Reflections on Krantz's "How to Teach Mathematics":
A Different View

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1 Introduction

When Steve Krantz asked me to write, for the appendix of this book, an essay expressing my views of teaching, I thought it would be useful to do this in relation to the book itself, rather than just give my views in isolation. So my first step was to figure out what the book was all about. I began, naturally enough, by reading it. And I discussed a lot of things with Steve. And I thought about it. Here is what I think the book is.

Krantz has produced a serious reflection on a traditional approach to teaching and learning. He acknowledges the existence of views other than his, but although the book goes far beyond a mere reference to these alternatives, it is considerably less than a balanced presentation of a variety of different approaches to collegiate mathematics education. Even with these appendix essays, the book retains its strong reflection of the very personal views of the author, based mainly on his many years of experience in the classroom together with some, not overly extensive, awareness of past, present, and future investigations into the teaching and learning enterprise.

We get two things from what Krantz has written and both are welcome. First, the text, together with the appendices (some of which I have seen as of this writing) contribute to moving our considerations of teaching and learning collegiate mathematics from an argument in which we are all fighting to defend our positions and attack those with whom we disagree, to a set of reasoned discussions in which we clarify points of agreement and disagreement, provide justifications for both, and ultimately search together for the syntheses that alone can make significant improvement in the breadth and depth of what mathematics our students learn.

Second, we learn from this book that although there really are identifiable categories of viewpoints such as traditionalists and reformers, it is not the case that a member of a category differs in all of her or his ideas from any member of a different category, nor do all members of a category agree with each other on every issue. What this realization helps us with is the need to focus on ideas and not on those who hold them. When we do this, we find many points of unexpected agreements and disagreements varying with the issue and the individual. I find that the most important precondition for me, in the words of Secretary Riley, to stop fighting and get on with the job of improving education, is to understand that Andrews, Askey, Hughes-Hallett, Krantz, Uhl and everyone else might, on any given issue, agree or disagree with me.

So, in the spirit of this latter point and in hopes of contributing to the former, I will try in this essay to reflect on a number of issues that arose for me in reading Krantz's book. My goal is, in each case, to explain and perhaps justify my position, to express my agreement or clarify and possibly justify my disagreement. The topics I will consider are: beliefs and theories about the nature of learning; teaching methodologies; the use of technology in mathematics education; applications and pedagogy; and some specific pedagogical issues.

2 Beliefs and theories about the nature of learning

In this section I would like to talk about traditional vs. reform approaches to mathematics education, the particular approach called constructivism and its relation to discovery learning, the relation between knowing how to calculate and understanding concepts, the intrinsic difficulty of undergraduate mathematics topics, and how we can try to decide what are the effects of a particular pedagogical method.

2.1 You @!∗#?! ∗∗ traditionalist!

Krantz believes that mathematical knowledge is something, in some sense or other, very definite and concrete, that exists in the minds of some people (usually called mathematicians) and that it is possible to take this entity and transfer it to the minds of other people (usually called students) using media like speaking, or writing, or showing pictures. I believe, on the other hand, that mathematical knowledge is much more elusive, being a characteristic of the behavior and thought, external and internal, active and reflective, of *all* people, in varying degrees of quality and sophistication, and that it is built by individuals, acting in social contexts as they try to make sense of certain situations in which they find themselves.

Something like Krantz's beliefs have been held by a large number of people for a very long time. My position is relatively new and although it has a name that is becoming, unfortunately, a buzzword (constructivism — there I said it, I even carry a card!), it is not a position really held by very many people. It is in this sense that I say Krantz is a traditionalist and I am a reformist. It only means that he is trying to hold on to what is best about a going concern, and I am trying to break some shells and make a new omelet.

Now you might think it the epitome of arrogance for *me* to say what *Krantz* thinks. Indeed, doing this contradicts one of my most deeply held beliefs (e.g., that one person can never really know what is in the mind of another person). But I only put it that way to catch your attention. What I really mean is that what I just wrote is one way that one might interpret many statements, such as the following, by Krantz in this book (the comments in square brackets are mine.)

> "...you [the teacher] must be conscious...of what are the key ideas you are trying to communicate to the students..."
>
> "If you [several bad things some teachers do sometimes] then you will not successfully convey the information."

"...the teacher who [several good things some teachers do sometimes] delivers a good class."

"Remember that you are delivering a product."

If we were to schematize the situation suggested by such comments, it is that of an active teacher trying to do something to passive students.

