The homotopy theory of ideals structured by operads

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Motivation: Ideals of ring spectra

The fundamental object of study in stable homotopy theory is the category of spectra, in which important computations of stable homotopy groups and cohomology groups takes place.

A spectrum X is a sequence of spaces $(X_n)_{n\in\mathbb{N}}$ plus morphisms $\Sigma X_n := S^1 \wedge X_n \longrightarrow X_{n+1}$, where $\wedge$ is the smash product of spaces $A \wedge B := (A \times B)/(A \vee B)$.

Examples: $S = (S^n)_{n\in\mathbb{N}}$. Stable homotopy groups $\pi_*^s(X) = [S, X]$.

For a given space Y, $(\Sigma^n Y)_{n\in\mathbb{N}}$ is a spectrum.

For a ring R, $(HR)_n = K(R, n)$, Eilenberg-Maclane spaces

Any generalized cohomology theory gives rise to a spectrum E, by the Brown representability theorem $h^n(X) \cong [X, E_n]$.

Note: can smash spectra levelwise and unit is S, but not symmetric monoidal because S is not a commutative monoid (the twist map on $S^1 \wedge S^1$ is not the identity).

Ring spectra

Symmetric spectra have symmetric group actions on each X_n.
The category of symmetric spectra is closed symmetric monoidal

$$(X \wedge Y)_n = \bigvee_{p+q=n} \Sigma(n)_+ \wedge_{\Sigma(p)\times\Sigma(q)} (X_p \wedge Y_q)$$

$$\mathrm{Hom}(X,Y)_n = \prod_{m\geq n} \mathrm{Map}_{\Sigma(m-n)}(X_{m-n}, Y_m)$$

A ring spectrum is a monoid. Analogy: an R-algebra is a monoid in R-modules; a DGA is a monoid in $Ch(R)$.
An ideal in algebra is a subgroup $(I,+) \subset (R,+)$ such that for all $r \in R, i \in I$, the product $i \cdot r \in I$.
Problem: ring spectra don't have elements!

Ideal of ring spectra

Instead of $I \subset R$, we should have something like a monomorphism $f : I \longrightarrow R$. That is, $f \in \mathrm{Arr}(\mathrm{M})$, with some kind of algebraic structure.

If $(\mathrm{M}, \otimes, 1)$ is a closed symmetric monoidal category, then $\mathrm{Arr}(\mathrm{M})$ has two monoidal structures:

1. Tensor monoidal structure: $f \otimes g : X_0 \otimes Y_0 \longrightarrow X_1 \otimes Y_1$, unit Id_1.
2. Pushout product monoidal structure (unit $\varnothing \longrightarrow 1$):

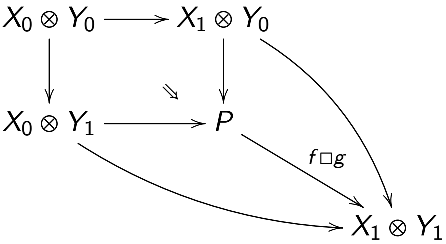
$$(X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \xrightarrow{f \square g} X_1 \otimes Y_1$$

Definition: A Smith ideal is a monoid in $\overrightarrow{\mathrm{M}}^{\square} := (\mathrm{Arr}(\mathrm{M}), \square)$

Monoidal Model Categories (implies Ho(M) monoidal)

$(\mathrm{M}, \otimes, 1)$ is a closed symmetric monoidal model category.

The pushout product of $f : X_0 \longrightarrow X_1$ and $g : Y_0 \longrightarrow Y_1$, is the corner map:



Pushout Product Axiom: If f and g are cofibrations, so is $f \square g$. If either is also a weak equivalence, so is $f \square g$.
Examples: *Set*, *Top*, *sSet*, $sMod_R$, spectra (symmetric, orthogonal, S-modules), equivariant/motivic, $Ch(R)$, StMod$(k[G])$, *Cat*, ...

