On the existence of N_∞ operads in equivariant homotopy theory

David White

Denison University

Joint with Javier Gutiérrez
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Exotic Smooth Structures on Spheres

Classical: If two smooth manifolds are homeomorphic, are they diffeomorphic?
Answer (Milnor, 1956): No! S^7 has exotic smooth structures.

Next question: Can we classify all exotic smooth structures on spheres, S^n? (First, assume $n \neq 4$).

Milnor and Kervaire (1963): the group of smooth n-dim manifolds homeomorphic to S^n (under connect sum operation) is isomorphic to the group Θ_n of h-cobordism classes of homotopy n-spheres.
Note: an h-cobordism is a cobordism $M \hookrightarrow W \hookleftarrow N$ where the inclusions are homotopy equivalences.
Note: M is a homotopy sphere iff M is an h-cobordism sphere (Smale et. al) iff M is a topological sphere (by Perelman).

Framed Manifolds and $bP^{n+1} \leq \Theta_n$

Θ_n = h-cobordism classes of homotopy n-spheres; finite, abelian. Cyclic subgroup $bP^{n+1} \leq \Theta_n$ of n-spheres that bound parallelizable manifolds. “Easy.”

Parallelizable manifold has trivial tangent bundle (hence also trivial normal bundle). **Framed** means it has a chosen trivialization of the normal bundle. Kervaire-Milnor; Levine:

$$bP^{n+1} = \begin{cases} 0 & \text{if n+1 is odd} \\ C_{Bernoulli} & \text{if } n+1 = 4k \\ 0 \text{ or } C_2 & \text{if } n+1 = 4k+2 \end{cases}$$

$bP^{n+1} \cong C_2 = \mathbb{Z}/2$ when $n+1 = 4k+2$, $2k+1 \neq 2^\ell - 1$

Kervaire Invariant and Framed Surgery

bP^{n+1} known except $n+1 = 2^{j+1} - 2$ (where 0 or $\mathbb{Z}/2$), i.e. dim 2, 6, 14, 30, 62, 126, 254, ...

Recall J-homomorphism $J : \pi_n(SO(k)) \to \pi_{n+k}(S^k)$

Framed surgery theory gives an injection (onto if n is odd) $\Theta_n/bP^{n+1} \overset{\psi}{\hookrightarrow} coker(J) = \pi_*^s/\operatorname{im}(J)$. It's an iso. iff Kervaire invariant in dim n is 0 (otherwise, image of index 2)

Kervaire invariant (of n-dim framed manifold) is Arf invariant of the skew-symmetric pairing on the middle-dimensional homology. It's an obstruction to framed surgery.

Pushing the problem into stable homotopy

Browder (1969): $K(M^n) = 1$ only possible if $n = 2^{j+1} - 2$. It's 1 iff the class $h_j^2 \in Ext_A^{2,2^{j+1}}(Z/2, Z/2)$ persists to the E_∞-page, i.e. represents an element $\theta_j \in \pi^s_{2^{j+1}-2}$

There are M with $K(M) = 1$ in dim 2, 6, 14, 30, and 62.

Exact sequence $0 \to bP^{n+1} \to \Theta_n \to \pi_*^s/\operatorname{im}(J) \to 0$ splits when $n \neq 2^\ell - 1$ or $2^\ell - 2$.

Other n: $0 = bP^{n+1} \to \Theta_n \to \pi_*^s/\operatorname{im}(J) \xrightarrow{\Phi} C_2 \to bP^n \to 0$

Extension problem: Φ iso. (if $K(M) = 1$) or $bP^n \cong 0$.

Enter Hill, Hopkins, Ravenel (2009)

Browder (1969 + computations in π_n^s): Can only have Kervaire invariant 1 if $n = 2^{j+1} - 2 = 2, 6, 14, \ldots$

HHR (2009): For $j \geq 7$, the element $h_j^2 \in Ext_A^{2,2^{j+1}}(Z/2, Z/2)$ does not represent an element $\theta_j \in \pi^s_{2^{j+1}-2}$

Corollary: Unless $n = 2, 6, 14, 30, 62, 126$, there is no manifold of Kervaire invariant 1. So, only $n = 126$ is left!