2.2 I admit it, I'm a constructivist!

I am convinced that learning does not happen in the way suggested by Krantz's comments. I believe that a person learns by actively trying to do something, or make sense of something and must, almost consciously, make fundamental changes in the make-up of that vague entity called her or his mind. It is making mental changes that I call constructing and it is my belief that the teacher can only act indirectly to get the students to construct these changes. That makes me a constructivist in my approach to mathematics education.

So what does a constructivist do by way of research in teaching and learning together with curriculum development? There are many people engaged in this enterprise and they have many different answers to these two questions. One pair of answers that a number of people have adopted in their work is this. Research means theoretical and empirical studies to understand the nature of the specific mental constructions that a person might make in order to understand a particular mathematical concept. These studies should point to pedagogical strategies that might help students make the mental constructions that are proposed as a result of this research. Then, of course, curriculum development involves the design and implementation of courses in which this pedagogy takes place. The instructional treatments should focus on getting students to make the specific constructions and it should also, guided by what we can learn from research, make use of innovative pedagogical approaches such as cooperative learning and writing computer programs as a way of making mental constructions.

Consider, for example, mathematical induction. In some research that was done over a decade ago, an important difference between students who were successful and those who were not with making proofs by mathematical induction appeared to emerge. It seemed that the successful students were constructing a mental process — or function — that took a positive integer and returned a value of true or false depending on some operation. For instance, if the problem was to show that for n sufficiently large, a casino with only \$3 and \$5 chips could accommodate any number n of dollars, then successful students seemed to be thinking of a function that acted on a positive integer n and returned the truth value of the proposition: there exists positive integers k, j such that $n = 3k + 5j$.

For many students this process conception represented serious progress from being restricted to thinking about a function in terms of plugging numbers into algebraic expression, which we would call an action. It was even harder for students to be conscious of an operation *on a function* which converted such a boolean valued function P of the positive integers into another such, say Q, in which $Q(n)$ is the truth value of the implication $P(n) \implies P(n+1)$. This requires thinking of a function as an object to which higher level actions can be performed.

A somewhat more detailed and extensive version of these comments came out of a research study that combined theoretical analysis with empirical results to obtain what was called a "genetic decomposition" or "cognitive model" of the concept of mathematical induction. The next stage of the study was to develop and implement an instructional treatment that helps students make such constructions. For example, to construct the above function P, we had them write and use the following computer program.

```
P := func(n);
        return exists k,j in [1..n] | n = 3*k + 5*j;
     end;
```

For converting the implication we first had them build a "machine" for doing this in general:

```
convert := func(F);
                return func(n);
                        return F(n) impl F(n+1);
                     end;
           end;
```

And then they could just do

```
Q := convert(P);
```

This code is written in the computer language **ISETL**, which is what we call a *mathematical programming language*. It's syntax is very similar to standard mathematical notation and it can express almost every finite mathematical construct (such as quantifications, functions as sets of ordered pairs, sets and subsets defined by conditions, finite sequences, etc.) in the language of mathematics. The idea is that by making mathematical constructions on the computer the student tends to make corresponding constructions in her or his mind. See [5] for a discussion of **ISETL** and its use in mathematics education.

The study found that using such an idea — together with some other pedagogical strategies, students developed what appeared to a surprisingly deep understanding of making proofs by mathematical induction — and they could make such proofs, even when the problems were somewhat different from what they had been practicing on!

You can see the details of this study in [3, 4]

This kind of approach has been applied to a fairly large number of concepts in undergraduate mathematics and courses (including textbooks, lesson plans, sample exams, etc.) have been developed in discrete mathematics, precalculus, calculus and abstract algebra. It is a general research program that I, together with several others, are trying to implement. An overall description of the research we are doing can be found in [2] and you can find links to specific published research reports and textbooks at the following WEB addresses:

http://rumec.cs.gsu.edu/

http://www.cs.gsu.edu/ matjbkx/edd/mypapers.html

Finally, since I am talking about constructivism, I think I need to say something about discovery learning. There seems to be some confusion here about what is being advocated. Krantz says that the "reform school of thought" favors discovery and that one of the "hallmarks of the new methodology" is that students should discover mathematical facts for themselves."

Not exactly. One does not have to be an educational traditionalist to realize that what took hundreds, if not thousands of years for mathematicians to discover is not likely to be figured out by very many students — even if they do have lot's of giant shoulders to stand on, powerful computers to work for them, and intelligent, sensitive teachers (all up to speed on existing research) to guide them. It is almost insulting to hear so many people suggest that reformers are so misguided as to not realize this. If we set before our students mathematical task to perform *before* they have learned the mathematics then they are not likely to invent a proof of the fundamental theorem of calculus.

