The homotopy theory of ideals in stable model categories

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Plan for this talk

- Brave New Algebra for stable homotopy theory.
- Explain Jeff Smith's notion of an ideal of ring spectra and connection to algebraic K-theory.
- This definition works in any nice stable monoidal model category; extends to operad-structured ideals.
- You don't need to be an expert in model categories to follow this talk. You can think of M as the category of symmetric spectra, $Ch(R)$, or R-mod (stable module category structure).
- Results also hold in ∞-category context.
- Thanks to Mark Hovey, Bob Bruner, and Dan Isaksen.

Brave New Algebra

Classical Algebra	Brave New Algebra
Ring of Integers $\mathbb{Z}$	Sphere Spectrum S
Abelian groups $=$ $\mathbb{Z}$-modules	Spectra $=$ S-modules
(Unital) Ring	Ring spectrum
Commutative ring	E_∞-ring spectrum
Both: projective/injective modules,	homological dim, semisimple, etc.
Ideal $I \subset R$ s.t. R/I is a ring	???

'Subobject' is the wrong idea. Better: an ideal is something you can quotient by $I \xrightarrow{j} R \xrightarrow{\text{coker}} R/I$. Jeff Smith (2006): an ideal is an arrow $j : I \longrightarrow R$ with extra structure.

Ideals are arrows with algebraic structure

If $(\mathrm{M}, \otimes, 1)$ is a closed symmetric monoidal category, then the arrow category $\mathrm{Arr}(\mathrm{M})$ has two monoidal structures:

1. Tensor monoidal structure: $f \otimes g : X_0 \otimes Y_0 \longrightarrow X_1 \otimes Y_1$, unit is $Id_1 : 1 \longrightarrow 1$.
2. Pushout product monoidal structure (unit $\varnothing \longrightarrow 1$):

$$(X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \xrightarrow{f \square g} X_1 \otimes Y_1$$

Definition: A Smith ideal is a monoid in $\overrightarrow{\mathrm{M}}^{\square} := (\mathrm{Arr}(\mathrm{M}), \square)$

Note: A monoid in $\overrightarrow{\mathrm{M}}^{\otimes}$ is a monoid homomorphism in M.

Unpacking definition of Smith ideal as monoid in $\overrightarrow{\mathrm{M}}^{\square}$

A Smith ideal is a monoid R, an R-bimodule I, and a map of R-bimodules $j : I \longrightarrow R$ such that $\mu(1 \otimes j) = \mu(j \otimes 1) : I \otimes I \longrightarrow I$.
Reason: $\eta : (\varnothing \longrightarrow 1) \longrightarrow j$ and unpack $\mu : j \square j \longrightarrow j$:

$$\begin{array}{ccc} (I \otimes R) \coprod_{I \otimes I} (R \otimes I) & \longrightarrow & R \otimes R \\ \downarrow & & \downarrow \\ I & \longrightarrow & R \end{array}$$

R/I is a monoid and $\mathrm{coker}(j) : R \longrightarrow R/I$ is a homomorphism.
Theorem (Hovey, 2014): The cokernel functor from $\overrightarrow{\mathrm{M}}^{\square}$ to $\overrightarrow{\mathrm{M}}^{\otimes}$ is strong symmetric monoidal ($j \mapsto (R \longrightarrow R/I)$), and right adjoint is the kernel. This forms a Quillen equivalence $\overrightarrow{\mathrm{M}}^{\square} \leftrightarrows \overrightarrow{\mathrm{M}}^{\otimes}$.

Smith ideals in algebraic contexts

- Representation theory: if $R = k[G]$ for field k and finite group G, then R-mod has stable module category structure. A monoid A is an R-algebra. A Smith ideal $j : I \longrightarrow A$ yields an ideal of A as $\mathrm{im}(j)$. Note: W.-Yau works out theory of operad-algebras in $StMod(k[G])$.
- Homological algebra: In $Ch(R)$, let $S^0(R)$ be the chain complex with R in degree zero and 0 elsewhere. A Smith ideal $j : I \longrightarrow S^0(R)$ yields an ideal of R as $\mathrm{im}(j)$.
- In $Ch(R)$, a monoid A is a DGA. A Smith ideal $j : I \longrightarrow A$ yields a homogeneous ideal of A via $\mathrm{im}(j)$.