Unpacking definition of Smith ideal

A Smith ideal is a monoid in $\overrightarrow{\mathrm{M}}^{\square}$. This means it's a monoid R, an R-bimodule I, and a map of R-bimodules $j : I \longrightarrow R$ such that $\mu(1 \otimes j) = \mu(j \otimes 1) : I \otimes I \longrightarrow I$. Reason: $\eta : (\varnothing \longrightarrow 1) \longrightarrow j$ and unpack $j \,\square\, j \longrightarrow j$:

$$\begin{array}{ccc} (I \otimes R) \coprod_{I \otimes I} (R \otimes I) & \longrightarrow & R \otimes R \\ \downarrow & & \downarrow \\ I & \longrightarrow & R \end{array}$$

A monoid in $\overrightarrow{\mathrm{M}}^{\otimes} := (\mathrm{Arr}(\mathrm{M}), \otimes, Id_1)$ is a monoid homomorphism.

Theorem (Hovey): The cokernel functor from $\overrightarrow{\mathrm{M}}^{\square}$ to $\overrightarrow{\mathrm{M}}^{\otimes}$ is strong symmetric monoidal $(j \mapsto (R \longrightarrow R/I))$, and right adjoint is the kernel. Goal: prove it's a Quillen equivalence.

A word on cokernel and kernel

Say M is pointed if $\varnothing \cong *$ (initial $\cong$ terminal).
Given $f : X \longrightarrow Y$, the cokernel $g : Y \longrightarrow Z$ is the pushout on the left, and the kernel $h : K \longrightarrow X$ is the pullback on the right

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow g \\
* & \longrightarrow & Z
\end{array}
\qquad\qquad
\begin{array}{ccc}
K & \longrightarrow & * \\
h \downarrow & & \downarrow \\
X & \xrightarrow[f]{} & Y
\end{array}
$$

Exercise: $\operatorname{coker}(f \,\square\, g) \cong \operatorname{coker} f \otimes \operatorname{coker} g$. Cokernel is strong symmetric monoidal so ker is lax symmetric monoidal.
Hence, there is a map $\ker f \,\square\, \ker g \longrightarrow \ker(f \otimes g)$ that's adjoint to $\operatorname{coker}(\ker f \,\square\, \ker g) \longrightarrow f \otimes g$.

Monoidal model structure for Arrow Category

The *projective model structure* on $\overrightarrow{\mathrm{M}}^{\square}$ has weak equivalences and fibrations defined levelwise. Use *injective model* on $\overrightarrow{\mathrm{M}}^{\otimes}$.

Theorem (Hovey, W.-Yau; Math Scandinavica 2018)

If M *is a monoidal model category, then so are* $\overrightarrow{\mathrm{M}}^{\square}$ *and* $\overrightarrow{\mathrm{M}}^{\otimes}$*.*

$$\begin{array}{ccc} V_0 & \xrightarrow{\alpha_0} & W_0 \\ {\scriptstyle f_V}\downarrow & & \downarrow{\scriptstyle f_W} \\ V_1 & \xrightarrow{\alpha_1} & W_1 \end{array} \qquad \begin{array}{ccc} X_0 & \xrightarrow{\beta_0} & Y_0 \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ X_1 & \xrightarrow{\beta_1} & Y_1 \end{array}$$

The pushout product in $\overrightarrow{\mathrm{M}}^{\square}$ is the map

$$(f_W \square f_X) \coprod_{f_V \square f_X} (f_V \square f_Y) \xrightarrow{\alpha \square_2 \beta} f_W \square f_Y$$

Hovey's helper results (exercises)

1. Let $L_0 : \mathrm{M} \longrightarrow \overrightarrow{\mathrm{M}}^{\otimes}$ be left adjoint to $Ev_0(j) = \mathrm{domain}(j)$, so $L_0(X) = 1_X$. Let $L_1 : \mathrm{M} \longrightarrow \overrightarrow{\mathrm{M}}^{\square}$ be left adj to Ev_1 (codomain) so $L_1(X) = (\varnothing \longrightarrow X)$. These are strict monoidal functors.
2. $\overrightarrow{\mathrm{M}}^{\square}$ and $\overrightarrow{\mathrm{M}}^{\otimes}$ satisfy the monoid axiom if M does, so monoids inherit a model structure.
3. Ev_1 is left Quillen from Smith ideals to monoids in M.
4. If M is pointed then coker: $\overrightarrow{\mathrm{M}}^{\square} \longrightarrow \overrightarrow{\mathrm{M}}^{\otimes}$ lifts to a Quillen functor from Smith ideals to monoid homomorphisms.
5. If M is a stable model category (and unit is cofibrant) then this is a Quillen equivalence. Need cofibrant monoid to forget to cofibrant object for the proof that Q.E. lifts to monoids.