So why do we do it? Why do we try to get students to work on mathematical problem situations that use mathematical concepts and methods they have not yet learned. It is not that, given the appropriate situations, they *are likely* to realize on their own that there is some kind of inverse relationship between velocity and area, between derivatives and limits of Riemann sums; or that they *can* (and I can tell you that many do) invent the chain rule after reflecting on some at first mysterious examples. All of these things happen in many reform courses and they happen with relatively small loss of "coverage". But they are not the most important reason for giving students an opportunity to discover various things in mathematics.

The real reason is the word I just used — opportunity — and it is an idea that goes back to Piaget. If you give your students the *opportunity* to discover some bit of mathematics, then whether they succeed or not, it seems to be a psychological fact that much of the attention and mental energy they put into figuring something out is directly transferred to understanding the explanation when it is given by a member of the student's group, or another group, or the teacher. And since you are not planning on waiting until everyone (anyone?) has made the discovery, you can cut it off and move on whenever you decide, thereby controlling the amount of time spent on *trying* to discover something.

2.3 A mathematician? You must be able to add up really long lists of numbers.

We all know what happens at cocktail parties when we own up to being mathematicians. People think that all we do is arithmetic operations with numbers — lot's of numbers and big ones, too. Traditionalists accuse reformers of ignoring calculations and reformers confuse traditionalists with mindless guests at a party. Both are wrong.

On one level, I am somewhat bemused that there is an argument here. How can anyone in the mathematics enterprise, from high school teacher, to industrial user, to researcher, even conceive of mathematics without lot's of heavy computations? On the other hand, how can anyone who actually understands mathematics think that calculation is the whole story. To be honest, I don't think I know any traditionalists or reformers who hold to either of these views. I think we all believe that understanding mathematics means that you are able to perform calculations and that you have some understanding of what those calculations are about.

You don't have to be a whiz at calculations, but you do have to be able to do them, however slowly and however many times you have to repeat them to avoid errors. Technological tools can and should be used — but as aids, not to replace the ability of the person who uses them. To be specific, you have to *be able* to take the derivative of any composition of elementary functions, even though you may often choose to use a computer algebra system to do the job, or to check your work.

As to understanding, you have to have the proverbial "feel" for calculations so as to have a rough idea of how they are going to come out without necessarily doing them, you have to be conscious of patterns, you have to know what the calculations are useful for and how they relate to other situations, mathematical and otherwise, and you have to know *why* particular calculations are made in particular ways.

The fact is that both calculations and conceptual understanding are essential parts of mathematics and they are completely dependent on each other. Differentiation is an incredibly useful tool for solving a multitude of problems from the everyday world. It is also an operator on certain function algebras that is linear but not exactly multiplicative — and it transforms composition to multiplication. I think it is important to understand these abstract properties of differentiation, but I can't imagine a very deep version of such understanding that is not based on a thorough knowledge of the rules for calculating derivatives.

Surely everyone agrees with what I have just written. So why is there a controversy? The controversy comes because there is a very large number of students who have difficulty with either calculations or conceptual understanding or both. Our differences here are about how to deal with this. I think it was a wonderful idea that maybe, if you could use a computer to do the calculations, the student would be freed to spend more time and energy understanding the concepts. Either that would be enough or it might turn out that going back later, the student could better learn how to calculate. So maybe we should stop insisting the students learn the calculations, let them use the technology and we can get on with the concepts

Unfortunately, it seems fairly clear to me that this does not work. Krantz is right when he points out that "...there is no evidence...that a person who is unable to use the quadratic formula" by hand but can do so with a machine will "...be able instead to analyze conceptual problems". My experience suggests that calculating derivatives is hard for many students but understanding differentiation as an operator on functions is much harder. There is simply no way around the pedagogical problem that we must find way to get our students to be better at both calculations and conceptual understanding.

I think, incidentally, there are ways to use technology to help do this and there are several references given by Krantz (for example, the paper by Heid and my own, joint, work in calculus with several others.)

An important point for teachers who plan to think about their students' work with calculations and conceptual understanding is my observation that the strongest resistance to de-emphasizing calculations can come from the students who (with considerable justification) feel extremely insecure when asked to reflect on calculations they can do by hand only haltingly. In the context of a theory I work with, there has to be a strong base ability to perform actions (calculations) before an individual is ready to move on to thinking about processes (reflection on the actions).

