The homotopy theory of ideals of ring spectra

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Brave New Algebra

Let M be a monoidal model category of spectra. Analogy:

Classical Algebra	Brave New Algebra
Ring of Integers $\mathbb{Z}$	Sphere Spectrum S
Abelian groups = $\mathbb{Z}$-modules	Spectra = S-modules
(Unital) Ring	Ring spectrum
Commutative ring	E_∞-ring spectrum
Both: projective/injective modules,	homological dim, semisimple, etc.
Ideal $I \subset R$ s.t. R/I is a ring	???

'Subobject' is the wrong idea. Better: an ideal is something you can quotient by $I \xrightarrow{j} R \xrightarrow{\text{coker}} R/I$. Jeff Smith (2006): an ideal is an arrow $j : I \longrightarrow R$ with extra structure.

Ideals are arrows with algebraic structure

If $(\mathrm{M}, \otimes, 1)$ is a closed symmetric monoidal category, then the arrow category $\mathrm{Arr}(\mathrm{M})$ has two monoidal structures:

1. Tensor monoidal structure: $f \otimes g : X_0 \otimes Y_0 \longrightarrow X_1 \otimes Y_1$, unit is $Id_1 : 1 \longrightarrow 1$.
2. Pushout product monoidal structure (unit $\varnothing \longrightarrow 1$):

$$(X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \xrightarrow{f \square g} X_1 \otimes Y_1$$

Definition: A Smith ideal is a monoid in $\overrightarrow{\mathrm{M}}^{\square} := (\mathrm{Arr}(\mathrm{M}), \square)$

Note: A monoid in $\overrightarrow{\mathrm{M}}^{\otimes}$ is a monoid homomorphism in M.

Unpacking definition of Smith ideal as monoid in $\overrightarrow{\mathrm{M}}^{\square}$

A Smith ideal is a monoid R, an R-bimodule I, and a map of R-bimodules $j : I \longrightarrow R$ such that $\mu(1 \otimes j) = \mu(j \otimes 1) : I \otimes I \longrightarrow I$.
Reason: $\eta : (\varnothing \longrightarrow 1) \longrightarrow j$ and unpack $\mu : j \square j \longrightarrow j$:

$$\begin{array}{ccc} (I \otimes R) \coprod_{I \otimes I} (R \otimes I) & \longrightarrow & R \otimes R \\ \downarrow & & \downarrow \\ I & \longrightarrow & R \end{array}$$

Note: R/I is a ring spectrum and $\mathrm{coker}(j) : R \longrightarrow R/I$ is a ring homomorphism.
Theorem (Hovey, 2014): The cokernel functor from $\overrightarrow{\mathrm{M}}^{\square}$ to $\overrightarrow{\mathrm{M}}^{\otimes}$ is strong symmetric monoidal ($j \mapsto (R \longrightarrow R/I)$), and right adjoint is the kernel. This forms a Quillen equivalence $\overrightarrow{\mathrm{M}}^{\square} \leftrightarrows \overrightarrow{\mathrm{M}}^{\otimes}$.

Smith's motivation: algebraic K-theory

Suppose that R is a ring spectrum with Smith ideals I and J. Define the Smith ideal $I \wedge_R J$. Let T be the homotopy pushout:

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & T \end{array}$$

in the category of ring spectra. Smith: "there is a fiber sequence $K(R/(I \wedge_R J)) \longrightarrow K(R/I) \otimes K(R/J) \longrightarrow K(T)$ of algebraic K-theory spectra." Proven by Land-Tamme, 2023; plus $T \cong R/I \odot_R^M R/J$, the $\odot$-ring from their 2019 Annals paper, for $M = (R/I) \wedge_R (R/J)$.

Operad structure matters: in E_∞ context, $A' \odot_A^M B \simeq B \odot_A^M A'$.

Our setup (W.-Yau)

Note: monoid morphisms are algebras over a 2-colored operad. Smith ideals are too. Generalize from *Ass* to operad O?

1. Goal: homotopy theory of ideals structured by an operad O, e.g., commutative ideals, A_∞-ideals, E_∞-ideals, E_n, Lie, L_∞, etc. Now $\mathrm{coker}(j) : R \longrightarrow R/I$ is O-alg morphism.
2. Let $L_0 \dashv Ev_0$, $L_1 \dashv Ev_1$. Given O, define $\overrightarrow{O}^{\otimes} = L_0 O$ (resp. $\overrightarrow{O}^{\square} = L_1 O$), C-colored operad in $\overrightarrow{\mathrm{M}}^{\otimes}$ (resp. $\overrightarrow{\mathrm{M}}^{\square}$).
3. A Smith O-ideal is an algebra over $\overrightarrow{O}^{\square}$; a morphism of O-algebras is an algebra over $\overrightarrow{O}^{\otimes}$.
4. coker is a Quillen equiv. $\mathrm{Alg}(\overrightarrow{O}^{\square}; \overrightarrow{\mathrm{M}}^{\square}) \leftrightarrows \mathrm{Alg}(\overrightarrow{O}^{\otimes}; \overrightarrow{\mathrm{M}}^{\otimes})$
5. There is a $(C \coprod C)$-colored operad O^s in M such that $\mathrm{Alg}(\overrightarrow{O}^{\square}; \overrightarrow{\mathrm{M}}^{\square}) \cong \mathrm{Alg}(O^s; \mathrm{M})$. Use to transfer model str.