How to prove Hovey's Quillen equivalence

If $\alpha : f \longrightarrow g$ is a (trivial) cofibration in $\overrightarrow{\mathrm{M}}^{\square}$ then induced map of colimits $\operatorname{coker} f \longrightarrow \operatorname{coker} g$ is a (trivial) cofibration.

If M is stable, f cofibrant in $\overrightarrow{\mathrm{M}}^{\square}$, p fibrant in $\overrightarrow{\mathrm{M}}^{\otimes}$, let $\alpha : \operatorname{coker} f \longrightarrow p$ and $\beta : f \longrightarrow \ker p$:

$$\begin{array}{ccc} B & \xrightarrow{g} & \operatorname{coker} f \\ {\scriptstyle \alpha_0}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\ X & \xrightarrow[p]{} & Y \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \beta_0}\downarrow & & \downarrow{\scriptstyle \alpha_0} \\ \ker p & \xrightarrow[q]{} & X \end{array}$$

Then α is a weak equivalence iff β is, using $A \longrightarrow B \longrightarrow \operatorname{coker} f \longrightarrow \Sigma A$ and $\Omega Y \longrightarrow \ker p \longrightarrow X \longrightarrow Y$ and every fiber sequence is isomorphic to a cofiber sequence. So: $\ker p \longrightarrow X \longrightarrow Y \longrightarrow \Sigma \ker p$ and use two out of three.

Our setup (W.-Yau)

Note: monoid morphisms are algebras over a 2-colored operad. Smith ideals are too. Generalize from *Ass* to operad O?

1. Goal: homotopy theory of ideals structured by an operad O, e.g., commutative ideals ($Ev_1(j)$ is a commutative monoid), A_∞-ideals, E_∞-ideals, E_n, Lie, L_∞, etc.
2. Given O, define $\overrightarrow{O}^{\otimes} = L_0 O$ (resp. $\overrightarrow{O}^{\square} = L_1 O$), C-colored operad in $\overrightarrow{\mathrm{M}}^{\otimes}$ (resp. $\overrightarrow{\mathrm{M}}^{\square}$).
3. A Smith O-ideal is an algebra over $\overrightarrow{O}^{\square}$; a morphism of O-algebras is an algebra over $\overrightarrow{O}^{\otimes}$.
4. There is a $(C \coprod C)$-colored operad O^s in M such that $\mathrm{Alg}(\overrightarrow{O}^{\square}; \overrightarrow{\mathrm{M}}^{\square}) \cong \mathrm{Alg}(O^s; \mathrm{M})$.
5. coker induces an adjunction $\mathrm{Alg}(\overrightarrow{O}^{\square}; \overrightarrow{\mathrm{M}}^{\square}) \leftrightarrows \mathrm{Alg}(\overrightarrow{O}^{\otimes}; \overrightarrow{\mathrm{M}}^{\otimes})$

Unpacking Smith O-ideal

Proposition (W.-Yau)

A Smith O-ideal in M *is precisely:*

- *an O-algebra* (A, λ_1) *in* M,
- *an A-bimodule* (X, λ_0) *in* M, *and*
- *an A-bimodule map* $f : (X, \lambda_0) \longrightarrow (A, \lambda_1)$

such that, for $1 \le i < j \le n$*, the following commutes*

$$\begin{array}{ccc}
O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{i-1}} X_{c_i} A_{c_{i+1}} \cdots X_{c_j} \cdots A_{c_n} & \xrightarrow{(\mathrm{Id}, f_{c_j}, \mathrm{Id})} & O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{i-1}} X_{c_i} A_{c_{i+1}} \cdots A_{c_n} \\
\Big\downarrow {\scriptstyle (\mathrm{Id}, f_{c_i}, \mathrm{Id})} & & \Big\downarrow {\scriptstyle \lambda_0} \\
O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{j-1}} X_{c_j} A_{c_{j+1}} \cdots A_{c_n} & \xrightarrow{\lambda_0} & X_d
\end{array}$$

What is this O^s?

