A diagrammatic approach to the homotopy theory of ideals in a monoidal model category

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Motivation: Ideals of ring spectra

The fundamental object of study in stable homotopy theory is the category of spectra, in which important computations of stable homotopy groups and cohomology groups takes place.

A spectrum X is a sequence of spaces $(X_n)_{n\in\mathbb{N}}$ plus morphisms $\Sigma X_n := S^1 \wedge X_n \longrightarrow X_{n+1}$, where $\wedge$ is the smash product of spaces $A \wedge B := (A \times B)/(A \vee B)$. Symmetric spectra have symmetric group actions on each X_n.

Examples: $X = (S^n)_{n\in\mathbb{N}}$.

For a given space Y, $(\Sigma^n Y)_{n\in\mathbb{N}}$ is a spectrum.

For a ring R, $(HR)_n = K(R, n)$, Eilenberg-Maclane spaces

Any generalized cohomology theory gives rise to a spectrum E, by the Brown representability theorem $h^n(X) \cong [X, E_n]$.

Ring spectra

The category of symmetric spectra is closed symmetric monoidal

$$(X \wedge Y)_n = \bigvee_{p+q=n} \Sigma(n)_+ \wedge_{\Sigma(p)\times\Sigma(q)} (X_p \wedge Y_q)$$

$$\mathrm{Hom}(X,Y)_n = \prod_{m\geq n} \mathrm{Map}_{\Sigma(m-n)}(X_{m-n}, Y_m)$$

A ring spectrum is a monoid. Analogy: an R-algebra is a monoid in R-modules; a DGA is a monoid in $Ch(R)$.
An ideal in algebra is a subgroup $(I,+) \subset (R,+)$ such that for all $r \in R, i \in I$, the product $i \cdot r \in I$.
Problem: ring spectra don't have elements!

Ideal of ring spectra

Instead of $I \subset R$, we should have something like a monomorphism $f : I \longrightarrow R$. That is, $f \in \mathrm{Arr}(\mathrm{M})$, with some kind of algebraic structure.

If $(\mathrm{M}, \otimes, 1)$ is a closed symmetric monoidal category, then $\mathrm{Arr}(\mathrm{M})$ has two monoidal structures:

1. Tensor monoidal structure: $f \otimes g : X_0 \otimes Y_0 \longrightarrow X_1 \otimes Y_1$, unit Id_1.
2. Pushout product monoidal structure (unit $\varnothing \longrightarrow 1$):

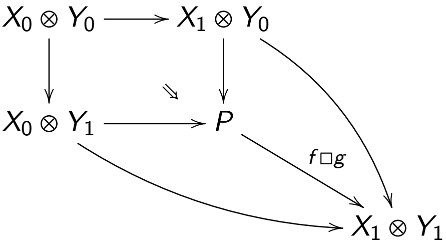
$$(X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \xrightarrow{f \square g} X_1 \otimes Y_1$$

Definition: A Smith ideal is a monoid in $\overrightarrow{\mathrm{M}}^{\square} := (\mathrm{Arr}(\mathrm{M}), \square)$

Monoidal Model Categories (implies Ho(M) monoidal)

$(\mathrm{M}, \otimes, 1)$ is a closed symmetric monoidal model category.

The pushout product of $f : X_0 \longrightarrow X_1$ and $g : Y_0 \longrightarrow Y_1$, is the corner map:



Pushout Product Axiom: If f and g are cofibrations, so is $f \square g$. If either is also a weak equivalence, so is $f \square g$.
Examples: *Set*, *Top*, *sSet*, $sMod_R$, spectra (symmetric, orthogonal, S-modules), equivariant/motivic, $Ch(R)$, $\mathrm{StMod}(k[G])$, *Cat*, ...

