The homotopy theory of ideals in stable model categories

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Joint with Donald Yau (Ohio State: Newark)
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http://personal.denison.edu/~whiteda/files/Slides/skye.pdf

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- Thanks to Mark Hovey, Bob Bruner, and Dan Isaksen.

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Definition: A Smith ideal is a monoid in $\overrightarrow{\mathsf{M}}^{\square} := (Arr(\mathsf{M}), \square)$

Note: A monoid in $\overrightarrow{\mathsf{M}}^{\otimes}$ is a monoid homomorphism in M.

Unpacking definition of Smith ideal as monoid in $\overrightarrow{\mathrm{M}}^{\square}$

A Smith ideal is a monoid R, an R-bimodule I, and a map of R-bimodules $j : I \longrightarrow R$ such that $\mu(1 \otimes j) = \mu(j \otimes 1) : I \otimes I \longrightarrow I$.

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Theorem (Hovey, 2014): The cokernel functor from $\overrightarrow{\mathrm{M}}^{\square}$ to $\overrightarrow{\mathrm{M}}^{\otimes}$ is strong symmetric monoidal ($j \mapsto (R \longrightarrow R/I)$), and right adjoint is the kernel. This forms a Quillen equivalence $\overrightarrow{\mathrm{M}}^{\square} \leftrightarrows \overrightarrow{\mathrm{M}}^{\otimes}$.

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- In $Ch(R)$, a monoid A is a DGA. A Smith ideal $j : I \longrightarrow A$ yields a homogeneous ideal of A via $\mathrm{im}(j)$.

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5. There is a $(\mathsf{C} \coprod \mathsf{C})$-colored operad O^s in M such that $\mathsf{Alg}\left(\overrightarrow{O}^{\square}; \overrightarrow{\mathsf{M}}^{\square}\right) \cong \mathsf{Alg}(O^s; \mathsf{M})$. Use to transfer model str.

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such that, for $1 \le i < j \le n$, *the following commutes*

$$\begin{array}{ccc} O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{i-1}} X_{c_i} A_{c_{i+1}} \cdots X_{c_j} \cdots A_{c_n} & \xrightarrow{(\mathrm{Id}, f_{c_j}, \mathrm{Id})} & O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{i-1}} X_{c_i} A_{c_{i+1}} \cdots A_{c_n} \\ \Big\downarrow {\scriptstyle (\mathrm{Id}, f_{c_i}, \mathrm{Id})} & & \Big\downarrow {\scriptstyle \lambda_0} \\ O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{j-1}} X_{c_j} A_{c_{j+1}} \cdots A_{c_n} & \xrightarrow{\lambda_0} & X_d \end{array}$$

What is this $\mathcal{O}^s$ with $\mathrm{Alg}(\overrightarrow{\mathcal{O}}^{\square}; \overrightarrow{\mathsf{M}}^{\square}) \cong \mathrm{Alg}(\mathcal{O}^s; \mathsf{M})$?

Given a C-colored operad $\mathcal{O}$, denote by C^0 (resp. C^1) the first (resp. second) copies of $C \coprod C$. Given $c \in C$, write $c^\epsilon \in C^\epsilon$ for the same c in each copy, for $\epsilon \in \{0, 1\}$. Define:

$$\mathcal{O}^s\binom{d^1}{c_1^{\epsilon_1}, \ldots, c_n^{\epsilon_n}} = \mathcal{O}\binom{d}{\underline{c}}$$

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An $\mathcal{O}^s$-algebra is a pair (A, X) of C-colored objects, plus structure maps making A into an $\mathcal{O}$-algebra, X into an A-bimodule, and $f : X \longrightarrow A$ into an A-bimodule map. This is similar to the two-colored operad for monoid maps.

Main theorem

Theorem (W.-Yau)

If M *is nice, and cofibrant Smith O-ideals are also entrywise cofibrant in* $\overrightarrow{\mathsf{M}}^{\square}$ *then there is a Quillen equivalence*

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For $O = Com$, M needs strong commutative monoid axiom.
For general O, need $X \otimes_{\Sigma_n} (-)^{\square n}$ and $f \square_{\Sigma_n} (-) : \mathsf{M}^{\Sigma_n} \longrightarrow \mathsf{M}$ homotopically well behaved, like pres. trivial cofibrations.

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Examples: symmetric/equivariant-orthgonal/motivic spectra, $Ch(k)$, StMod$(k[G])$, enriched functors, S-modules, etc.

Comparison with ∞-operads

Theorem (W.-Yau)

If M *is cof. gen.,* $\mathsf{M}^\flat \subset \mathsf{M}$*, and* *$O$ is Σ_C-cofibrant* *(symmetric) C-colored operad. Denote by:*

- $\mathsf{Alg}(O;\mathsf{M})^c[W_O^{-1}]$*, the* ∞*-category obtained from the semi-model category* $\mathsf{Alg}(O;\mathsf{M})$.
- $\mathsf{Alg}(O;\mathsf{M}[W^{-1}])$*, the* ∞*-category obtained by first passing from* M *to the (symmetric) monoidal* ∞*-category* $\mathsf{M}[W^{-1}]$ *and then passing to O-algebras.*

Then $\mathsf{Alg}(O;\mathsf{M})^c[W_O^{-1}] \simeq \mathsf{Alg}(O;\mathsf{M}[W^{-1}])$ *as* ∞*-categories.*

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3. Every ring spectrum is weakly equivalent to a quotient of the sphere spectrum by some Smith ideal. Define a monoid homomorphism $p : R \longrightarrow S$ to be a strong quotient if $S \otimes_R QN \longrightarrow N$ is a w.e. for all fibrant N (and cof. rep. Q).

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4. Connection to Prasma's 'homotopy normal maps'?

Open Problems 2

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9. Section 6 of White-Yau lists conjectures and open problems related to Smith O-ideal theory in: positive flat model on symmetric spectra and equivariant orthogonal spectra, positive complete model structure, global equivariant, injective model structures, and S-modules.

Open problems relating K-theory and ideals

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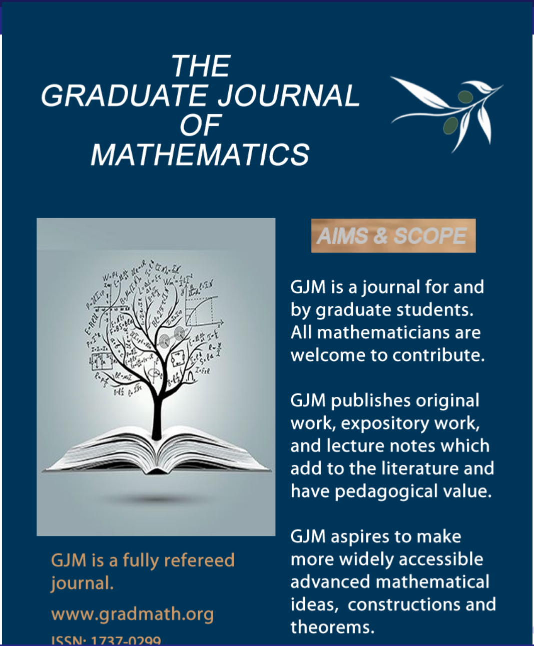
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