Given a C-colored operad O, denote by C^0 (resp. C^1) the first (resp. second) copies of $C \cup C$. Given $c \in C$, write $c^\epsilon \in C^\epsilon$ for the same c in each copy, for $\epsilon \in \{0,1\}$. Define:

$$O^s\binom{d^1}{c_1^{\epsilon_1},\ldots,c_n^{\epsilon_n}} = O\binom{d}{\underline{c}}$$

$$O^s\binom{d^0}{c_1^{\epsilon_1},\ldots,c_n^{\epsilon_n}} = \begin{cases} O\binom{d}{\underline{c}} & \text{if at least one } \epsilon_i = 0 \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

So, an O^s-algebra is a pair (A, X) of C-colored objects, plus structure maps making A into an O-algebra, X into an A-bimodule, and $f : X \longrightarrow A$ into an A-bimodule map.
This is similar to the two-colored operad for monoid maps.

Main theorem

Theorem (W.-Yau)

If M *is nice, and cofibrant Smith O-ideals are also entrywise cofibrant in* $\overrightarrow{\mathrm{M}}^{\square}$ *then there is a Quillen equivalence*

$$\{\textit{Smith O-Ideals}\} \underset{\text{ker}}{\overset{\text{coker}}{\rightleftarrows}} \{\textit{O-Algebra Maps}\}$$

For Σ-cofibrant O, just need M stable, monoidal, cof gen.
For $O = Com$, M needs strong commutative monoid axiom.
For general O, need good behavior of $X \otimes_{\Sigma_n} (-)^{\square n}$ and $f \square_{\Sigma_n} (-) : \mathrm{M}^{\Sigma_n} \longrightarrow \mathrm{M}$
Examples: symmetric spectra, $Ch(k)$, StMod($k[G]$), motivic, equivariant orthogonal spectra, enriched functors, S-modules, etc.

Comparison with ∞-operads

Theorem (W.-Yau)

If M is cof. gen., $\mathrm{M}^\flat \subset \mathrm{M}$, and $\mathcal{O}$ is Σ_C-cofibrant (symmetric) C-colored operad.

- *Denote by $\mathrm{Alg}(\mathcal{O};\mathrm{M})^c[W_{\mathcal{O}}^{-1}]$ the ∞-category obtained from the semi-model category $\mathrm{Alg}(\mathcal{O};\mathrm{M})$, by first passing to the subcategory of cofibrant objects, and then inverting the weak equivalences between $\mathcal{O}$-algebras.*
- *Denote by $\mathrm{Alg}(\mathcal{O};\mathrm{M}[W^{-1}])$ the ∞-category obtained by first passing from M to the (symmetric) monoidal category $\mathrm{M}[W^{-1}]$ and then passing to $\mathcal{O}$-algebras, where $\mathcal{O}$ is viewed as a colored operad in $\mathrm{M}[W^{-1}] \simeq \mathrm{M}^\flat[W^{-1}]$.*

Then $\mathrm{Alg}(\mathcal{O};\mathrm{M})^c[W_{\mathcal{O}}^{-1}] \simeq \mathrm{Alg}(\mathcal{O};\mathrm{M}[W^{-1}])$ as ∞-categories.

More details

Definition (Haugseng)

A *subcategory of flat objects* is a full symmetric monoidal subcategory $\mathrm{M}^\flat$ s.t.:

1. All cofibrant objects are flat (i.e., in $\mathrm{M}^\flat$).
2. If X is flat and f is w.e. in $\mathrm{M}^\flat$, then $X \otimes f$ is w.e.

Proposition (W.-Yau)

Suppose M *is a cofibrantly generated monoidal model category and* O *is a* Σ*-cofibrant* C*-colored operad valued in* M*. Then the forgetful functor* $U : \mathrm{Alg}(O; \mathrm{M}) \longrightarrow \mathrm{M}^C$ *preserves and reflects homotopy sifted colimits.*

Cor: story works for $\overrightarrow{\mathrm{M}}^\square, \overrightarrow{\mathrm{M}}^\otimes$. So $\mathrm{Alg}\left(\overrightarrow{O}^\otimes; \overrightarrow{\mathrm{M}}^\otimes\right) \simeq \mathrm{Alg}\left(\overrightarrow{O}^\square; \overrightarrow{\mathrm{M}}^\square\right)$

Open Problems

Almost every question you can ask, e.g., tensor prod of ideals? What is the relationship between ideals of $\pi_*(R)$ and ideals of ring spectra? If $R = S$, the sphere spectrum, and $2 \in \pi_0 S$ is the cofiber of the 'times 2' map, then (2) is an ideal of $\pi_* S$ but the mod 2 Moore spectrum is not a ring spectrum, even up to homotopy. So what is the ring spectrum quotient of S by 2?