Corollary: In most dimensions, $\Theta_n / bP^{n+1} \to \pi_*^s / \operatorname{im}(J)$ is an isomorphism.

Corollary: Except in dimensions 2, 6, 14, 30, 62, and maybe 126, every stably framed smooth manifold is framed cobordant to a homotopy sphere. Surgery works!

Back to Exotic Smooth Structures

For $n \neq 4, 125, 126$, if the order of π_n^s is known, we can compute the number of exotic n-spheres. Except for n of the form $2^k - 3 \geq 125$, we can also describe the group Θ_n precisely.

Example: For dimension $n = 7$, the group Θ_7 is the cyclic group $\mathbb{Z}Z/28$

Theorem (HHR): Unless $n = 2, 6, 14, 30, 62, 126$,

- when $n = 4k + 2$, $\Theta_{4k+2} \cong \pi^s_{4k+2}$, and
- when $n = 4k + 1$, $|\Theta_{4k+1}| = a_k|\pi^s_{4k+1}|$ where $a_k = 1$ if k even and 2 if k odd.

Theorem (Wang-Xu): no exotic smooth structures in dim 5, 6, 12, 56, 61. Proof by computing π_n^s.

HHR proof sketch

To show $h_j^2 \in Ext_A^{2,2^{j+1}}(Z/2, Z/2)$ does NOT represent $\theta_j \in \pi^s_{2^{j+1}-2}$:

1. Create **the 256-periodic spectrum** (generalized cohomology theory) $\Omega = D^{-1}MU^{\wedge 4}$.
2. The Detection Theorem - can see if θ_j is zero or not via its Hurewicz image in $\Omega^{2-2^{j+1}}(pt)$
3. The Periodicity Theorem: $\Omega^{*+256}(X) \cong \Omega^*(X)$
4. The Gap Theorem: $\Omega^i(pt) = 0$ for $-4 < i < 0$

Proof relies on Slice Spectral Sequence in G-spectra ($G = \mathbb{Z}/8$).

Orthogonal G-spectra

An orthogonal G-spectrum is a sequence (X_n) of $G \times O(n)$-spaces, with $\sigma_n : \Sigma X_n \to X_{n+1}$. Structure maps $X_n \wedge S^k \to X_{n+k}$ are $G \times O(n) \times O(k)$-equivariant. Denote Sp^G.

Topological closed symmetric monoidal model category with $\mathrm{Hom}(X, Y)_n = \prod_{m \geq n} \mathrm{Map}_{O(m-n)}(X_{m-n}, Y_m)$.

$$(X \wedge Y)_n = \bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} (X_p \wedge Y_q)$$

E is a commutative ring G-spectrum if $\tau : E \wedge E \to E \wedge E$, $\eta : S \to E$, and associative, unital, commutative $\mu : E \wedge E \to E$ (via commutative diagrams). Denote $CAlg(Sp^G)$.

Multiplicative Norms for Commutative G-spectra

Adjoint $\mathrm{ind}_H^G(X) \dashv \mathrm{res}_H^G : G\text{-set} \to H\text{-set}$; $\mathrm{ind}_H^G(X) = \coprod_{G/H} X$

For G-spectra, $ind_H^G(X) = \bigvee_{i \in G/H} (H_i)_+ \wedge_H X$

Can also define $N_H^G(X) = \wedge_{i \in G/H} (H_i)_+ \wedge_H X$. For any finite G-set T, can define $N^T X = \bigwedge_T X$.

Adjunction $(N_H^G \dashv res_H^G) : CAlg(Sp^H) \leftrightarrows CAlg(Sp^G)$

Commutative ring G-spectra X have multiplicative norm maps $N^T X \to X$ for all T. These are used in the HHR computations that resolve the Kervaire Invariant One problem.

Every homomorphism $\rho : G \to \Sigma_{|T|}$ gives $G \rtimes \Sigma$ action on $N^T X$. Norm maps via $G_+ \wedge_H N^T(\mathrm{res}_H X) \cong (G \times \Sigma_n)/\Gamma_T \wedge_{\Sigma_{|T|}} X^{\wedge |T|}$ and $X^{\wedge |T|} \to X$.

