A unified approach to the homotopy theory of operads and algebras

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Motivation: operads, algebras, modules

1. Operads encode algebraic structures: Com, Lie, As, etc.
2. We want a homotopy theory for operads and algebras, to know when two are weakly equivalent, to be able to cofibrantly replace, Bousfield localization, hocolim, etc.
3. Transfer model structures to operads and algebras, with weak equivalences and fibrations underlying.
4. The category of operads is itself algebras over a colored operad (indeed, a polynomial monad).
5. Flavors and generalizations of operads: symmetric, non-symmetric, constant free, reduced, cyclic, modular, n-operads, PROPs, properads, wheeled variants. Different filtration arguments and different proofs in each variant.
6. Want a unified way to study operads, algebras, modules, bimodules, infinitessimal bimodules, etc.

A zoo of monoidal model categories

1. $(\mathrm{sSet}, \times, *)$ or Top
2. $(\mathrm{sSet}_*, \wedge, S^0)$ or Top_*
3. $(\mathrm{Ch}(R), \otimes_R, S^0(R))$
4. $(k[G]-\mathrm{mod}, \otimes_k, k)$ with stable module category
5. (Categories of presheaves or functors, Day convolution)
6. Spectra, G-spectra, motivic spectra
7. Graphs, groupoids, categories, 2-cat, etc.

Some of these have a structured interval object. Others require a hard filtration working cell-by-cell.

Different flavors of operads have different categories of trees, so different cells. We want all as special case of one theorem.

Definition and Motivation

- Given $\Phi : \mathcal{B}^{op} \to CAT$, form the Grothendieck construction $\int \Phi$ whose objects are pairs (O, A) where $O \in \mathcal{B}$ and $A \in \Phi(O)$, e.g., $O =$ operad and A is O-algebra.
- Morphisms $(\phi, f) : (O, A) \to (O', A')$ has $\phi : O \to O'$ and $f : A \to \phi^*(A')$.
- Global setting of $\int \Phi$ is a convenient place to study all 'fibers' (a.k.a. vertical structures) simultaneously.
- Examples: $\mathcal{B} =$ monoids, and $\Phi(R) :=$ R-modules.
- $\mathcal{B} =$ operads, and $\Phi(O) := Alg_O$ or $\Phi(O) := Mod_O$.
- $\mathcal{B} =$pairs of operads, and $\Phi(O, P) := (O, P)-$bimodules.
- $\mathcal{B} = (O, P, A, B, X)$, and A, B are (O, P)-bimod, and X infinitesimal (A, B)-bimod.
- Goal: connect homotopy theory of $\mathcal{B}, \Phi(O), \int \Phi$.

History of the problem

- Previous papers assumed $\mathcal{B}$ and all $\Phi(\mathrm{O})$ have model str's + more, then induced model str on $\int \Phi$, whose weak equivalences and fibrations match those in $\mathcal{B}$ and $\Phi(\mathrm{O})$'s.
- Roig 1994, Stanculescu 2012. Assume:
  1. for every w.e. ϕ in $\mathcal{B}$, then ϕ^* preserves and reflects w.e.'s.
  2. for every triv cof u in $\mathcal{B}$, the unit of $(\mathrm{u}_!, \mathrm{u}^*)$ is a w.e.
- Harpaz-Prasma 2015:
  1. for every trivial cof. u or triv. fib. v in $\mathcal{B}$, then $\mathrm{u}_!$ and v^* preserve weak equivalences.
  2. for every weak equiv. ϕ, $(\phi_!, \phi^*)$ is a Quillen equivalence.
- Cagne-Mellies 2020 (conditions imply HP2):
  1. HP1 but now $\mathrm{u}_!$ and v^* preserve and reflect w.e.'s
  2. Beck-Chevalley. Given $\mathrm{u} \circ \mathrm{v} = \mathrm{v}' \circ \mathrm{u}$ in $\mathcal{B}$, then $\mu : (\mathrm{u}')_! \mathrm{v}^* \to (\mathrm{v}')^* \mathrm{u}_!$ is w.e. in $\Phi(\mathrm{dom}(\mathrm{v}'))$.