Let $f : I \longrightarrow R$ be any map. What is the Smith ideal generated by f? The free functor T yields an ideal of $T(R)$ not R.

Every ring spectrum is weakly equivalent to a quotient of the sphere spectrum by some Smith ideal. Define a monoid homomorphism $p : R \longrightarrow S$ to be a strong quotient if $S \otimes_R QN \longrightarrow N$ is a w.e. for all fibrant N (and cof. rep. Q). Can we classify strong quotients of ring spectra?

What is the connection to the 'homotopy normal maps' of Prasma?

Connection to algebraic K-theory

Suppose that R is a ring spectrum with Smith ideals I and J. Define the Smith ideal $I \wedge_R J$. Let T be the homotopy pushout of $R/I \leftarrow R \longrightarrow R/J$ in the category of E_1-algebras. There is a fiber sequence of algebraic K-theory spectra: $K(R/(I \wedge_R J)) \longrightarrow K(R/I) \otimes K(R/J) \longrightarrow K(T)$. This was Smith's original motivation, recently proven by Land-Tamme, 2023. Their $\widetilde{R}$ is $R/(I \wedge_R J)$ where $I = fib(R \longrightarrow R')$ and $J = fib(R \longrightarrow S)$ are ideals. They compute $K(R \longrightarrow R/(I \wedge_R J))$ and prove $T \cong R/I \odot_R^M R/J$, the $\odot$-ring from their Annals paper, for $M = (R/I) \wedge_R (R/J)$. Their work applies to any localizing invariant, not just K-theory. They recover results of Waldhausen on K-theory of pushouts of group rings, and of Burghelea (1985) for periodic cyclic homology, plus much more.

Operad structure matters: in E_∞ context, $A' \odot_A^M B \simeq B \odot_A^M A'$.

Work to do relating K-theory and ideals

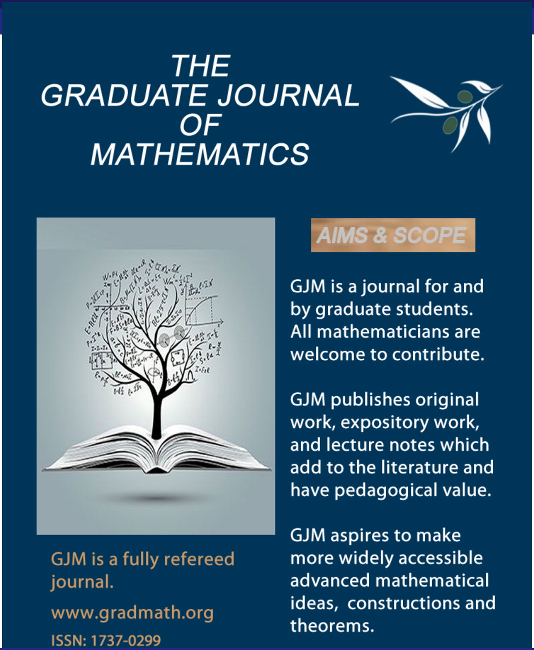
Now is a great time to compute examples of various R/I, $R/(I \wedge_R J)$, and $A' \odot_A^M B$.

Land-Tamme is about ring spectra, but Smith ideals work in general stable model categories. Can you prove Smith's vision regarding $E(R/(I \wedge_R J))$ for motivic spectra, equivariant spectra, chain complexes, and the stable module category?

Section 6 of White-Yau lists conjectures and open problems related to Smith O-ideal theory in: positive flat model on symmetric spectra and equivariant orthogonal spectra, positive complete model structure, global equivariant, injective model structures, and S-modules.

References

- Hovey: Smith ideals of structured ring spectra, arXiv:1401.2850.
- White-Yau: Arrow Categories of Monoidal Model Categories, arXiv:1703.05359, Mathematica Scandinavica 2018.
- White-Yau: Smith Ideals of Operadic Algebras in Monoidal Model Categories, arXiv:1703.05377, AGT 2024.
- White: Model Structures on Commutative Monoids in General Model Categories, arXiv:1403.6759, JPAA 2017.
- White-Yau: Bousfield localizations and algebras over colored operads, arXiv:1503.06720, ACS 2018.
- White-Yau: Homotopical adjoint lifting theorem, arXiv:1606.01803, ACS 2019.
- Haugseng: Algebras for enriched ∞-operads, arXiv:1909.10042.
- Pavlov-Scholbach: Homotopy theory of symmetric powers, HHA 2018.
- Mao: Revisiting derived crystalline cohomology, arXiv:2107.0292.
- Bruner-Isaksen: Jeff Smith's Theory of Ideals, arXiv:2208.07941.
- Land-Tamme: On the K-theory of pushouts, arXiv:2304.12812.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Q & A: Transferring Model Structures

