The homotopy theory of substitudes

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The Baez-Dolan Stabilization Hypothesis

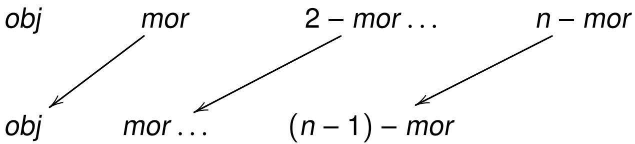
'Higher Dimensional Algebra and Topological Quantum Field Theory' (1995) by Baez and Dolan studied n-dimensional TQFTs via n-category representations.

Look for relationships between categories of weak n-categories as n varies.

Many definitions of weak n-category, by Rezk, Tamsamani, Simpson's higher Segal categories, Ara's n-quasi-categories, Bergner-Rezk models on simplicial presheaves, etc.

Let Sp_k be k-truncated spaces, modeling k-types ($\pi_{>k} = 0$). Rezk's $\Theta_n Sp_k$ models $(n+k, n)$-categories so $nCat \cong \Theta_n Sp_0$. The Θ construction encodes shapes of pasting diagrams.

Consider the reindexing functor U : n-cat→ $(n-1)$-cat:



Now objects have a composition law, morphisms have vertical and horizontal composition, etc. That's extra structure!

Example: start with a 2-category $C = (x, 1_x, hom(1_x, 1_x))$ with 1 object and 1 morphism, reindex twice: 2-cat→ 0-cat.
Eckmann-Hilton: $hom(1_x, 1_x)$ is a commutative monoid.
3-cat→ 0-cat gives no further structure; we say that reindexing stabilized. What about 3-cat to 1-cat, 4-cat to 1-cat, etc?

Forget k levels: $n + k \to n$

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	sets	categories	2-categories
$k = 1$	monoids	monoidal categories	monoidal 2-categories
$k = 2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories
$k = 3$	''	symmetric monoidal categories	weakly involutory monoidal 2-categories
$k = 4$	''	''	strongly involutory monoidal 2-categories
$k = 5$	''	''	''

k-tuply monoidal weak n-categories

If a $(n+k)$-cat $\mathcal{C}$ is trivial up to k, reindex to $\mathcal{D}$, an n-cat with extra structure. $nCat_k$ is the category of such $\mathcal{D}$. Forgetful $U : nCat_k \to nCat_{k-1}$ has a left adjoint S called suspension.

Conjecture (Baez-Dolan Stabilization Hypothesis)

If $k \geq n+2$ then $S : nCat_k \to nCat_{k+1}$ is an equivalence.

Batanin: the extra structure on $\mathcal{D}$ is that of an algebra in $nCat$ over the k-operad G_k, the cofibrant replacement of 1_k in Op_k.
Stabilization Hypothesis (Equivalent Formulation): For $k \geq n+2$, $\text{Alg}_{G_k}(nCat) \leftrightarrows \text{Alg}_{G_{k+1}}(nCat)$ is a Quillen equivalence. We'll deduce from a Q.E. $Op_k^{loc}(nCat) \leftrightarrows Op_{k+1}^{loc}(nCat) \leftrightarrows SO$.

k-Operads I: k-ordinals

Definition (T is a k-ordinal)

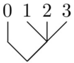
Let $T \in FinSet$, with k binary relations $<_0, \ldots, <_{k-1}$ s.t.:

1. $<_p$ *is nonreflexive;*
2. *for every pair $a, b \in T$, there exists exactly one p such that $a <_p b$ or $b <_p a$*
3. *if $a <_p b$ and $b <_q c$ then $a <_{min(p,q)} c$.*

Every n-ordinal can be represented as a pruned planar tree with n levels. For example, the 2-ordinal

$$0 <_0 1,\ 0 <_0 2,\ 0 <_0 3,\ 1 <_1 2,\ 2 <_1 3$$

is represented by the following pruned tree



k-Operads II: the definition

$\mathbb{V}$ is symmetric monoidal, U_k = terminal k-ordinal.

Definition (A_T is a k-operad in $\mathbb{V}$)

$\forall T \in Ord(k)$, a collection A_T of objects of $\mathbb{V}$ with:

- *a morphism $e : I \to A_{U_k}$ (the unit);*
- *for every order preserving $\sigma : T \to S$, a morphism*

$$m_\sigma : A_S \otimes A_{T_0} \otimes \cdots \otimes A_{T_j} \to A_T \quad (\textit{the multiplication}).$$

- *Associativity given $T \to S \to R$*
- *Coherent w.r.t identity $T \to T$ and unique mor $T \to U_k$.*

where $T_i = \sigma^{-1}(i)$ for $\sigma : T \to S$. Batanin provided the following adjunction $\text{des}_k : SO(\mathbb{V}) \leftrightarrows Op_k(\mathbb{V}) : \text{sym}_k$

k-Operads III: quasibijections

A morphism of k-ordinals (order-preserving map) is a quasibijection if it is a bijection of underlying sets. Let Q_k be the subcategory of quasibijections of $Ord(k)$. Have $U : Op_k(\mathbb{V}) \to [Q_k^{op}, \mathbb{V}]$. Think: collections. Q_k^{op} acts on Op_k, but not invertibly.