2.4 Yes, Barbie, math is tough!

I think Krantz is wrong when he says: "There is no topic in the [calculus] course that is intrinsically difficult. We merely need to train our students to do it." Aside from the fact that his second sentence here is more of a focus on actions as opposed to reflection than I think wise, I am not sure what could possibly be meant by the term "intrinsically difficult". Given any topic (in calculus or any mathematics subject), any students, and any point in their development, there will be some for whom the topic is hard and some for whom it is easy. Both my experience and my research suggests very strongly that the idea of a function as an object, so that the derivative of a function is a function, or the solution to a problem can be a function, is hard. It was hard for entire societies and required a long historical development. I think that almost everybody has to make some serious mental changes to move from thinking of a function as something that does something (process) to thinking of it as something to which something is done (action on an object). Whatever the phrase means, I think that developing the ability to understand a function as an object is "intrinsically difficult".

The point is important because we really have to decide, regarding such examples, whether we can get by with pedagogy focused on just training our students to do it, or if we have to develop substantial pedagogical strategies to help our students overcome a major obstacle.

2.5 So who's right?

I mean who is right in all controversies about how people learn and what teachers can do to help? I think Krantz's answer to this question expresses the situation exquisitely in four words: "Frankly, we don't know".

But we had better find out! I think that research is one way to do so, but we cannot expect the kind of research that makes a small number of studies (or even a single one) and allows us to say that this or that pedagogical approach is best — or even better. Our domain is much too difficult, compared for example, with medical research and even there we have extensive studies over long periods of time that tell us things like thalidomide is relatively harmless and watching atomic explosions can be done with impunity.

The kind of research we need is basic, long term and must relate to theoretical analyses in addition to the gathering of data. Research can begin to tell us a few things, such as that it helps students understand and succeed with proofs by induction if they are able to think in terms of functions whose domains are the set of positive integers and whose ranges are the set of two boolean values. But we need to integrate this research gradually and intelligently with our teaching and this is one reason I hope that the fledging field of research in undergraduate mathematics education becomes accepted as one of the mathematical sciences.

But research will not be enough. We need discussions on which people, no matter how much they disagree are working together to find solutions to our educational problems and not just score points off each other. We need working together in modes such as this book and its appendices. Indeed, one of the most important contributions of Krantz's book is those four words above, together with the clear thrust of doing something to find out. Perhaps here is a place to point out, as I have done elsewhere, that Piaget (who is a great role model for me) habitually dealt with major figures who disagreed with him by inviting them for a year or so to Geneva to work together with him on a project — not to decide who is right, but to synthesize their opposing views (the book by Beth and Piaget listed in Krantz's bibliography is one product of this custom). As far as I can tell, in the field of collegiate mathematics education, the only person who has really done anything that even moves in that direction is Krantz in writing this book, in inviting these appendices, and discussing with many people what are their objections to what he is writing.

3 Teaching methodologies

Throw out the old and bring in the new! Well, perhaps, but let's be careful. I would like to talk about why I don't think the lecture method is a very good idea. I would also like to describe some possible alternatives. This brings up the question of how to decide what kind if teaching one should do? Having made that decision, it turns out that the work is not over, it has just begun. Learning how to use a particular teaching methodology is not automatic. We all went through many years (decades for some of us!) being subjected to and using what very roughly might be called a traditional pedagogy. So we will be prepared to use that if it is our choice. I hope it won't be. But that means that there is a lot of work to be done in developing the ability to use a pedagogical strategy on your students that is different from what you experienced.

3.1 Do lectures work?

Krantz says that he is not ready to give up on lectures because:

> "They have worked for thousands of years, in many different societies, and in many different contexts. And they have worked for me."

Is this really true? Forget about "working" for a moment, and let me ask if lecturing as we know it has been happening for "thousands of years". Did Socrates or Plato lecture? Is this what the mathematical monks of the Middle Ages were doing? What was Fermat's classroom presence like? Was Newton effective in his presentations to students? How large a class size did Galois have to deal with? Was lecturing the way European students of the 18th and 19th learned mathematics? Or maybe Krantz is referring to thousands of years of lectures on mathematics in China?

I am not an historian and I don't know. From what little I have heard, however, I think that the practice of lecturing as the main teaching methodology for university mathematics is much more recent. I suspect that instead of "thousands of years", we may be talking about hundreds of years, or perhaps even decades. Whatever is the extent to which lectures have worked, and whatever is meant here by "work", I am not so sure there is an extensive history supporting this particular strategy.

What cannot be doubted, of course, is Krantz's claim that they have worked for him. Well, something worked for Krantz and the multitude of other successful research mathematicians we have produced, let us say in this century. It is possible that for these particular individuals, just about anything would have worked (even cooperative learning!) Even they did learn mathematics by attending lectures, this collection of individuals is hardly typical and certainly not very representative of the student body we are working with in our society's great experiment with mass education at the post-secondary level.