Our strategy is the opposite: get $\int \Phi$ first

- The assumptions in previous work often fail, e.g., the weak equivalence $O \to Com$ for an E_∞-operad O does not induce a Quillen equivalence on algebras in spaces.
- For operad-style settings, we induce a model structure on $\int \Phi$ from the base $\mathcal{M}$, then deduce model structures on $\mathcal{B}$ and all $\Phi(O)$. Often $\int \Phi$ is alg over $\mathbb{N}+1$ colored operad.
- New filtration for operads and algebras simultaneously. Think: trees with extra markings for algebras.
- For operad-style settings, we always get a semi-model structure on $\int \Phi$ and hence induce same on $\mathcal{B}$ and on $\Phi(O)$'s for cofibrant O.
- Recover all known results about (semi-)model structures on operads and algebras, plus new results, from one theorem.
- Key is polynomial monads and quasi-tame notion.

How to do homotopy theory in $\int \Phi$

Lemma (well-known)

If $\mathrm{T} = \mathrm{UF}$ is monad on cofibrantly gen. $\mathcal{N} = \mathrm{Coll}(\mathcal{M}) \times \mathcal{M}$ and if for all generating trivial cofibrations $\mathrm{j} : \mathrm{K} \to \mathrm{L}$ in $\mathcal{N}$, transfinite compositions of pushouts in $\mathrm{Alg_T}(\mathcal{N})$:

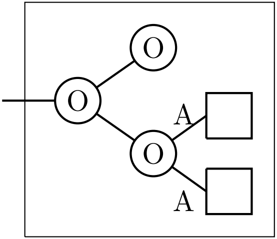
$$\begin{array}{ccc} \mathrm{F(K)} & \longrightarrow & \mathrm{F(L)} \\ \downarrow & & \downarrow \\ & \searrow\!\!\searrow & \\ (\mathrm{O}, \mathrm{A}) & \longrightarrow & (\mathrm{O}', \mathrm{B}) \end{array}$$

are weak equivalences then $\mathrm{Alg_T}(\mathcal{N})$ has transferred model structure, with weak equivalences and fibrations defined in $\mathcal{N}$.

If above works only for (O, A) cofibrant (resp. $\mathrm{U}((\mathrm{O}, \mathrm{A}))$ cofibrant) then get semi-model structure (resp. semi over $\mathcal{N}$).

Filtration to compute the pushout $(O, A) \to (O', B)$

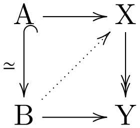
New filtration covers operads and algebras at once via trees with marked (boxed) vertices plus rules for composition:



Now $(O, A) \to (O', B)$ is a transfinite composition of simpler pushouts in $\mathcal{N}$. Hence, $(O, A) \to (O', B)$ is a weak equivalence. If polynomial monad P is quasi-tame, we get a model structure on $\int \Phi$. Otherwise, we get a semi-model structure.

Semi-model categories, given $F : \mathcal{M} \leftrightarrows \mathcal{D} : U$

Definition: $(\mathcal{D}, W, Q, F)$ satisfies all model category axioms except we only require the following for A and K cofibrant (resp. cof in $\mathcal{M}$):



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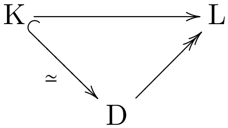


Still have cofibrant replacement. All model category results have semi-model category analogues (often cofibrantly replace first): Ken Brown lemma, cylinders and path objects, cube lemma, Quillen equivalences, Reedy model structures, (co)simplicial frames, homotopy (co)limits, simplicial mapping spaces, Bousfield localization, etc. Combinatorial semi is Quillen equiv. to combinatorial model category.

Polynomial monads

A polynomial P from I to J in Set is a diagram of sets of the form

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

Such a diagram generates a functor:

$$\mathbb{P} : Set/I \to Set/J.$$

$$\mathbb{P}(X)_j = \coprod_{b \in t^{-1}(j)} \prod_{e \in p^{-1}(b)} X_{s(e)},$$

Definition (A theorem of Gambino and Kock)

A polynomial monad is a polynomial P from I to I together with a cartesian monad structure on $\mathbb{P}$. Say P is finitary if $p^{-1}(b)$ is finite for any $b \in B$.