Jeff Smith's motivation: algebraic K-theory

Suppose that R is a ring spectrum with Smith ideals I and J. Define the Smith ideal $I \wedge_R J$. Let T be the homotopy pushout:

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & T \end{array}$$

in the category of ring spectra. Smith: "there is a fiber sequence $K(R/(I \wedge_R J)) \longrightarrow K(R/I) \otimes K(R/J) \longrightarrow K(T)$ of algebraic K-theory spectra." Proven by Land-Tamme, 2023; plus $T \cong R/I \odot_R^M R/J$, the $\odot$-ring from their 2019 Annals paper, for $M = (R/I) \wedge_R (R/J)$. The E_∞-operad algebra structure matters.

Our setup (W.-Yau)

Note: monoid morphisms are algebras over a 2-colored operad. Smith ideals are too. We generalize from monoids to operad O.

1. Examples: commutative ideals, A_∞, E_∞, E_n, Lie, L_∞, etc. Now $\text{coker}(j) : R \longrightarrow R/I$ is O-alg morphism.
2. Let $L_0 \dashv Ev_0$, $L_1 \dashv Ev_1$. Given O, define $\overrightarrow{O}^{\otimes} = L_0 O$ (resp. $\overrightarrow{O}^{\square} = L_1 O$), C-colored operad in $\overrightarrow{\mathrm{M}}^{\otimes}$ (resp. $\overrightarrow{\mathrm{M}}^{\square}$).
3. A Smith O-ideal is an algebra over $\overrightarrow{O}^{\square}$; a morphism of O-algebras is an algebra over $\overrightarrow{O}^{\otimes}$.
4. coker is a Quillen equiv. $\mathrm{Alg}\left(\overrightarrow{O}^{\square}; \overrightarrow{\mathrm{M}}^{\square}\right) \leftrightarrows \mathrm{Alg}\left(\overrightarrow{O}^{\otimes}; \overrightarrow{\mathrm{M}}^{\otimes}\right)$
5. There is a $(C \coprod C)$-colored operad O^s in M such that $\mathrm{Alg}\left(\overrightarrow{O}^{\square}; \overrightarrow{\mathrm{M}}^{\square}\right) \cong \mathrm{Alg}(O^s; \mathrm{M})$. Use to transfer model str.

Unpacking Smith O-ideal $j : X \longrightarrow A$ s.t. A/X is O-algebra

Proposition (W.-Yau)

A Smith O-ideal in M *is precisely:*

- *an O-algebra* (A, λ_1), *an A-bimodule* (X, λ_0) *in* M, *and*
- *an A-bimodule map* $f : (X, \lambda_0) \longrightarrow (A, \lambda_1)$

such that, for $1 \le i < j \le n$, *the following commutes*

$$\begin{array}{ccc} O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{i-1}} X_{c_i} A_{c_{i+1}} \cdots X_{c_j} \cdots A_{c_n} & \xrightarrow{(\mathrm{Id}, f_{c_j}, \mathrm{Id})} & O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{i-1}} X_{c_i} A_{c_{i+1}} \cdots A_{c_n} \\ \downarrow{\scriptstyle (\mathrm{Id}, f_{c_i}, \mathrm{Id})} & & \downarrow{\scriptstyle \lambda_0} \\ O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{j-1}} X_{c_j} A_{c_{j+1}} \cdots A_{c_n} & \xrightarrow{\lambda_0} & X_d \end{array}$$

What is this O^s with $\mathrm{Alg}(\overrightarrow{O}^{\square}; \overrightarrow{\mathrm{M}}^{\square}) \cong \mathrm{Alg}(O^s; \mathrm{M})$?

Given a C-colored operad O, denote by C^0 (resp. C^1) the first (resp. second) copies of $C \coprod C$. Given $c \in C$, write $c^\epsilon \in C^\epsilon$ for the same c in each copy, for $\epsilon \in \{0,1\}$. Define:

$$O^s\binom{d^1}{c_1^{\epsilon_1}, \ldots, c_n^{\epsilon_n}} = O\binom{d}{\underline{c}}$$

$$O^s\binom{d^0}{c_1^{\epsilon_1}, \ldots, c_n^{\epsilon_n}} = \begin{cases} O\binom{d}{\underline{c}} & \text{if at least one } \epsilon_i = 0 \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

An O^s-algebra is a pair (A, X) of C-colored objects, plus structure maps making A into an O-algebra, X into an A-bimodule, and $f : X \longrightarrow A$ into an A-bimodule map.
This is similar to the two-colored operad for monoid maps.