What does seem clear now and for this student body is that lecturing is not working. This is attested to by reports of minuscule attendance at the lectures, poor performance in tests based on those lectures, dissatisfaction on the part of teachers of these students in subsequent courses (in mathematics and courses for which mathematics is a prerequisite) and what seems to be a decline in the rate of successful completion of mathematics courses.

If we can all agree that this problem exists, there are certainly very different views about its causes, and what to do about it. Krantz takes the position that lecturing would be sufficiently effective if we were better at it. I differ with that on purely personal grounds. I think I am an excellent lecturer — but I don't think my students got nearly as much out of my lecture courses as they do today in my courses which use other pedagogies.

Others argue that it is the students' fault. The background they obtained in high school (where they were taught by teachers that are largely products of lecture-pedagogy) is too weak, or their attitudes are all wrong, or the conditions under which we teach are inadequate. Be all that as it may, if we have a huge enterprise (collegiate mathematics education) which is not working, then it is pretty unlikely that all of the fault lies in places other than the methods teachers use in that enterprise. I think that as mathematicians we are responsible for working to make changes in the overall system to help deal with our problems, but that is a long-term operation. In the meantime, we must figure out how to do the best we can with the students we have and the conditions under which we work. In my view, that means we must look for alternatives to the lecture method.

3.2 What are the alternatives to traditional pedagogical methods?

There is a multitude of new pedagogical strategies in collegiate mathematics education that many people have been developing and implementing over the last decade. They include various ways in which technology can be used, cooperative learning, replacing some lecturing with methods in which students are more active, writing, and the use of history.

It would be very nice if I could point to a place where all of these methods are described. Now is the wrong time for this. We are too much in a period of new ideas, revising first attempts, discarding some approaches and pursuing others that seem more promising. Krantz gives some information on what is available and MAA

OnLine is another source. But I am afraid that the interested faculty member must do a lot of digging in the library, read publications of the MAA and AMS, attend sectional and national meetings where some of these approaches are discussed, and generally be on the lookout for new ideas as they emerge and are reported.

There is some discussion above in my section on constructivism where I talk about some of the pedagogical approaches I have been working with. This is in the context of pedagogy that supports a constructivist view of how learning takes place.

At some point in the future, we will have to think about producing compendia of new pedagogies. Amongst other issues this will raise is the question of effectiveness. What do we have to say to the working teacher about the relative effectiveness of these new methods?

3.3 How does a teacher decide on what pedagogical approach to use?

There is not a lot that can be said about this today. As I commented earlier in agreement with Krantz, we simply do not know how to decide with any great degree of certainty on the effectiveness of a particular teaching method. It is very much like parenting. There are lot's of views and many things to read, but in the end, each teacher, like each parent, must decide as best he or she can what pedagogical approaches to adopt.

I must, however, insert a word of caution here. Some people might interpret the previous paragraph as stating that, for a given teacher, whatever "works" for her or him is the method that should be adopted. The danger here is in restricting the concern to the teacher. As Krantz puts it, "...we should each choose those methods that work for us and for our students."

This is not so easy as it sounds. Aside from the difficulty in deciding what works for our students, we must acknowledge that sometimes, what works best for the teacher, may not work very well for the students. To put it extremely, the teacher who faces the blackboard speaks in a monotone, writes out the text on the board, works a few illustrative exercises, and assigns homework, may be using a methodology that "works best" for her or him (in the sense, for example, of minimizing the distractions from other issues in the teacher's life). But I hope we can all agree that this is not likely to be an approach that works very well, much less best for the students involved. At the very least, it is not at all clear that a teacher alone can always determine how effective is her or his approach to teaching.

3.4 What about implementing a new pedagogy?

Here, I agree with Krantz that, as hard as it is to decide to implement a new pedagogical approach, this is only the beginning. As Finkel and Monk [7] point out, it is very difficult for a mathematician, with no background in educational methodology, only the experiences he or she had as a student, and in some cases, long years of practice with a method he or she has decided to replace, to actually change the way he or she teaches.

First, one has to make the decision. Then you need to learn about the method you have chosen to implement — how it works, what you can use from previous methods, what changes you need to make. For most people, this is still not enough. You need some kind of mentored experience in actually using the new methodology.

