**
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- **Driven by numerical concerns**
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- **Numerical Form:**

$$(-1)^s \, M \; 2^E$$

  - **Sign bit *s*** determines whether number is negative or positive
  - **Significand *M*** normally a fractional value in range [1.0,2.0).
  - **Exponent *E*** weights value by power of two

- **Encoding**
  - MSB `s` is sign bit ***s***
  - `exp` field encodes ***E*** (but is not equal to E)
  - `frac` field encodes ***M*** (but is not equal to M)

s	exp	frac

Precisions

- **Single precision: 32 bits**

s	exp	frac
1	8-bits	23-bits

- **Double precision: 64 bits**

s	exp	frac
1	11-bits	52-bits

- **Extended precision: 80 bits (Intel only)**

s	exp	frac
1	15-bits	63 or 64-bits

Normalized Values

- **Condition: exp ≠ 000…0 and exp ≠ 111…1**

- **Exponent coded as *biased* value: *E* = *Exp* – *Bias***
  - *Exp*: unsigned value `exp`
  - *Bias* = 2^{k-1} - 1, where *k* is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- **Significand coded with implied leading 1: *M* = $1.xxx…x_2$**
  - `xxx…x`: bits of `frac`
  - Minimum when `000…0` (*M* = 1.0)
  - Maximum when `111…1` (*M* = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

- Value: `Float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- Significand

  M = `1.1101101101101`$_2$

  frac= `11011011011010000000000`$_2$

- Exponent

  E = 13

  Bias = 127

  Exp = 140 = `10001100`$_2$

- Result:

s	exp	frac
0	10001100	11011011011010000000000

Denormalized Values

- **Condition:** exp = 000…0

- **Exponent value: *E* = –*Bias* + 1 (instead of *E* = 0 – *Bias*)**
- **Significand coded with implied leading 0: *M* = 0.xxx…x$_2$**
  - **xxx…x**: bits of **frac**
- **Cases**
  - **exp** = 000…0, **frac** = 000…0
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - **exp** = 000…0, **frac** ≠ 000…0
    - Numbers very close to 0.0
    - Lose precision as get smaller
    - Equispaced

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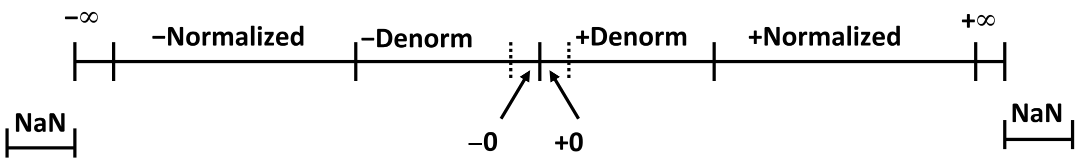
Special Values

- **Condition: exp =** 111…1

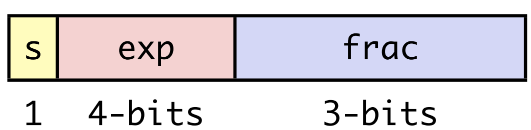
- **Case: exp =** 111…1, **frac =** 000…0
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = −1.0/−0.0 = +∞, 1.0/−0.0 = −∞

- **Case: exp =** 111…1, **frac ≠** 000…0
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ − ∞, ∞ × 0

Visualization: Floating Point Encodings



Tiny Floating Point Example



- **8-bit Floating Point Representation**
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the **`frac`**

- **Same general form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

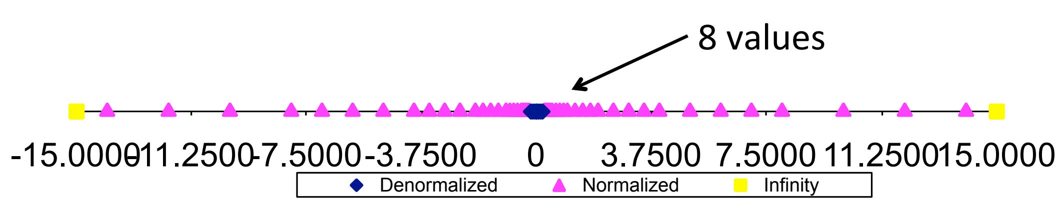
	s	exp	frac	E	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 23-1-1 = 3

s	exp	frac
1	3-bits	2-bits

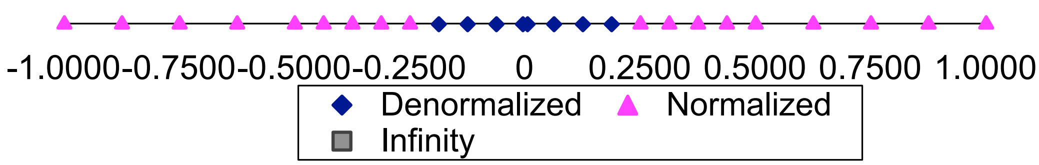
- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

