Localization and Ring Objects in Model Categories

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Localization in Algebra

Localization: systematically adjoin multiplicative inverses

Setup: R = ring, $S \subset R$ multiplicatively closed

Get: $S^{-1}R = R \times S/\sim$, e.g. $(\mathbb{Z}^{\times})^{-1}\mathbb{Z} = \mathbb{Q}$, $\langle 2\rangle^{-1}\mathbb{Z} = \mathbb{Z}_{(2)}$

Also get: universal ring homomorphism $R \to S^{-1}R$ taking S to units ,
i.e. for any $f : R \to E$ taking S to units

$\exists\, !g$ making diagram commute:

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[d, "f"'] & S^{-1}R \arrow[dl, dashed, "g"] \\ E & \end{tikzcd}$$

How to generalize to categories? (No mult. inverses)

Inverting s is the same as inverting the map $\mu_s(r) = s \cdot r$

Localization in Categories

Setup: $\mathcal{C}$ = category, T = set of morphisms. Get: $\mathcal{C}[T^{-1}]$ and universal $\mathcal{C} \to \mathcal{C}[T^{-1}]$ taking T to isomorphisms.

$$
\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}[T^{-1}] \arrow[dl, dashed] \\
\mathcal{D} &
\end{tikzcd}
$$

$$obj(\mathcal{C}[T^{-1}]) = obj(\mathcal{C})$$

Example: Top[{homotopy equivalences}$^{-1}$] = HoTop

Adjoining f^{-1} forces us to adjoin many $g \circ f^{-1}$ & $f^{-1} \circ h$

$\mathcal{C}[T^{-1}](X, Y) = \text{Zigzags}/\sim$

$$
\begin{tikzcd}
\bullet \arrow[d, "f_1"] \arrow[dr, "g_1"] & & \bullet \arrow[dl, "f_2"'] \arrow[dr, "g_2"] & \\
X & \bullet & & Y
\end{tikzcd}
$$

Oops! Zigzags is not a set

Model Categories

Can't localize an arbitrary C at an arbitrary T

Let $C = \mathcal{M}$ have all small (co)limits and distinguished classes of maps $\mathcal{W}, \mathcal{F}, \mathcal{Q}$ satisfying some axioms.

Called: weak equivalences, fibrations (e.g. $F \to E \to B$), cofibrations (e.g. satisfying homotopy extension property)

If we set $T = \mathcal{W}$ then $\mathcal{M}[\mathcal{W}^{-1}] = \mathrm{Ho}(\mathcal{M})$ exists and has the desired universal property

Some model categories: Spaces, Spectra, Ch(R), G-spectra (many model category structures)

(Left) Bousfield Localization

Suppose we want to invert $f \notin \mathcal{W}$. Because Ho($\mathcal{M}$) is nice:

$$\begin{tikzcd} \mathcal{M} \arrow[r, dotted] \arrow[d] & \boxed{L_f\mathcal{M}} \arrow[d, dotted] \\ \mathrm{Ho}(\mathcal{M}) \arrow[r] & \mathrm{Ho}(\mathcal{M})[f^{-1}] \end{tikzcd}$$

$$obj(L_f\mathcal{M}) = obj(\mathcal{M})$$

$$L_f\mathcal{M}(X,Y) = \mathcal{M}(X,Y)$$

Under standard hypotheses on $\mathcal{M}$, $L_f\mathcal{M}$ = model category.
$\mathcal{W}_f = \langle f \cup \mathcal{W} \rangle \supset \mathcal{W}$, $\mathcal{Q}_f = \mathcal{Q}$, $\mathcal{F}_f \subset \mathcal{F}$

Note: localizing a set T of maps is the same as localizing $f = \coprod_{g \in T} g$, so it's fine to look at just L_f

A question

L_f preserves many standard properties of model categories. Does it preserve monoids? Yes for A_∞ and E_∞. No for strict commutative (Hill, 2011). Goal: Figure out when it does