G-operads

Operads encode algebraic structure. An operad P is a collection of sets (or spaces or G-spaces) $P(n)$ parameterizing n-ary operations $f : X^{\wedge n} \to X$ for all n. Action of Σ_n on $P(n)$, unit $1 \in P(1)$, and composition $\circ : P(k) \times (P(n_1) \times \cdots \times P(n_k)) \to P(n)$ for $n = \sum_{i=1}^{k} n_i$.

Algebras X have $P(n) \wedge_{\Sigma_n} X^n \to X$ for all n. Examples:

1. *Com* has $Com(n) = *$ for all n. Algebras = $CAlg(Sp^G)$.
2. In *Top*, $E_\infty(n) = E\Sigma_n$ (free Σ_n-action and contractible), and E_∞-operads parameterize "homotopy coherent" commutativity.
3. In Top^G, N_∞-operads encode E_∞ plus multiplicative norms.

N_∞-operads (Blumberg-Hill 2015)

An N_∞ **operad** is a G-operad P such that $P(0)$ is G-contractible, the action of Σ_n on $P(n)$ is free, and $P(n)$ is the universal space for a family $\mathcal{F}_n = \mathcal{N}_n(P)$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times 1$.
Here $P(n)^\Gamma = \varnothing$ if $\Gamma \notin \mathcal{F}_n$, and $P(n)^\Gamma = *$ otherwise.

If $\mathcal{F}_n$ is all subgroups of $G \times \Sigma_n$ that contain all subgroups of the form $H \times 1$, then you have all norms, and it's *complete* N_∞. These operads are G-weakly equivalent to *Com*.

If $\mathcal{F}_n = \{H \times 1\}$, then N_∞ is the same as E_∞ in Top^G.

Motivating Question: Which collections $\mathcal{F} = (\mathcal{F}_n)$ have associated N_∞-operads?

Model Categories

A **model category** is a setting for abstract homotopy theory. Examples: Top, sSet, Ch(R), stable module cat, Spectra, G-spectra, motivic spectra, operads, categories, graphs, flows, ...

Formally, a bicomplete category $\mathcal{M}$ and classes of maps $\mathcal{W}, \mathcal{F}, \mathcal{Q}$ (= weak equivalences, fibrations, cofibrations) satisfying axioms to behave like Top. Lifting, factorization, 2 out of 3, retracts.

An object X is **cofibrant** if $\varnothing \to X$ is a cofibration (where $\varnothing$ is initial). The **cofibrant replacement** QY of Y is the result of factoring $\varnothing \to Y$ into cofibration followed by trivial fibration $QY \to Y$. Ex: CW approximation, Projective Resolution.

Existence of N_∞-operads (idea)

Non-equivariantly, $E\Sigma_n$ is the cofibrant replacement of $*$ in Top^{Σ_n} (with the projective model structure). Think: free Σ_n-action and contractible.

So an E_∞-operad P is cofibrant in $Coll = \prod_{n=0}^{\infty} Top^{\Sigma_n}$.

Given a family $\mathcal{F}_n$ of subgroups of $G \times \Sigma_n$, a universal classifying space $E\mathcal{F}_n$ is a cofibrant replacement of $*$ in the *fixed-point* model structure $Top_{\mathcal{F}_n}^{G\times\Sigma_n}$, where f is a weak equivalence (resp. fibration) iff f^Γ is for all $\Gamma \in \mathcal{F}_n$. Think: good fixed point behavior.

So, given $\mathcal{F} = (\mathcal{F}_n)$, an N_∞-operad associated to $\mathcal{F}$ (if it exists) is cofibrant in $Coll_{\mathcal{F}} = \prod_{n=0}^{\infty} Top_{\mathcal{F}_n}^{G\times\Sigma_n}$.

Existence of N_∞-operads (proof)

Given $\mathcal{F}$, transfer a model structure along the free-operad functor $F : Coll_{\mathcal{F}} \leftrightarrows Op^G_{\mathcal{F}} : U$. A map of operads f is a weak equivalence (resp. fibration) iff $U(f)$ is.