Unpacking definition of Smith ideal

A Smith ideal is a monoid in $\overrightarrow{\mathrm{M}}^{\square}$. This means it's a monoid R, an R-bimodule I, and a map of R-bimodules $j : I \longrightarrow R$ such that $\mu(1 \otimes j) = \mu(j \otimes 1) : I \otimes I \longrightarrow I$. Reason: $\eta : (\varnothing \longrightarrow 1) \longrightarrow j$ and unpack $j \square j \longrightarrow j$:

$$\begin{array}{ccc} (I \otimes R) \coprod_{I \otimes I} (R \otimes I) & \longrightarrow & R \otimes R \\ \downarrow & & \downarrow \\ I & \longrightarrow & R \end{array}$$

A monoid in $\overrightarrow{\mathrm{M}}^{\otimes} := (\mathrm{Arr}(\mathrm{M}), \otimes, Id_1)$ is a monoid homomorphism.

Theorem (Hovey): The cokernel functor from $\overrightarrow{\mathrm{M}}^{\square}$ to $\overrightarrow{\mathrm{M}}^{\otimes}$ is strong symmetric monoidal ($j \mapsto (R \longrightarrow R/I)$), and right adjoint is the kernel. Goal: prove it's a Quillen equivalence.

Monoidal model structure for Arrow Category

The *projective model structure* on $\overrightarrow{\mathrm{M}}^{\square}$ has weak equivalences and fibrations defined levelwise. Use *injective model* on $\overrightarrow{\mathrm{M}}^{\otimes}$.

Theorem (W.-Yau; arXiv:1703.05359; Math Scandinavica 2018)

If M *is a monoidal model category, then so is* $\overrightarrow{\mathrm{M}}^{\square}$*.*

$$\begin{array}{ccc} V_0 & \xrightarrow{\alpha_0} & W_0 \\ {\scriptstyle f_V}\downarrow & & \downarrow{\scriptstyle f_W} \\ V_1 & \xrightarrow{\alpha_1} & W_1 \end{array} \qquad \begin{array}{ccc} X_0 & \xrightarrow{\beta_0} & Y_0 \\ {\scriptstyle f_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ X_1 & \xrightarrow{\beta_1} & Y_1 \end{array}$$

The pushout product in $\overrightarrow{\mathrm{M}}^{\square}$ is the map

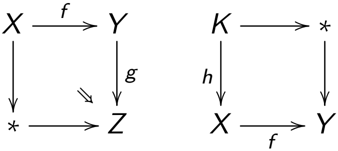
$$(f_W \square f_X) \coprod_{f_V \square f_X} (f_V \square f_Y) \xrightarrow{\alpha \square_2 \beta} f_W \square f_Y$$

Hovey's results

1. Let $L_0 : \mathsf{M} \longrightarrow \overrightarrow{\mathsf{M}}^{\otimes}$ be left adjoint to Ev_0, so $L_0(X) = 1_X$. Let $L_1 : \mathsf{M} \longrightarrow \overrightarrow{\mathsf{M}}^{\square}$ be left adj to Ev_1 so $L_1(X) = (\varnothing \longrightarrow X)$. These are strict monoidal functors.
2. $\overrightarrow{\mathsf{M}}^{\square}$ and $\overrightarrow{\mathsf{M}}^{\otimes}$ satisfy the monoid axiom if M does, so monoids inherit a model structure.
3. Ev_1 is left Quillen from Smith ideals to monoids in M.
4. If M is pointed then coker: $\overrightarrow{\mathsf{M}}^{\square} \longrightarrow \overrightarrow{\mathsf{M}}^{\otimes}$ lifts to a Quillen functor from Smith ideals to monoid homomorphisms.
5. If M is a stable model category (and unit is cofibrant) then this is a Quillen equivalence. Need cofibrant monoid to forget to cofibrant object for the proof.

A word on cokernel and kernel

Say M is pointed if $\varnothing \cong *$ (initial $\cong$ terminal).
Given $f : X \longrightarrow Y$, the cokernel $g : Y \longrightarrow Z$ is the pushout on the left, and the kernel $h : K \longrightarrow X$ is the pullback on the right



Exercise: $\operatorname{coker}(f \,\square\, g) \cong \operatorname{coker} f \otimes \operatorname{coker} g$. Cokernel is strong symmetric monoidal so ker is lax symmetric monoidal.
Hence, there is a map $\ker f \,\square\, \ker g \longrightarrow \ker(f \otimes g)$ that's adjoint to $\operatorname{coker}(\ker f \,\square\, \ker g) \longrightarrow f \otimes g$.