Polynomial monads encode flavors of operads

A finitary polynomial monad P has a category of algebras $\mathrm{Alg}_{\mathrm{P}}(\mathcal{M})$ (with I colors). Plug in your flavor of trees.

- Free monoid monad. $\mathrm{M}(\mathrm{X}) = \coprod_{\mathrm{n}} \mathrm{X}^{\mathrm{n}}$. The corresponding polynomial is
$$1 \leftarrow \mathrm{LTr}^* \rightarrow \mathrm{LTr} \rightarrow 1$$
Where LTr^* is linear rooted trees with one vertex marked.
- Free non symmetric operad monad. The corresponding polynomial is
$$\mathbb{N} \leftarrow \mathrm{PTr}^* \rightarrow \mathrm{PTr} \rightarrow \mathbb{N}$$
Where PTr^* is planar rooted trees with one vertex marked.
- Free symmetric operad monad. The corresponding polynomial is
$$\mathbb{N} \leftarrow \mathrm{ORTr}^* \rightarrow \mathrm{ORTr} \rightarrow \mathbb{N}$$
Where ORTr is ordered rooted trees.

Examples of polynomial monads

- Free symmetric operad monad; also non-symmetric operads, presheaves, monoids, enriched categories;
- Monads for cyclic and modular operads;
- Dioperads, properads, (generalized) PROPs, and wheeled and colored versions of all monads above.
- Free n-operad monad (see Batanin-Berger 'Tame' paper).

Theorem (Kock, Zawadowski)

The category of finitary polynomial monads and their cartesian morphisms is equivalent to the category of colored symmetric operads in Set with free action of symmetric groups.

Polynomial monads and the Grothendieck construction

- Goal: If $\mathcal{B}$ is the category of algebras over a polynomial monad, then so is $\int \Phi$.
- Let T be an I-colored symmetric operad in $\mathcal{M}$, equipped with a morphism of operads $\phi : \mathrm{T} \to \mathrm{SO(J)}$. Note: ϕ induces $\phi^* : \mathrm{Alg}_{\mathrm{SOp(J)}}(\mathcal{M}) \to \mathrm{Alg}_{\mathrm{T}}(\mathcal{M})$.
- Let O be an algebra of T. An algebra of O in $\mathcal{M}$ is a J-collection $\mathrm{C} = \{\mathrm{C_j} \mid \mathrm{j} \in \mathrm{J}\}$ of objects of $\mathcal{M}$ equipped with a map of T-algebras $\mathrm{O} \to \phi^*(\mathrm{End(C)})$ where End(C) is the endomorphism operad of C.
- The category of O-algebras is isomorphic to the category $\Phi(\mathrm{O}) := \mathrm{Alg}_{\phi_!(\mathrm{O})}(\mathcal{M})$.
- Theorem: If ϕ is a map of polynomial monads in Set then there exists a polynomial monad Gr(T) such that the category $\int \Phi$ is isomorphic to the category $\mathrm{Alg}_{\mathrm{Gr(T)}}(\mathcal{M})$.

Polynomial monad Gr(T)

We are given $\phi : \mathrm{T} \to \mathrm{SO}(\mathrm{J})$, displayed vertically:

$$\begin{array}{ccccccc}
\mathrm{I} & \xleftarrow{\ \mathrm{s}\ } & \mathrm{E} & \xrightarrow{\ \mathrm{p}\ } & \mathrm{B} & \xrightarrow{\ \mathrm{t}\ } & \mathrm{I} \\
\downarrow{\scriptstyle \mathrm{c}} & & \downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{c}} \\
\mathrm{Bq}(\mathrm{J}) & \longleftarrow & \mathrm{ORTr}^*(\mathrm{J}) & \longrightarrow & \mathrm{ORTr}(\mathrm{J}) & \longrightarrow & \mathrm{Bq}(\mathrm{J})
\end{array}$$

We define a polynomial monad

$$\mathrm{I} \sqcup \mathrm{J} \xleftarrow{\ \mathrm{S}\ } \mathrm{D}^* \longrightarrow \mathrm{D} \xrightarrow{\ \mathrm{T}\ } \mathrm{I} \sqcup \mathrm{J}$$

where $\mathrm{D} = \mathrm{B} \sqcup \mathrm{B} = \mathrm{B} \sqcup \{(\mathrm{b}, \sigma) \mid \sigma \in \mathrm{ORTr}(\mathrm{J}), \mathrm{b} \in \psi^{-1}(\sigma)\}$ and S and T are induced by $\mathrm{s}, \mathrm{t}, \mathrm{c}, \psi$ above.