In $Op^G_{\mathcal{F}}$, define P to be the cofibrant replacement of Com.

Prove $U(P)$ is still cofibrant in $Coll_{\mathcal{F}}$. This is hard!

Note: highly non-constructive. Related work of Bonventre-Pereira and Rubin.

Obstruction: Composition $\circ : P(k) \times (P(n_1) \times \cdots \times P(n_k)) \to P(n)$ could become $* \to \varnothing$ after taking Γ-fixed points, for $\Gamma \notin \mathcal{F}_n$

Existence of N_∞-operads (formal statement)

Definition (Realizable sequence of families of subgroups)

A sequence $\mathcal{F} = (\mathcal{F}_n)$ is *realizable* if, for each decomposition $n = n_1 + \cdots + n_k$,

$$\mathcal{F}_k \wr (\mathcal{F}_{n_1} \times \cdots \times \mathcal{F}_{n_k}) \subset \mathcal{F}_n,$$

i.e. every subgroup of $G \times \Sigma_n$ "built from" subgroups of $G \times \Sigma_{n_i}$ via blocks twisted by $G \times \Sigma_k$ is already in $\mathcal{F}_n$.

Theorem (Gutiérrez-W.)

A sequence $\mathcal{F} = (\mathcal{F}_n)$ is realizable if and only if there is an N_∞-operad P such that $P(n)$ is a universal classifying space for the family $\mathcal{F}_n$.

Existence Proof (the hard work)

To show P cofibrant in $Op_{\mathcal{F}}^G$ implies $U(P)$ cofibrant in $Coll_{\mathcal{F}}$, prove that for every cofibration $K \to L$ in $Coll_{\mathcal{F}}$, and every cofibrant $P \in Op_{\mathcal{F}}^G$, then the pushout $P \to P[u]$ is a cofibration in $Op_{\mathcal{F}}^G$.

$$\begin{array}{ccc} F(K) & \to & F(L) \\ \downarrow & & \downarrow \\ P & \to & P[u] \end{array}$$

Use tree-decomposition of F due to Berger-Moerdijk (2003).

Model Structure on Algebras over N_∞-operads

For $\mathcal{H}$ a family of subgroups of G, a $\mathcal{H}$-N_∞-operad has families with all $H \times 1$ for $H \in \mathcal{H}$. These are realizable too.

Theorem

1. *(W.-Yau) For every operad P in Top^G, P-algebras in Sp^G have a model structure where f is a weak equivalence (resp. fibration) if and only if $U(f)$ is in Sp^G.*
2. *(Gutiérrez-W.) In the positive (complete) model structure on $Sp^{\mathcal{H}}$, a weak equivalence $f : P_{\mathcal{F}} \to P'_{\mathcal{F}'}$ in $Op^G_{\mathcal{F}'}$, induces a Quillen equivalence $Alg_P \leftrightarrows Alg_{P'}$.*
3. *(Gutiérrez-W.) For complete $\mathcal{H}$-N_∞-operads P, the unique map $P \to Com$ induces a Quillen equivalence $Alg_P \leftrightarrows CAlg$.*

Left Bousfield Localization, $L_{\mathcal{C}}$

Given $\mathcal{C} \subset mor(Sp^G)$, $L_{\mathcal{C}}Sp^G$ is a universal model structure where $\mathcal{C}$ are weak equivalences.

Theorem (W.)

1. *$L_{\mathcal{C}}Sp^G$ is a monoidal model category iff $\mathcal{C} \otimes (G/H)_+$ is a new weak equivalence for all H.*
2. *$L_{\mathcal{C}}Sp^G$ satisfies the commutative monoid axiom (so $CAlg(L_{\mathcal{C}}Sp^G)$ has a transferred model structure) if and only if $\mathrm{Sym}(\mathcal{C})$ consists of new weak equivalences.*
3. *Such localizations $L_{\mathcal{C}}$ preserve all N_∞-operad algebras and commutative ring G-spectra.*

Relevance: HHR needed their $\Omega = L_{\mathcal{C}}(MU^{\wedge 4})$ to be commutative!

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