Main theorem

Theorem (W.-Yau)

If M *is nice, and cofibrant Smith O-ideals are also entrywise cofibrant in* $\overrightarrow{\mathrm{M}}^{\square}$ *then there is a Quillen equivalence*

$$\{\textit{Smith O-Ideals}\} \underset{\ker}{\overset{\text{coker}}{\rightleftarrows}} \{\textit{O-Algebra Maps}\}$$

For Σ-cofibrant O, just need M stable, monoidal, cof gen.
For $O = Com$, M needs strong commutative monoid axiom.
For general O, need $X \otimes_{\Sigma_n} (-)^{\square n}$ and $f \square_{\Sigma_n} (-) : \mathrm{M}^{\Sigma_n} \longrightarrow \mathrm{M}$ homotopically well behaved, like preserving trivial cofibrations.
Examples: symmetric spectra, $Ch(k)$, $\mathrm{StMod}(k[G])$, motivic, equivariant orthogonal spectra, enriched functors, S-modules, etc.

Comparison with ∞-operads

Theorem (W.-Yau)

If M *is cof. gen.,* $\mathsf{M}^\flat \subset \mathsf{M}$*, and* O *is* Σ_C*-cofibrant (symmetric)* C*-colored operad. Denote by:*

- $\mathsf{Alg}(O;\mathsf{M})^c[W_O^{-1}]$*, the* ∞*-category obtained from the semi-model category* $\mathsf{Alg}(O;\mathsf{M})$*.*
- $\mathsf{Alg}(O;\mathsf{M}[W^{-1}])$*, the* ∞*-category obtained by first passing from* M *to the (symmetric) monoidal* ∞*-category* $\mathsf{M}[W^{-1}]$ *and then passing to* O*-algebras.*

Then $\mathsf{Alg}(O;\mathsf{M})^c[W_O^{-1}] \simeq \mathsf{Alg}(O;\mathsf{M}[W^{-1}])$ *as* ∞*-categories.*

Almost every question you can ask is open:

1. Relationship between ideals of $\pi_*(R)$ and ideals of R?
2. If $R = S$, the sphere spectrum, and $2 \in \pi_0 S$ is the cofiber of the 'times 2' map, then (2) is an ideal of $\pi_* S$ but the mod 2 Moore spectrum is not a ring spectrum, even up to homotopy. What is the ring spectrum quotient of S by 2?
3. Every ring spectrum is weakly equivalent to a quotient of the sphere spectrum by some Smith ideal. Define a monoid homomorphism $p : R \longrightarrow S$ to be a strong quotient if $S \otimes_R QN \longrightarrow N$ is a w.e. for all fibrant N (and cof. rep. Q). Can we classify strong quotients of ring spectra?
4. Connection to Prasma's 'homotopy normal maps'?

Open Problems 2

5. Let $f : I \longrightarrow R$ be any map. What is the Smith ideal generated by f? The free functor T yields an ideal of $T(R)$ not R.
6. Principle ideals? Maximal ideals? Regular sequences?
7. Depth? Krull dimension? Cohen-Macaulay modules?
8. Non-commutative version with left/right ideals.
9. Section 6 of White-Yau lists conjectures and open problems related to Smith O-ideal theory in: positive flat model on symmetric spectra and equivariant orthogonal spectra, positive complete model structure, global equivariant, injective model structures, and S-modules.

Open problems relating K-theory and ideals

10. Compute examples of various R/I, $R/(I \wedge_R J)$, and $A' \odot_A^M B$.
11. Use Smith ideals for new computations in algebraic K-theory, following Smith's original vision.
12. Land-Tamme is about ring spectra, but Smith ideals work in general stable model categories. Can you prove Smith's vision regarding $E(R/(I \wedge_R J))$ for motivic spectra, equivariant spectra, chain complexes, and the stable module category?
13. Land-Tamme works for general localizing invariants, like Waldhausen K-theory of pushouts of group rings, and Burghelea's work on periodic cyclic homology. Can you extend Smith's vision and get fiber sequences for general localizing invariants, using Smith ideals?

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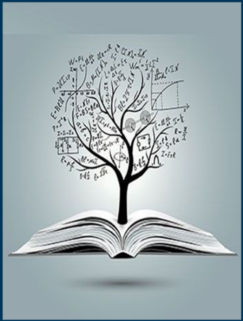

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