Given associative $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ with unit S, a monoid E has $\mu : E \otimes E \to E$, $\eta : S \to E$, commutative diagrams

$$\begin{tikzcd} E \otimes E \otimes E \arrow[r] \arrow[d] & E \otimes E \arrow[d] \\ E \otimes E \arrow[r] & E \end{tikzcd} \qquad \begin{tikzcd} S \otimes E \arrow[r] \arrow[dr] & E \otimes E \arrow[d] & E \otimes S \arrow[l] \arrow[dl] \\ & E & \end{tikzcd}$$

Morally: $a(bc) = (ab)c$ and $1 \cdot a = a = a \cdot 1$

Commutative E also has twist $\tau : E \otimes E \to E \otimes E$.

Monoidal Model Categories

1. Pushout Product Axiom: Given $f : A \to B$ and $g : X \to Y$ cofibrations, $f \square g$ is a cofibration. If $f \in \mathcal{W}$ then $f \square g \in \mathcal{W}$.

$$
\begin{tikzcd}
A \otimes X \arrow[r] \arrow[d] & B \otimes X \arrow[d] \arrow[ddr, bend left] & \\
A \otimes Y \arrow[r] \arrow[ur, phantom, "\searrow"] \arrow[drr, bend right] & P \arrow[dr, "f \square g"] & \\
& & B \otimes Y
\end{tikzcd}
$$

2. Unit Axiom: For cofibrant X, $QS \otimes X \to S \otimes X \cong X$ is in $\mathcal{W}$

3. Monoid Axiom: Transfinite compositions of pushouts of maps in $\{\text{Trivial-Cofibrations } \otimes id_X\}$ are weak equivalences.

Preservation of Strict Monoids

(1) & (2) $\Rightarrow$ $\mathrm{Ho}(\mathcal{M})$ is monoidal ($\otimes$ is a Quillen bifunctor)
(3) implies the monoids $\mathrm{Mon}(\mathcal{M})$ form a model category.

$X \in \mathrm{Ho}(\mathcal{M})$ is a *strict monoid* if there is a monoid $R \in \mathcal{M}$ commuting "on the nose" such that $R \cong X$ in $\mathrm{Ho}(\mathcal{M})$.

Localization *preserves strict monoids* if the composition $\mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(L_f\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$ takes X to a strict monoid

Theorem

If $L_f\mathcal{M}$ satisfies (1)-(3) then L_f preserves strict monoids

$L_f\mathcal{M}$ can fail Pushout Product Axiom: $\mathcal{M} = \mathbb{F}_2[\Sigma_3]$-mod and $f : \mathbb{F}_2 \to \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$ taking 1 to $(1, 1, 1)$

Preservation of Monoidal Structure

The Unit Axiom is trivially preserved by L_f because $Q_f = Q$

Theorem

If $\mathcal{M}$ is a cofibrantly generated, left proper, monoidal model category with cofibrant objects flat and generating (trivial) cofibrations I and J having cofibrant domains, and if $f \otimes K$ is an f-local equivalence for all (co)domains K of maps in $I \cup J$, ***then $L_f M$ is a monoidal model category*** *with cofibrant objects flat and domains of $I_f \cup J_f$ cofibrant.*

Theorem

Assuming further that $\mathcal{M}$ is weakly finitely generated, that f has SSet-small (co)domain, and a technical condition on $Q \otimes -$, then $L_f \mathcal{M}$ ***satisfies the monoid axiom.***

Preservation of Strict Commutative Monoids

Theorem

If $L_f\mathcal{M}$ is a monoidal model category with CommMon$(L_f\mathcal{M})$ a model category, then L_f preserves strict commutative monoids

John Harper suggested a Σ_n-*equivariant monoid axiom*

This gets CommMon$(-)$ to be a model category, and should work for more general coloured operads

Next: L_f preserves Σ_n-equivariant monoid axiom

After that: Applying results to examples, especially G-spectra.

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