Unpacking Smith O-ideal $j : X \longrightarrow A$ s.t. A/X is O-algebra

Proposition (W.-Yau)

A Smith O-ideal in M *is precisely:*

- *an O-algebra* (A, λ_1) *in* M*,*
- *an A-bimodule* (X, λ_0) *in* M*, and*
- *an A-bimodule map* $f : (X, \lambda_0) \longrightarrow (A, \lambda_1)$

such that, for $1 \le i < j \le n$*, the following commutes*

$$\begin{array}{ccc}
O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{i-1}} X_{c_i} A_{c_{i+1}} \cdots X_{c_j} \cdots A_{c_n} & \xrightarrow{(\mathrm{Id}, f_{c_j}, \mathrm{Id})} & O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{i-1}} X_{c_i} A_{c_{i+1}} \cdots A_{c_n} \\
\Big\downarrow {\scriptstyle (\mathrm{Id}, f_{c_i}, \mathrm{Id})} & & \Big\downarrow {\scriptstyle \lambda_0} \\
O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{j-1}} X_{c_j} A_{c_{j+1}} \cdots A_{c_n} & \xrightarrow{\lambda_0} & X_d
\end{array}$$

What is this O^s with $\mathrm{Alg}(\overrightarrow{O}^{\square}; \overrightarrow{\mathrm{M}}^{\square}) \cong \mathrm{Alg}(O^s; \mathrm{M})$?

Given a C-colored operad O, denote by C^0 (resp. C^1) the first (resp. second) copies of $C \coprod C$. Given $c \in C$, write $c^\epsilon \in C^\epsilon$ for the same c in each copy, for $\epsilon \in \{0,1\}$. Define:

$$O^s\binom{d^1}{c_1^{\epsilon_1}, \ldots, c_n^{\epsilon_n}} = O\binom{d}{\underline{c}}$$

$$O^s\binom{d^0}{c_1^{\epsilon_1}, \ldots, c_n^{\epsilon_n}} = \begin{cases} O\binom{d}{\underline{c}} & \text{if at least one } \epsilon_i = 0 \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

So, an O^s-algebra is a pair (A, X) of C-colored objects, plus structure maps making A into an O-algebra, X into an A-bimodule, and $f : X \longrightarrow A$ into an A-bimodule map.

This is similar to the two-colored operad for monoid maps.

Main theorem

Theorem (W.-Yau)

If M *is nice, and cofibrant Smith O-ideals are also entrywise cofibrant in* $\overrightarrow{\mathrm{M}}^{\square}$ *then there is a Quillen equivalence*

$$\{\textit{Smith O-Ideals}\} \underset{\mathrm{ker}}{\overset{\mathrm{coker}}{\rightleftarrows}} \{\textit{O-Algebra Maps}\}$$

For Σ-cofibrant O, just need M stable, monoidal, cof gen.
For $O = Com$, M needs strong commutative monoid axiom.
For general O, need $X \otimes_{\Sigma_n} (-)^{\square n}$ and $f \square_{\Sigma_n} (-) : \mathrm{M}^{\Sigma_n} \longrightarrow \mathrm{M}$ homotopically well behaved, like preserving trivial cofibrations.
Examples: symmetric spectra, $Ch(k)$, $\mathrm{StMod}(k[G])$, motivic, equivariant orthogonal spectra, enriched functors, S-modules, etc.

Comparison with ∞-operads

Theorem (W.-Yau)

If M is cof. gen., $\mathrm{M}^\flat \subset \mathrm{M}$, and O is Σ_C-cofibrant (symmetric) C-colored operad.

- *Denote by $\mathrm{Alg}(O;\mathrm{M})^c[W_O^{-1}]$ the ∞-category obtained from the semi-model category $\mathrm{Alg}(O;\mathrm{M})$, by first passing to the subcategory of cofibrant objects, and then inverting the weak equivalences between O-algebras.*
- *Denote by $\mathrm{Alg}(O;\mathrm{M}[W^{-1}])$ the ∞-category obtained by first passing from M to the (symmetric) monoidal category $\mathrm{M}[W^{-1}]$ and then passing to O-algebras, where O is viewed as a colored operad in $\mathrm{M}[W^{-1}] \simeq \mathrm{M}^\flat[W^{-1}]$.*

Then $\mathrm{Alg}(O;\mathrm{M})^c[W_O^{-1}] \simeq \mathrm{Alg}(O;\mathrm{M}[W^{-1}])$ as ∞-categories.

Open Problems

Almost every question you can ask, e.g., What is the relationship between ideals of $\pi_*(R)$ and ideals of ring spectra? If $R = S$, the sphere spectrum, and $2 \in \pi_0 S$ is the cofiber of the 'times 2' map, then (2) is an ideal of $\pi_* S$ but the mod 2 Moore spectrum is not a ring spectrum, even up to homotopy. So what is the ring spectrum quotient of S by 2?

Let $f : I \longrightarrow R$ be any map. What is the Smith ideal generated by f? The free functor T yields an ideal of $T(R)$ not R.

Every ring spectrum is weakly equivalent to a quotient of the sphere spectrum by some Smith ideal. Define a monoid homomorphism $p : R \longrightarrow S$ to be a strong quotient if $S \otimes_R QN \longrightarrow N$ is a w.e. for all fibrant N (and cof. rep. Q). Can we classify strong quotients of ring spectra?

What is the connection to the 'homotopy normal maps' of Prasma?

Work to do relating K-theory and ideals

Now is a great time to compute examples of various R/I, $R/(I \wedge_R J)$, and $A' \odot_A^M B$.

Land-Tamme is about ring spectra, but Smith ideals work in general stable model categories. Can you prove Smith's vision regarding $E(R/(I \wedge_R J))$ for motivic spectra, equivariant spectra, chain complexes, and the stable module category?

Section 6 of White-Yau lists conjectures and open problems related to Smith O-ideal theory in: positive flat model on symmetric spectra and equivariant orthogonal spectra, positive complete model structure, global equivariant, injective model structures, and S-modules.

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