A word on Quillen equivalences

If $\alpha : f \longrightarrow g$ is a (trivial) cofibration in $\overrightarrow{\mathrm{M}}^{\square}$ then induced map of colimits $\operatorname{coker} f \longrightarrow \operatorname{coker} g$ is a (trivial) cofibration.

If M is stable, f cofibrant in $\overrightarrow{\mathrm{M}}^{\square}$, p fibrant in $\overrightarrow{\mathrm{M}}^{\otimes}$, let $\alpha : \operatorname{coker} f \longrightarrow p$ and $\beta : f \longrightarrow \ker p$:

$$
\begin{array}{ccc}
B & \xrightarrow{g} & \operatorname{coker} f \\
{\scriptstyle \alpha_0}\downarrow & & \downarrow{\scriptstyle \alpha_1} \\
X & \xrightarrow[p]{} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \beta_0}\downarrow & & \downarrow{\scriptstyle \alpha_0} \\
\ker p & \xrightarrow[q]{} & X
\end{array}
$$

Then α is a weak equivalence iff β is, using $A \longrightarrow B \longrightarrow \operatorname{coker} f \longrightarrow \Sigma A$ and $\Omega Y \longrightarrow \ker p \longrightarrow X \longrightarrow Y$ and every fiber sequence is isomorphic to a cofiber sequence. So: $\ker p \longrightarrow X \longrightarrow Y \longrightarrow \Sigma \ker p$ and use two out of three.

Our setup (W.-Yau)

Note: monoid morphisms are algebras over a 2-colored operad. Smith ideals are too. Generalize from *Ass* to operad O?

1. Goal: homotopy theory of ideals structured by an operad O, e.g., commutative ideals ($Ev_1(j)$ is a commutative monoid), A_∞-ideals, E_∞-ideals, E_n, Lie, L_∞, etc.
2. Given O, define $\overrightarrow{O}^{\otimes} = L_0 O$ (resp. $\overrightarrow{O}^{\square} = L_1 O$), C-colored operad in $\overrightarrow{\mathsf{M}}^{\otimes}$ (resp. $\overrightarrow{\mathsf{M}}^{\square}$).
3. A Smith O-ideal is an algebra over $\overrightarrow{O}^{\square}$; a morphism of O-algebras is an algebra over $\overrightarrow{O}^{\otimes}$.
4. There is a $(C \coprod C)$-colored operad O^s in M such that $\mathrm{Alg}(\overrightarrow{O}^{\square}; \overrightarrow{\mathsf{M}}^{\square}) \cong \mathrm{Alg}(O^s; \mathsf{M})$.
5. coker induces an adjunction $\mathrm{Alg}(\overrightarrow{O}^{\square}; \overrightarrow{\mathsf{M}}^{\square}) \leftrightarrows \mathrm{Alg}(\overrightarrow{O}^{\otimes}; \overrightarrow{\mathsf{M}}^{\otimes})$

Unpacking Smith O-ideal

Proposition (W.-Yau)

A Smith O-ideal in M *is precisely:*

- *an O-algebra* (A, λ_1) *in* M,
- *an A-bimodule* (X, λ_0) *in* M, *and*
- *an A-bimodule map* $f : (X, \lambda_0) \longrightarrow (A, \lambda_1)$

such that, for $1 \le i < j \le n$, *the following commutes*

$$\begin{array}{ccc}
O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{i-1}} X_{c_i} A_{c_{i+1}} \cdots X_{c_j} \cdots A_{c_n} & \xrightarrow{(\mathrm{Id}, f_{c_j}, \mathrm{Id})} & O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{i-1}} X_{c_i} A_{c_{i+1}} \cdots A_{c_n} \\
\Big\downarrow {\scriptstyle (\mathrm{Id}, f_{c_i}, \mathrm{Id})} & & \Big\downarrow {\scriptstyle \lambda_0} \\
O\binom{d}{\underline{c}} \otimes A_{c_1} \cdots A_{c_{j-1}} X_{c_j} A_{c_{j+1}} \cdots A_{c_n} & \xrightarrow{\lambda_0} & X_d
\end{array}$$