Upshot: If T is a polynomial monad then so is Gr(T), hence we have a semi-model structure on $\int \Phi$ always.

Tame vs Quasitame

- Analyze the pushout P of $X \stackrel{g}{\leftarrow} F(K) \stackrel{F(f)}{\rightarrow} F(L)$ via classifiers.
- Construct a monad $T_{f,g}$ whose algebras are 5-tuples (X, K, L, f, g). There's a map of monads $a : T_{f,g} \to T$ s.t. $a_! : Alg_{T_{f,g}}(\mathcal{M}) \to Alg_T(\mathcal{M})$ is exactly the pushout P.
- U(P) is the colimit over the classifier $T^{T_{f,g}}$. So $U(X) \to U(P)$ is a transfinite composition of pushouts of morphisms in $\mathcal{M}$.
- T is tame if T^{T+1} is a coproduct of categories with terminal object. This implies $T^{T_{f,g}}$ has a final subcategory; colimit easy to compute. Full model structure on Alg_T.
- T is quasi-tame if $\pi_1(T^{T+1})$ is equivalent to a discrete groupoid. This implies $T^{T_{f,g}}$ decomposes into two pieces, which we can analyze separately. Full model structure.

Model category conditions on monoidal model cat $\mathcal{M}$

- Monoid axiom: (Triv. Cof. $\otimes\mathcal{M}$)-cell $\subset$ w.e.'s.
- h-monoidal: for each (trivial) cofibration $f : X \to Y$ and each object Z, the map $f \otimes Z$ is a (trivial) h-cofibration, i.e.,

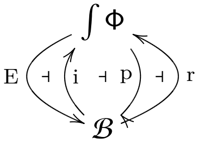
$$\begin{array}{ccccc} X\otimes Z & \longrightarrow & A & \xrightarrow{\simeq} & B \\ {\scriptstyle f\otimes Z}\downarrow & & \downarrow & & \downarrow \\ Y\otimes Z & \longrightarrow & A' & \xrightarrow[\therefore\simeq]{} & B' \end{array}$$

- Compactly generated: all objects are small relative to $I^{\otimes}$-cell, and weak equivalences are closed under filtered colimits along morphisms in $I^{\otimes}$.
- Commutative monoid axiom: if f is a trivial cof then so is $f^{\square n}/\Sigma_n$. Ex: sSet, Top, Ch(k), StMod(k[G]), spectra^{+}.
Yields (R, M) model str for R commutative monoid.

We get model structures on:

- Pairs (R, M) where R is a monoid and M is an R-module.
- Pairs (O, A) where O is a nonsymmetric operad and A is an O-algebra. Same for (O, M) with left O-module.
- Triples (O, P, M) where M is an (O, P)-bimodule. Infinitessimal, too, via (O, P, A, B, M).
- Pairs (O, M) where O is a constant-free symmetric operad (or n-operad) and M is a constant-free module.
- Semi on (O, A) where O is a symmetric operad (or hyperoperad) and A is an O-algebra.
- Let $\mathcal{M} = Ch(k)$, k characteristic zero. Get full (vertical) model structure on category of twisted modular operads of Ginzberg and Kapranov (1998). This is new.
- Note: monad T for nonsymmetric operads is tame but Gr(T) only quasi-tame.

Structure of Grothendieck construction



- Assume everything is bicomplete (need to make them model categories) and all ϕ^* preserve w.e.'s.
- Projection $\mathrm{p} : \int \Phi \to \mathcal{B}$, $\mathrm{p}(\mathrm{O}, \mathrm{A}) = \mathrm{O}$.
- Right adjoint $\mathrm{r} : \mathcal{B} \to \int \Phi$, $\mathrm{r}(\mathrm{O}) = (\mathrm{O}, \mathrm{t}_{\mathrm{O}})$.
- Left adjoint $\mathrm{i} : \mathcal{B} \to \int \Phi$, $\mathrm{i}(\mathrm{O}) = (\mathrm{O}, \mathrm{i}_{\mathrm{O}})$.
- Sometimes $\mathrm{E} : \int \Phi \to \mathcal{B}$ like enveloping algebra or enveloping operad $(\mathrm{O}, \mathrm{A}) \mapsto \mathrm{O}_{\mathrm{A}}$.