Lemma (well-known)

If O is a C-colored operad in M *and if for all generating trivial cofibrations $j : K \longrightarrow L$ in* M*, transfinite compositions of pushouts in $Alg_O(\mathrm{M})$:*

$$\begin{array}{ccc} F(K) & \longrightarrow & F(L) \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

are weak equivalences then $Alg_O(\mathrm{M})$ has transferred model structure, with weak equivalences and fibrations defined in M.

If above works only for A cofibrant then get transferred semi-model structure. Note: $A \longrightarrow B$ is transfinite composition of pushouts of $O_A\binom{d}{\underline{c}} \otimes_{\Sigma_n} j^{\square n}$. If O is Σ-cof, A cof, then O_A is Σ-cof.

Q & A: Proof that Arr(M) has pushout product axiom

We'll focus on $\overrightarrow{\mathrm{M}}^{\square}$. To get $\mathrm{M}^{I^{\times n}}$ and $\mathrm{M}^{\square^{op}}$, iterate.

To save space, write $W_1 X_0$ for $W_1 \otimes X_0$, etc. Let $f_V : V_0 \longrightarrow V_1$, and f_W, f_X, f_Y similarly.

If $\alpha : f_V \longrightarrow f_W$ and $\beta : f_X \longrightarrow f_Y$, then $\alpha \square_2 \beta$ is the following commutative square in M:

$$\begin{array}{ccc}
\left(W_1X_0 \coprod_{W_0X_0} W_0X_1\right) \coprod_{\left(V_1X_0 \coprod_{V_0X_0} V_0X_1\right)} \left(V_1Y_0 \coprod_{V_0Y_0} V_0Y_1\right) & \xrightarrow{\zeta} & W_1Y_0 \coprod_{W_0Y_0} W_0Y_1 \\
{\scriptstyle (f_W\square f_X) \coprod_{f_V\square f_X} (f_V\square f_Y)} \Big\downarrow & & \Big\downarrow {\scriptstyle f_W\square f_Y} \\
W_1X_1 \coprod_{V_1X_1} V_1Y_1 & \xrightarrow{\alpha_1\square\beta_1} & W_1Y_1
\end{array}$$

Proof (cont)

Lemma (Hovey): In $\overrightarrow{\mathrm{M}}^{\square}$, γ from $f : X_0 \longrightarrow X_1$ to $g : Y_0 \longrightarrow Y_1$ is a (trivial) cofibration iff γ_0 and $\gamma_1 \circledast g : X_1 \coprod_{X_0} Y_0 \longrightarrow Y_1$ are.

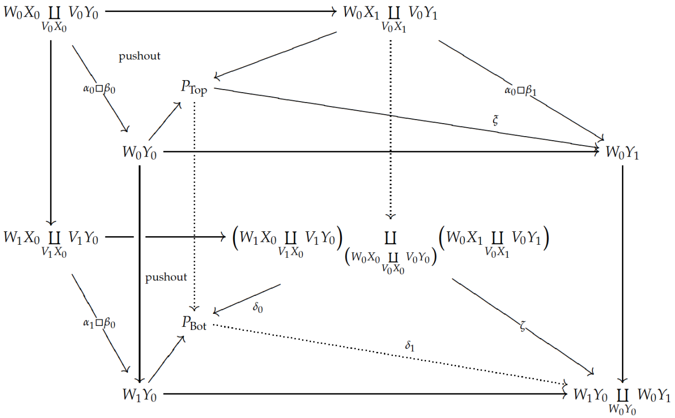
Assume α is a cofibration and β is a (trivial) cofibration in $\overrightarrow{\mathrm{M}}^{\square}$

We must prove ζ is a (trivial) cofibration and the pushout corner map

$$\left(W_1 X_1 \coprod_{V_1 X_1} V_1 Y_1\right) \coprod_{Z} \left(W_1 Y_0 \coprod_{W_0 Y_0} W_0 Y_1\right) \xrightarrow{(\alpha_1 \square \beta_1) \circledast (f_W \square f_Y)} W_1 Y_1$$

is a (trivial) cofibration.