- **6-bit IEEE-like format**
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3

s	exp	frac
1	3-bits	2-bits



Interesting Numbers

`{single,double}`

Description	*exp*	*frac*	*Numeric Value*
Zero	**00…00**	**00…00**	**0.0**
Smallest Pos. Denorm.	**00…00**	**00…01**	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
Single ≈ 1.4×10^{-45}			
Double ≈ 4.9×10^{-324}			
Largest Denormalized	**00…00**	**11…11**	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
Single ≈ 1.18×10^{-38}			
Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	**00…01**	**00…00**	$1.0 \times 2^{-\{126,1022\}}$
Just larger than largest denormalized			
One	**01…11**	**00…00**	**1.0**
Largest Normalized	**11…10**	**11…11**	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
Single ≈ 3.4×10^{38}			
Double ≈ 1.8×10^{308}			

Special Properties of Encoding

- **FP Zero Same as Integer Zero**
  - All bits = 0
- **Can (Almost) Use Unsigned Integer Comparison**
  - Must first compare sign bits
  - Must consider −0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

Floating Point Operations: Basic Idea

- **x $+_f$ y = Round(x + y)**
- **x $\times_f$ y = Round(x × y)**
- **Basic idea**
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- **Rounding Modes (illustrate with $ rounding)**

	$1.40	**$1.60**	**$1.50**	**$2.50**	**–$1.50**
Towards zero	$1	$1	$1	$2	–$1
Round down (−∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

- **What are the advantages of the modes?**

Closer Look at Round-To-Even

- **Default Rounding Mode**
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated
- **Applying to Other Decimal Places / Bit Positions**
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

- **Binary Fractional Numbers**
  - "Even" when least significant bit is `0`
  - "Half way" when bits to right of rounding position = $100\ldots_2$

- **Examples**
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

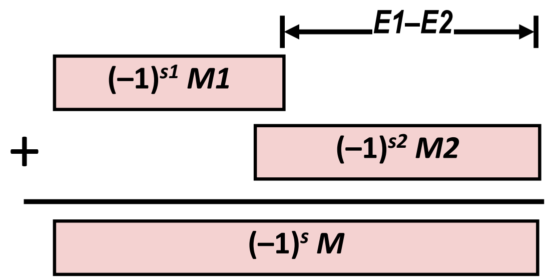
- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ \mathrm{x}\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign *s*: *s1* ^ *s2*
  - Significand *M*: *M1* x *M2*
  - Exponent *E*: *E1* + *E2*

- **Fixing**
  - If *M* ≥ 2, shift *M* right, increment *E*
  - If *E* out of range, overflow
  - Round *M* to fit `frac` precision

- **Implementation**
  - Biggest chore is multiplying significands

Floating Point Addition

- $(-1)^{s1}\ M1\ \ 2^{E1}\ \ +\ \ (-1)^{s2}\ M2\ \ 2^{E2}$
  - Assume *E1* > *E2*



- **Exact Result:** $(-1)^{s}\ M\ \ 2^{E}$
  - Sign *s*, significand *M*:
    - Result of signed align & add
  - Exponent *E*: *E1*

- **Fixing**
  - If *M* ≥ 2, shift *M* right, increment *E*
  - if *M* < 1, shift *M* left *k* positions, decrement *E* by *k*
  - Overflow if *E* out of range
  - Round *M* to fit `frac` precision

Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- **Floating point in C**
- Summary

Floating Point in C

- **C Guarantees Two Levels**
  - **`float`** single precision
  - **`double`** double precision
- **Conversions/Casting**
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`/`float` → `int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int` → `double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int` → `float`**
    - Will round according to rounding mode

Floating Point Puzzles

- **For each of the following C expressions, either:**
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither **`d`** nor **`f`** is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0` ⇒ `((d*2) < 0.0)`
- `d > f` ⇒ `-f > -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

Floating Point

- Background: Fractional binary numbers
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Summary

- **IEEE Floating Point has clear mathematical properties**
- **Represents numbers of form M x 2^E**
- **One can reason about operations independent of implementation**
  - As if computed with perfect precision and then rounded
- **Not the same as real arithmetic**
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

Creating Floating Point Number

s	exp	frac
1	4-bits	3-bits

- **Steps**
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding

- **Case Study**
  - Convert 8-bit unsigned numbers to tiny floating point format

Example Numbers

128	10000000
15	00001101
33	00010001
35	00010011
138	10001010
63	00111111

Normalize

s	exp	frac
1	4-bits	3-bits

- **Requirement**
  - Set binary point so that numbers of form 1.xxxxx
  - Adjust all to have leading one
    - Decrement exponent as shift left

Value	*Binary*	*Fraction*	*Exponent*
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	4
19	00010011	1.0011000	4
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- **Round up conditions**
  - Round = 1, Sticky = 1 → > 0.5
  - Guard = 1, Round = 1, Sticky = 0 → Round to even

Value	*Fraction*	*GRS*	*Incr?*	*Rounded*
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

- **Issue**
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	*Rounded*	*Exp*	*Adjusted*	*Result*
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64