There are some indications that mathematics departments are beginning to introduce courses in pedagogical methods for graduate students who will have a career in college teaching. There are also a number of mini courses and workshops organized by the professional societies. Krantz has mentioned MAA's Project CLUME and, in fact, MAA has a program of Professional Development. Finally, those of us working in educational reform produce a great deal of written material that can be helpful and Krantz has referred to these.

All of this is a good start, but I am afraid it is still not enough to make systemic change in teaching and learning collegiate mathematics. I hope that readers of this book will not only pursue the opportunities for professional development that do exist, but also will push for an extension of these activities to a sufficient level.

4 The use of technology in mathematics education

The use (or not) of technology is one of the most controversial issues in mathematics education today. We appear to be totally polarized — there are those for and those against. To me, this is completely ridiculous because there are many ways in which technology can be used in mathematics education and I differ with some people who advocate some of those ways at least as much as I disagree with those who are more or less against any significant use of computers.

For example, both Krantz and I deprecate what is called programmed learning, or Interactive Tutoring Systems. Krantz worries, correctly in my view, that some of these systems may not provide enough opportunity for the student to ask questions and the teacher to respond. I am also concerned about the ways in which these systems try to get students to think about concepts. I do not believe that mathematics can be reduced to a collection of goals and subgoals together with not very rich ways of connecting them.

I also have a lot of concerns about the use of today's sophisticated calculators. In earlier times, I did not hear students say that the limit of a function at a point is the value of that function at a nearby point. Is this new misconception due to certain ways in which calculators tend to be used? I have similar troubles with the graphing capabilities of hand-held calculators because I think they can focus the students' attention on the least interesting examples. But this particular issue will soon go away. Hand-held computers are rapidly approaching the functionality of desk-top computers so that soon we will have to look for something else to fight about.

So we have to talk about the ways in which technology can be used and I also want to explain the way I think is the most effective and why.

4.1 What are the different ways in which technology can be used to help students learn mathematics?

I have written elsewhere [6] about the different ways that I think technology can be effective. They are, roughly: using graphics capabilities to *show* mathematical phenomena; using the computational abilities of a computer algebra system to *do* mathematics; and using the expressibility of a programming language to *construct* mathematical entities on the computer.

4.1.1 Using technology to show mathematics

There is no question that we can produce incredibly wonderful pictures on a screen using today's technology. On the one hand, I think that makes many mathematical situations more real and accessible to students — at least as phenomena to be explained, manipulated, and perhaps understood. This can be very helpful but, in my view, entering an expression, pressing buttons and looking at a screen, or even manipulating the screen with a mouse is a little too passive in terms of the mathematics involved in producing the picture.

To take a simple example, consider entering an expression that defines a function and having the computer produce a graph. Suppose even that a table of values is produced, and that it is possible to manipulate one or more of these "representations" (expression, picture, table) and have the others change correspondingly. Even in such a sophisticated system, I do not see that the student is helped in any way to understand the connection between the various processes of plugging a number into an expression to get a result, looking down one column of a table for a number to see what is in the corresponding place on the other column, or locating a point on the horizontal axis and seeing how far up you have to move until you hit the curve. Conceptually these are all the same process and it is very important for students to understand that. I am not sure they do. Yes, students can learn from such technological systems that if you add a constant to the expression for the function the graph goes up. But what is it that helps them realize that the reason is that you are still computing the same values to place on the graph, but now every answer you get is increased by the constant, and that means higher up on the vertical axis?

My conclusion from all this is that using technology to show mathematics to students can be helpful, but it will be much more effective if used in conjunction with other activities.

4.1.2 Using a computer algebra system to do mathematics

Using the power of **Maple** or **Mathematica**, students can learn to perform highly sophisticated and powerful algorithms. The idea is that making use of mathematics in this way is going to help the students learn elementary versions of the mathematics involved. Perhaps, but I am not yet convinced that using a computer algebra system to do applications that involve Padé approximations or Tchebycheff polynomials is going to help the first year calculus student understand how to compute the McLaurin series expansion of $\sin x$ and the issues that arise when you reflect on what relationship that series has to the sin function. It is possible, but there is absolutely no evidence in favor of this approach — even less so than evidence for other claims, both traditional and reform.

Some people take the very opposite view of using a computer algebra system to reduce the time and effort spent on learning standard manipulations. Please pay very careful attention to the fact that I said "reduce" and not eliminate. My remarks earlier about the importance of calculations still stand. What I am saying is that we can use a computer algebra system to do as well or better than in traditional classes — in less time. Here there is some research here for example, the study by Kathy Heid [8].