Main theorem

Theorem (W.-Yau)

If M *is nice, and cofibrant Smith O-ideals are also entrywise cofibrant in* $\overrightarrow{\mathsf{M}}^{\square}$ *then there is a Quillen equivalence*

$$\{\textit{Smith O-Ideals}\} \underset{\text{ker}}{\overset{\text{coker}}{\rightleftarrows}} \{\textit{O-Algebra Maps}\}$$

For Σ-cofibrant O, just need M stable, monoidal, cof gen.
For $O = Com$, M needs strong commutative monoid axiom.
For general O, need good behavior of $X \otimes_{\Sigma_n} (-)^{\square n}$ and $f \square_{\Sigma_n} (-) : \mathsf{M}^{\Sigma_n} \longrightarrow \mathsf{M}$
Examples: symmetric spectra, $Ch(k)$, StMod$(k[G])$ (note: if $char(k) || G|$ then works for all O)

Comparison with ∞-operads

Theorem (W.-Yau)

If M is cof. gen., $\mathsf{M}^\flat \subset \mathsf{M}$, and O is Σ_C-cofibrant (symmetric) C-colored operad.

- *Denote by $\mathsf{Alg}(O;\mathsf{M})^c[W_O^{-1}]$ the ∞-category obtained from the semi-model category $\mathsf{Alg}(O;\mathsf{M})$, by first passing to the subcategory of cofibrant objects, and then inverting the weak equivalences between O-algebras.*
- *Denote by $\mathsf{Alg}(O;\mathsf{M}[W^{-1}])$ the ∞-category obtained by first passing from M to the (symmetric) monoidal category $\mathsf{M}[W^{-1}]$ and then passing to O-algebras, where O is viewed as a colored operad in $\mathsf{M}[W^{-1}] \simeq \mathsf{M}^\flat[W^{-1}]$.*

Then $\mathsf{Alg}(O;\mathsf{M})^c[W_O^{-1}] \simeq \mathsf{Alg}(O;\mathsf{M}[W^{-1}])$ as ∞-categories.

More details

Definition (Haugseng)

A *subcategory of flat objects* is a full symmetric monoidal subcategory $\mathsf{M}^\flat$ s.t.:

1. All cofibrant objects are flat (i.e., in $\mathsf{M}^\flat$).
2. If X is flat and f is w.e. in $\mathsf{M}^\flat$, then $X \otimes f$ is w.e.

Proposition (W.-Yau)

Suppose M *is a cofibrantly generated monoidal model category and* O *is a* Σ*-cofibrant* C*-colored operad valued in* M*. Then the forgetful functor* $U : \mathrm{Alg}(O;\mathsf{M}) \longrightarrow \mathsf{M}^C$ *preserves and reflects homotopy sifted colimits.*

Cor: story works for $\overrightarrow{\mathsf{M}}^\square, \overrightarrow{\mathsf{M}}^\otimes$. So $\mathrm{Alg}(\overrightarrow{O}^\otimes; \overrightarrow{\mathsf{M}}^\otimes) \simeq \mathrm{Alg}(\overrightarrow{O}^\square; \overrightarrow{\mathsf{M}}^\square)$

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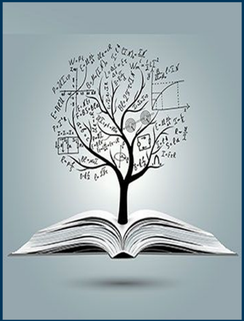

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Transferring Model Structures

Lemma (well-known)

If O is a C-colored operad in M and if for all generating trivial cofibrations $j : K \longrightarrow L$ in M, transfinite compositions of pushouts in $Alg_O(\mathrm{M})$:

$$\begin{array}{ccc} F(K) & \longrightarrow & F(L) \\ \downarrow & & \downarrow \\ & \searrow & \\ A & \longrightarrow & B \end{array}$$

are weak equivalences then $Alg_O(\mathrm{M})$ has transferred model structure, with weak equivalences and fibrations defined in M.