$Q_k \cong \coprod Q_k(m)$ where $m = |Ord(k)|$, just like $\Sigma \cong \coprod \Sigma_n$.

Q_k^{op} are the unary operations of the substitude we use to encode k-operads. Q_k^{op} is not a groupoid, so we don't use Feynman categories or colored operads to encode k-operads.

Idea: localize to force Q_k^{op} to act invertibly, up to homotopy.

Higher Braided Operads

Let Op_k^{loc} be a localization so that Q_k^{op} acts invertibly. Call them locally constant k-operads, a.k.a. higher braided operads.

1. For $k = 1, 2, \infty$, Ho(locally constant k-operads) $\simeq$ Ho(nonsymmetric), Ho(braided), and Ho(symmetric).
2. Contractible operads detect 1-fold, 2-fold, and infinite loop spaces. Cofibrant replacement of the terminal higher braided operad is an E_n-operad, so detects n-fold loop spaces.
3. The nerve of $Q_k(m)$ is ho. equiv. to unordered configuration space of m points in $\mathbb{R}^k$, a $K(\pi, 1)$ only for $n = 1, 2, \infty$.
4. Fund. grpd: $\pi_1(Q_\infty) \simeq \Sigma$ (sym gps), $\pi_1(Q_2) \simeq Br_2$ (braid gps), $\pi_1(Q_1)$ is contractible (non-sym operads).
5. $Q_k \to \Sigma$ is iso on $\pi_{\leq k}$

Substitudes (Day-Street)

Let $\mathbb{V}$ be a symmetric monoidal category, e.g., $\Theta_n Sp_0$.

A $\mathbb{V}$-substitude (P, A) is a small $\mathbb{V}$-category A together with a sequence of $\mathbb{V}$-functors: $P_n : \underbrace{A^{op} \otimes \cdots \otimes A^{op}}_{n-times} \otimes A \to \mathbb{V},\ n \geq 0,$

equipped naturally with associative, unital, and equivariant:

1. substitution operations (like operad composition)
2. unit morphisms $\eta : A(a_1, a_2) \to P_1(a_1; a_2)$
3. isos $\gamma_\sigma : P(a_1, \ldots, a_n; a) \to P(a_{\sigma(1)}, \ldots, a_{\sigma(n)}; a)$

Think: colored operad $\mathscr{E}$ with identity-on-objects $\mathbb{V}$-functor $\eta : A \to U(\mathscr{E})$=unary ops. Example: $(O^{(k)}, Q_k^{op})$ for Op_k. $\mathbb{V}$-substitudes are equivalent to regular patterns (Getzler), category-colored operads (Petersen)

Transferring Model Structures

Lemma (well-known)

If $T = UF$ is a monad on cofibrantly generated $\mathcal{M}$ and if for all generating trivial cofibrations $j : K \to L$ in $\mathcal{M}$, transfinite compositions of pushouts in $Alg_T(\mathcal{M})$:

$$\begin{array}{ccc} F(K) & \longrightarrow & F(L) \\ \downarrow & & \downarrow \\ \mathscr{O} & \longrightarrow & \mathscr{P} \end{array} \quad (\text{pushout } \searrow)$$

are weak equivalences then $Alg_T(\mathcal{M})$ has transferred model structure, with weak equivalences and fibrations defined in $\mathcal{M}$.

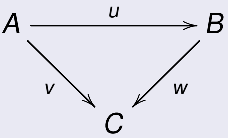
If above works only for $\mathscr{O}$ cofibrant then get transferred semi-model structure. Apply to Σ-free unary tame substitudes (P, A) whose unit is faithful, where $\mathcal{M} = [A, \mathbb{V}]$. Get $Op_k(\Theta_n Sp_0)$.

Fundamental localizers (Grothendieck; Cisinski)

Let W be a class of functors between small categories.

Definition (W is a fundamental localizer)

1. *W contains all identities; satisfies two out of three;*
2. *If $i : A \to B$ has a retraction $r : B \to A$ and $r \cdot i : B \to B$ is in W then i is in W;*
3. *If $A \in Cat_*$ (has terminal obj) then $A \to \mathbf{1}$ is in W;*
4. *If $u/c : A/c \to B/c$ is in W for each object $c \in C$ in:*



then u is in W. Call W's elements W-equivalences.

W-asphericity

Fix a fundamental localizer W. A small category A is W-aspherical if the unique functor $! : A \to 1$ is in W. Like being nullhomotopic.

W is a proper fundamental localizer if there exists a set $\mathcal{S}$ of small categories such that W is minimal in making elements of $\mathcal{S}$ aspherical, in the sense that any $A \in \mathcal{S}$ is in W-aspherical. We write $W(\mathcal{S}) := W$ in this case.