Yet another approach is to use a computer algebra system to provide data on the basis of which students can discover mathematical relationships. For example, before talking at all about the rules for differentiating combinations of functions, I ask students to use **Maple** to calculate about a page (closely spaced, I admit) of derivatives of elementary functions using **Maple**. Then, in class discussions, I can get most students to invent the rules for derivatives of sum, difference, scalar multiple; many will invent the product rule; and a few will come up with the chain rule.

Of course all of that is preliminary to a class discussion of why these rules hold and some elementary proofs. But the computer work helps them understand these rules and also seems to get them reasonably good at doing the manipulations by hand.

So I think that, like visual effects, using computers to do mathematics is a good educational strategy. But it is not the best way to use technology.

4.1.3 Using a programming language to construct mathematical concepts

Let me begin with the biggest objection to having students write programs in order to learn mathematics. A couple of decades ago, this was a very popular idea and there was even a project (I believe its acronym was CRICISAM) to foster it. The trouble is that, in those days, programming was done in FORTRAN and there was so much extra effort in learning the programming language, dealing with bugs and other *non-mathematical* issues that any benefit that might accrue was canceled out.

I think this was a very accurate assessment. But that was then and this is now. The syntax of programming languages today can be quite simple and I hope that my illustrations in the section above on constructivism will suggest to the reader that rather sophisticated programs can be written with little syntactical difficulty. In fact, my experience with students is that although they continue to complain that the syntax gives them difficulty, I find that in just about every case the problem is either a mathematical concept that is not understood (like a function returning a function) or has to do with very inefficient work habits and methods of organizing material. My experience over the last decade or so has been that there are reasonable ways of using an appropriate computer language so that, in writing programs, syntax and system issues are minimized and the focus of the students can be almost entirely on the mathematics.

There are several reports of how we go about this and the reader can consult my WEB page at http://www.cs.gsu.edu/ẽdd/ for details. One thing I can say here is that our approach very much makes mathematics a laboratory course and, as Krantz suggests, we have the students meet, certain days of the week, in a computer lab where they work in cooperative groups on computer activities designed to foster the mental constructions we think they need to make in order to develop an understanding of the mathematics being studied. On the other days of the week, they meet in a classroom where the computer activities are discussed and work is focused on using the mental constructions they made in the computer lab to develop mathematical understandings.

4.2 Which is best?

I think that all three of the above ways of using computers are effective in various ways, but I think that overall, the third is more effective than the other two. The most effective, however, is when all three approaches are used so that the student constructs mathematical concepts on the computer and then uses these constructs, or fancy versions of them found in a computer algebra system to do mathematics and to produce visualizations on a computer screen.

My argument for the effectiveness of writing programs in learning mathematics is two-fold. First of all, it relates to the theory in that specific mental constructs seem to arise out figuring out how to perform certain computer tasks. For example, if the student understands a certain mathematical procedure as an action or externally driven activity, then asking her or him to implement the procedure as a computer program tends to lead her or him to interiorize this action to a process. Moreover, I know of no more effective way of learning to encapsulate a process to an object than implementing the process as a computer program and then writing a program that uses that process as input and/or output. Thus writing a program that accepts two functions, constructs a program implementing the composition of those two functions and returns this program and then applying this tool in various situations helps a great deal in developing an understanding of functions as objects.

Again, there are reports of our research in which these effects are described and they can be found by consulting the above WEB page.

5 Applications and pedagogy

Here we come to an issue on which I differ with my fellow reformers — and many traditionalists as well — perhaps more than on any other issue. Like Krantz, I feel that the wrong kinds of applications used in the wrong way can provide distractions to the mathematical issues on which we want the students to focus. Krantz refers to a problem about the destruction of trees in a tropical rain forest and suggests that a multitude of details about the situation tends to obscure the fact that what is needed here is to construct a function, take its derivative, set it equal to zero and solve. I think he is right and details that distract should be avoided, but I would go even further.

One argument is that is that a "real-world" context will make the problem more interesting for students and motivate them to use more energy in dealing with the situation. Unfortunately, one-person's real world is another's vague abstraction. This was brought home to me very sharply about 25 years ago when I was teaching (more or less traditionally) a unit on permutations and combinations. I asked the students how many starting line-ups could be made from a basketball squad with 12 members. I thought I was using a "real-world" example, but after class a student approached me and asked how many players there were on a starting line-up in basketball! It is true that this was at a "hockey school" and the student was a woman at a time when women were more or less excluded from basketball. Nevertheless, I realized that at least for this student, the application was not very helpful.

But even for students who do find a particular application context interesting, I question whether the resulting motivation really relates to the mathematics as opposed to the context — which can take them away from the mathematics! At the very least, I do not find any examples at all in the literature of research providing even a suggestion that using contexts that are interesting for students helps them understand the mathematics that *we* see in the context.