If above works only for A cofibrant then get transferred semi-model structure. Note: $A \longrightarrow B$ is transfinite composition of pushouts of maps $O_A\binom{d}{\underline{c}} \otimes_{\Sigma_n} j^{\square n}$. If O is Σ-cof, A cof, then O_A is Σ-cof.

Proof that Arr(M) has pushout product axiom

We'll focus on $\overrightarrow{\mathsf{M}}^{\square}$. To get $\mathsf{M}^{I^{\times n}}$ and $\mathsf{M}^{\square^{op}}$, iterate.

To save space, write W_1X_0 for $W_1 \otimes X_0$, etc. Let $f_V : V_0 \longrightarrow V_1$, and f_W, f_X, f_Y similarly.

If $\alpha : f_V \longrightarrow f_W$ and $\beta : f_X \longrightarrow f_Y$, then $\alpha \,\square_2\, \beta$ is the following commutative square in M:

$$\begin{array}{ccc}
\left(W_1X_0 \coprod_{W_0X_0} W_0X_1\right) \coprod_{\left(V_1X_0 \coprod_{V_0X_0} V_0X_1\right)} \left(V_1Y_0 \coprod_{V_0Y_0} V_0Y_1\right) & \xrightarrow{\zeta} & W_1Y_0 \coprod_{W_0Y_0} W_0Y_1 \\
\Big\downarrow{\scriptstyle (f_W\square f_X) \coprod_{f_V\square f_X} (f_V\square f_Y)} & & \Big\downarrow{\scriptstyle f_W\square f_Y} \\
W_1X_1 \coprod_{V_1X_1} V_1Y_1 & \xrightarrow{\alpha_1\square\beta_1} & W_1Y_1
\end{array}$$

Proof (cont)

Lemma (Hovey): In $\overrightarrow{\mathrm{M}}^{\square}$, γ from $f : X_0 \longrightarrow X_1$ to $g : Y_0 \longrightarrow Y_1$ is a (trivial) cofibration iff γ_0 and $\gamma_1 \circledast g : X_1 \coprod_{X_0} Y_0 \longrightarrow Y_1$ are.

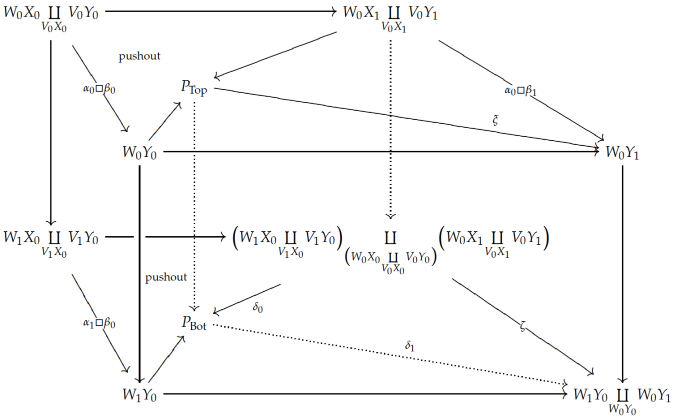
Assume α is a cofibration and β is a (trivial) cofibration in $\overrightarrow{\mathrm{M}}^{\square}$

We must prove ζ is a (trivial) cofibration and the pushout corner map

$$\Big(W_1 X_1 \coprod_{V_1 X_1} V_1 Y_1 \Big) \coprod_{Z} \Big(W_1 Y_0 \coprod_{W_0 Y_0} W_0 Y_1 \Big) \xrightarrow{(\alpha_1 \square \beta_1) \circledast (f_W \square f_Y)} W_1 Y_1$$

is a (trivial) cofibration.