Global to horizontal and vertical

- All (semi-)model structures are transferred, so we know the weak equivalences and fibrations in $\mathcal{B}, \Phi(\mathrm{O}), \int \Phi$.
- Note (ϕ, f) is a w.e. (resp. fib) iff ϕ and f are. Define cofibrations by the lifting property.
- Lemma: $(\phi, \mathrm{f}) : (\mathrm{O}, \mathrm{A}) \to (\mathrm{P}, \mathrm{B})$ is a global (trivial) cofibration if and only if ϕ is a horizontal (trivial) cofibration and $\mathrm{f}^* : \phi_!(\mathrm{A}) \to \mathrm{B}$ is a vertical (trivial) cofibration.
- Theorem: If $\int \Phi$ admits global (semi-)model structure then $\mathcal{B}$ admits horizontal (semi-)model structure, $\Phi(\mathrm{O})$ admits vertical model structure for each $\mathrm{O} \in \mathcal{B}$ (semi-model structure for a cofibrant O), and $\mathrm{p} : \int \Phi \to \mathcal{B}$ is left and right Quillen.
- If $\phi : \mathrm{O} \to \mathrm{O}'$, then ϕ^* and $\phi_!$ form a Quillen pair.

Cofibrant generation, properness, and rectification

- If $\int \Phi$ is left (resp. right) proper then $\mathcal{B}$ and $\Phi(\mathrm{O})$ are left (right) proper for any O. Same for cofibrantly generated.
- Same for relatively left/right proper.
- Application: for $\mathcal{M} = \mathrm{Ch}(k)$, characteristic zero, then O-alg is left proper for all O.
- If $\int \Phi$ is left proper, and ϕ is w.e., ϕ^* reflects w.e.'s, and the unique map $\tau : \mathrm{i_O} \to \phi^*(\mathrm{i_{O'}})$ is a w.e., then $(\phi^*, \phi_!)$ is a Quillen equivalence (this is rectification). Application: strictification in Lack's model structure for 2Cat.
- Relative version if relatively left proper and O, O' are u-cofibrant, for $(\mathrm{U}, \mathrm{u}) : \int \Phi \to \int \Psi$. Like Σ-cofibrant.
- If O, O' are cofibrant then get Q.E. even if $\int \Phi$ is not (relatively) left proper.

From semi to full

Lemma

If $\mathcal{M}$ is a semi-model and any morphism admits (triv. cof, fib.) factorization then $\mathcal{M}$ is a model category.

So: lifting follows from factorization.

Lemma

Let $\int \Phi$ be semi and all ϕ^* preserve fibrations and weak equivalences. If, for any $\mathrm{f} : \mathrm{A} \to \mathrm{B}$ in $\Phi(\mathrm{O})$, the induced map $\mathrm{E}(\mathrm{O}, \mathrm{f}) : \mathrm{E}(\mathrm{O}, \mathrm{A}) \to \mathrm{E}(\mathrm{O}, \mathrm{B})$ admits a (triv. cof., fib.) factorization in $\mathcal{B}$, then so does f in $\Phi(\mathrm{O})$.

Upshot: if $\mathcal{B}$ is full model structure then so are the $\Phi(\mathrm{O})$'s.

Future Work

- Florian's work deducing consequences of the new model structures on (infinitessimal) bimodules.
- Bousfield localization for $\int \Phi$.
- For n-operad global model structure, force quasi-bijections act invertibly. Prove global Baez-Dolan stabilization result.
- Generalize Braun, Chuang, Lazarev work on derived localizations of (A, M) where A is an algebra and M is an A-module, to global setting $\int \Phi$, e.g., (O, A) where O is operad and A is O-algebra.
- Determine how localizations of operads and algebras are related.

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