Proof (cont)

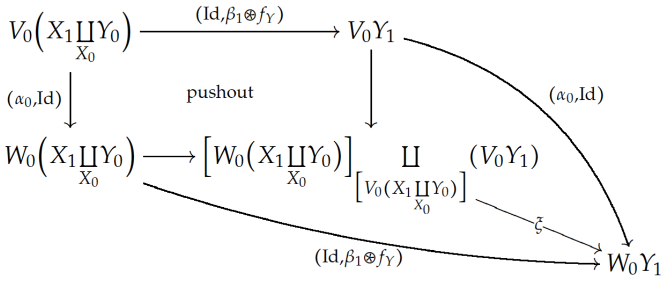


$W_0X_0 \coprod_{V_0X_0} V_0Y_0$
$W_0X_1 \coprod_{V_0X_1} V_0Y_1$
pushout
$\alpha_0\Box\beta_0$
P_{Top}
$\alpha_0\Box\beta_1$
ξ
W_0Y_0
W_0Y_1
$W_1X_0 \coprod_{V_1X_0} V_1Y_0$
$\left(W_1X_0 \coprod_{V_1X_0} V_1Y_0\right) \coprod_{\left(W_0X_0 \coprod_{V_0X_0} V_0Y_0\right)} \left(W_0X_1 \coprod_{V_0X_1} V_0Y_1\right)$
pushout
δ_0
$\alpha_1\Box\beta_0$
P_{Bot}
δ_1
ζ
W_1Y_0
$W_1Y_0 \coprod_{W_0Y_0} W_0Y_1$

Proof (cont)

$\zeta = \delta_1 \circ \delta_0$, and δ_0 is a pushout of $\alpha_1 \square \beta_0$ so is a (trivial) cofibration.

δ_1 is a pushout of ξ, which we rewrite as the pushout product $\alpha_0 \square (\beta_1 \circledast f_Y)$, below, so both are (trivial) cofibrations.



Proof (cont)

To finish, rewrite

$$\left(W_1X_1 \coprod_{V_1X_1} V_1Y_1\right) \coprod_Z \left(W_1Y_0 \coprod_{W_0Y_0} W_0Y_1\right) \xrightarrow{(\alpha_1 \square \beta_1) \circledast (f_W \square f_Y)} W_1Y_1$$

as:

$$\begin{array}{ccc}
\left(W_1X_1 \coprod\limits_{V_1X_1} V_1Y_1\right) \coprod\limits_Z \left(W_1Y_0 \coprod\limits_{W_0Y_0} W_0Y_1\right) & \xrightarrow{(\alpha_1 \square \beta_1) \circledast (f_W \square f_Y)} & W_1Y_1 \\
\downarrow \cong & & \downarrow = \\
W_1\left(X_1 \coprod\limits_{X_0} Y_0\right) \coprod\limits_{\left(V_1 \coprod\limits_{V_0} W_0\right)\left(X_1 \coprod\limits_{X_0} Y_0\right)} \left(V_1 \coprod\limits_{V_0} W_0\right)Y_1 & \xrightarrow{(\alpha_1 \circledast f_W) \square (\beta_1 \circledast f_Y)} & W_1Y_1
\end{array}$$

Recap

Theorem (W.-Yau; arXiv:1703.05359; Math Scandinavica 2018)

If M *is a monoidal model category, then so are* $\overrightarrow{\mathrm{M}}^{\square}$, $\mathrm{M}^{I^{\times n}}$, *and* $\mathrm{M}^{\square^{op}}$.

Lemma (Hovey): In $\overrightarrow{\mathrm{M}}^{\square}$, γ from $f : X_0 \longrightarrow X_1$ to $g : Y_0 \longrightarrow Y_1$ is a (trivial) cofibration iff γ_0 and $\gamma_1 \circledast g : X_1 \coprod_{X_0} Y_0 \longrightarrow Y_1$ are.

If α is cof and β is (triv) cof, then let $\gamma = \alpha \,\square_2\, \beta$.

We proved $\gamma_0 = \zeta$ and $\gamma_1 \circledast g$ from previous slide, are (triv) cof's.