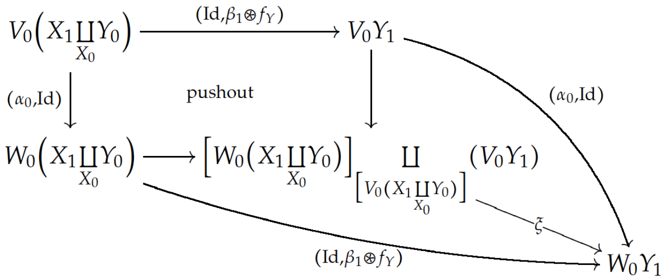
Proof (cont)


$W_0X_0 \coprod_{V_0X_0} V_0Y_0$
$W_0X_1 \coprod_{V_0X_1} V_0Y_1$
pushout
$\alpha_0 \square \beta_0$
P_{Top}
$\alpha_0 \square \beta_1$
ξ
W_0Y_0
W_0Y_1
$W_1X_0 \coprod_{V_1X_0} V_1Y_0$
$\left(W_1X_0 \coprod_{V_1X_0} V_1Y_0\right) \coprod_{\left(W_0X_0 \coprod_{V_0X_0} V_0Y_0\right)} \left(W_0X_1 \coprod_{V_0X_1} V_0Y_1\right)$
pushout
δ_0
$\alpha_1 \square \beta_0$
P_{Bot}
δ_1
ζ
W_1Y_0
$W_1Y_0 \coprod_{W_0Y_0} W_0Y_1$

Proof (cont)

$\zeta = \delta_1 \circ \delta_0$, and δ_0 is a pushout of $\alpha_1 \square \beta_0$ so is a (trivial) cofibration.

δ_1 is a pushout of ξ, which we rewrite as the pushout product $\alpha_0 \square (\beta_1 \circledast f_Y)$, below, so both are (trivial) cofibrations.



Proof (cont)

To finish, rewrite

$$\Big(W_1X_1 \coprod_{V_1X_1} V_1Y_1\Big)\coprod_Z\Big(W_1Y_0 \coprod_{W_0Y_0} W_0Y_1\Big) \xrightarrow{(\alpha_1\square\beta_1)\circledast(f_W\square f_Y)} W_1Y_1$$

as:

$$\begin{array}{ccc}
\Big(W_1X_1 \coprod_{V_1X_1} V_1Y_1\Big)\coprod_Z\Big(W_1Y_0 \coprod_{W_0Y_0} W_0Y_1\Big) & \xrightarrow{(\alpha_1\square\beta_1)\circledast(f_W\square f_Y)} & W_1Y_1 \\
\downarrow \cong & & \downarrow = \\
W_1\Big(X_1\coprod_{X_0}Y_0\Big) \coprod_{\big(V_1\coprod_{V_0}W_0\big)\big(X_1\coprod_{X_0}Y_0\big)} \Big(V_1\coprod_{V_0}W_0\Big)Y_1 & \xrightarrow{(\alpha_1\circledast f_W)\square(\beta_1\circledast f_Y)} & W_1Y_1
\end{array}$$

Recap

Theorem (W.-Yau; arXiv:1703.05359; Math Scandinavica 2018)

If M *is a monoidal model category, then so are* $\overrightarrow{\mathrm{M}}^{\square}$, $\mathrm{M}^{I^{\times n}}$, *and* $\mathrm{M}^{\square^{op}}$.

Lemma (Hovey): In $\overrightarrow{\mathrm{M}}^{\square}$, γ from $f : X_0 \longrightarrow X_1$ to $g : Y_0 \longrightarrow Y_1$ is a (trivial) cofibration iff γ_0 and $\gamma_1 \circledast g : X_1 \coprod_{X_0} Y_0 \longrightarrow Y_1$ are.

If α is cof and β is (triv) cof, then let $\gamma = \alpha \square_2 \beta$.

We proved $\gamma_0 = \zeta$ and $\gamma_1 \circledast g$ from previous slide, are (triv) cof's.