Given the lack of information that applications are helpful and the concern that the student will miss the mathematics in a context, or even be turned away from it, I think we should seriously reconsider our enthusiasm for the use of applications in helping students understand mathematics.

6 Some specific pedagogical issues

Let me close with some brief comments about several issues that are, in fact, more at the heart of Krantz's book than perhaps are the matters I have been discussing at (probably too much) length. In spite of his very laudatory decision to relate his book to current issues of teaching methodology, research in learning and curriculum reform, Krantz has written a book on how to teach (traditionally). As such, he provides a lot of useful suggestions for beginners, and he presents his views on just about every educational issue one can imagine. In addition, he has provided a few of us with a platform on which to state our own views on some of these topics. Who can resist such an invitation? I will try to be brief.

Beginning with the most important, let me say that I agree with Krantz completely when he says that there is nothing essentially wrong with the content of any standard lower-division math course. I have a few quibbles such as too much emphasis on the analytic as opposed to the algebraic and geometric or the importance of adding courses that make mathematics a service course not only for the physical sciences, but the social and computer sciences. But for the most part, I think we have the content about right. This is my belief as a mathematician. I have also surveyed faculty who teach courses for which mathematics is a prerequisite and they tend to confirm this view. Perhaps the most telling argument is to take a look at the reform textbooks that are emerging and notice that the content is not really very different. After all, I am told that even the Babylonians asked their students about the rate of descent of the top of a ladder leaning against a building!

I think Krantz is right that students are not really ready for formal proofs until they have completed calculus and are taking one of the transition courses that have emerged in recent years. I think that forcing formalisms on students too early can contribute to our society's turning away from studying mathematics.

In my opinion, Krantz is too even-handed about so-called objective examinations such as multiple-choice. No exam is really objective. Even a multiple choice examination makes a selection of material and what could be more subjective than making up the incorrect choices on a multiple choice exam? I think the overwhelming weight of arguments for such exams is the convenience of the person who grades the exam. A similar point can be made for timed exams. Would we really use them if it were not inconvenient to give students as much time (within reason) as they need? Do we really want to know only what they know so well that they have it on their finger tips and can produce it in high-pressure, emotionally tense situations? Don't we also want to know what they say when they have a chance to relax and reflect on a mathematical issue?

I think Krantz is very wrong when he says that "hard copy textbooks, more or less of traditional form, work. My 42 years of teaching experience tells me that traditional textbooks are essentially unread by students who use them mainly to find template solutions for problems that will be assigned for homework and given on the test. Moreover, I find that the greatest unanimity in all of education is found in the community's reaction to any attempt to vary from this norm. Such attempts are resisted, if not rejected, by students, publishers, and the overwhelming majority of faculty. This has been my experience with the textbooks I have written that are designed to support the ideas I have expressed here and I hear the same story from other writers of "different" textbooks. It seems to me that if we are to have real improvement in teaching and learning mathematics, either this situation has to change, or alternatives to textbooks must be found.

There is an orthodoxy about class size neatly expressed by Krantz who says that "We all know, deep in our guts, that small classes are a much more effective venue for learning than large classes." I sometimes wonder if what we really know deep in our guts is how to count and the salutary effect small classes will have on the job market. I am not sure we can't do really wonderful things with large classes. Personally, I feel that the most effective teaching I have ever done was with a calculus class in which there were 74 students. Not very large, but not exactly small either. My personal opinion is that using the methods referred to in this essay, I think we could learn to teach mathematics as well to classes of 200 as 20. Indeed, certain groups in France report success with new pedagogical strategies they are using in classes this large [1]. This is a largely unexamined issue. The research is, at best, mixed on the effects of class size on learning. In spite of the economics of the situation, I think we should take a good look at it and see if we can find pedagogical strategies that are both learning-effective and cost-effective.

Well, what better topic to end these ramblings with than the authority of the teacher and what is its source. Krantz suggests it comes from how you dress, from maintaining a certain distance, and from not being too chummy with the students. I think he has this completely wrong. Authority in teaching as in anything else comes from a strong, secure knowledge of and satisfaction with who you are and how you want to be, to dress, to talk and to move. This varies with people and the ones who get the most respect from their students are the ones who remain completely true to themselves, their nature and their personality. I have run many academic programs and often I am asked about a dress code. My answer always is: you dress to please yourself and anyone else you feel like pleasing.

Thank you Steve, for this opportunity to spout off. Let's get a cup of coffee.

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