Example: W_∞ is the class of functors whose nerve is a weak equivalence of simplicial sets. Cisinski proved: W_∞ is the minimal fundamental localizer. This implies, among other things, that $W_\infty = W(\{A\})$ for any A with a terminal object.

Fundamental localizer for truncation

Let $W_n = W(\{S^{n+1}\})$, where S^{n+1} is a small category which has the homotopy type of $(n+1)$-sphere. W_n-equivalences are n-equivalences (functors inducing isos on $\pi_{\leq n}N(-)$).

Ex: W_0 is functors that induce isomorphism on connected components, and W_0-aspherical categories are connected.

Ex: For $k \geq 3$, the total order functor induces an n-equivalence $[-] : Q_k \to \Sigma$ for $1 \leq n+1 \leq k$, by a classifying space computation. Hence Baez-Dolan stabilization theorem needs $n+1 \leq k$, for the critical Quillen equivalence. Also, $[-]_2 : Q_2 \to Br$ is an n-equivalence for $1 \leq n \leq \infty$.

Homotopy Theories: Ho_W = $Cat[W^{-1}]$, Ho_{W_∞} is HoTop, Ho_{W_n} is n-truncated homotopy types.

W-locally constant presheaves

Let A be a small category and $\mathbb{V}$ a model category. Let $Ho[A, \mathbb{V}]$ be the localization of $[A, \mathbb{V}]$ with respect to levelwise weak equivalences. Let W be a proper fundamental localizer.

Definition (Cisinski): A presheaf $F : A \to \mathbb{V}$, is called W-locally constant if for any W-aspherical small category A' and any functor $u : A' \to A$ the presheaf $u^*(F) : A' \to \mathbb{V}$ is isomorphic to a constant presheaf in $Ho[A', \mathbb{V}]$. Denote them $LC_W[A, \mathbb{V}]$.

Ex: F is W_∞-locally constant if and only if for any $f : a \to b$ in A, $F(f)$ is a weak equivalence in $\mathbb{V}$. Because $W_\infty = W(0 \to 1)$

Say u is a local W-equivalence if u^* is an equivalence of categories on $LC_W[-, \mathcal{M}]$ for any model category $\mathcal{M}$.

Cisinski localization

Theorem (Cisinski, Batanin – W.)

Let W be a proper fundamental localizer and $\mathbb{V}$ a combinatorial model category. Then:

1. *For $A \in Cat$ there exists a left Bousfield localization (of proj, inj, Reedy) $[A, \mathbb{V}]^W$ such that its local objects are levelwise fibrant and W-locally constant presheaves.*
2. *For a local W-equivalence $u : A \to B$ between small categories, the restriction functor*

$$u^* : [B, \mathbb{V}]^W_{proj} \to [A, \mathbb{V}]^W_{proj}$$

is a right Quillen equivalence.

Related: Homotopy theory of homotopy functors (Chorny-W)

Left Bousfield localization $L_{\mathcal{C}}\mathcal{M}$

Given $\mathcal{C} \subset mor(\mathcal{M})$, $L_{\mathcal{C}}\mathcal{M}$ is a universal model structure where $\mathcal{C}$ are weak equivalences and $id : \mathcal{M} \to L_{\mathcal{C}}\mathcal{M}$ is left Quillen. Same cofibrations as $\mathcal{M}$, more weak equivalences. Say W is a $\mathcal{C}$-local object if $map(B, W) \to map(A, W)$ is a w.e. in sSet for all $f : A \to B$ in $\mathcal{C}$. Say $g : X \to Y$ is a $\mathcal{C}$-local equivalence if $map(Y, W) \to map(X, W)$ is a w.e. for all $\mathcal{C}$-local W.

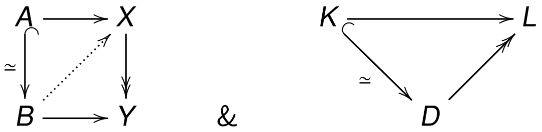
Goal: lift localizations from $[A, \mathbb{V}]^W_{proj}$ to Op^W_k

Problem: Op_k is not known to be left proper.

Solution: semi-model categories.

Semi-model categories (Spitzweck; Fresse; many others)

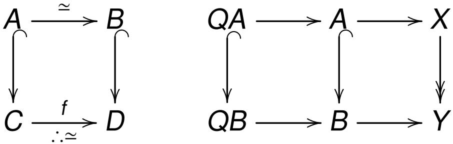
Definition: $(\mathcal{M}, W, Q, F)$ satisfies all model category axioms except we only require the following *for A and K cofibrant*:



Still have cofibrant replacement. All model category results have semi-model category analogues (often cofibrantly replace first): Ken Brown lemma, cylinders and path objects, cube lemma, Quillen equivalences, Reedy model structures, (co)simplicial frames, homotopy (co)limits, simplicial mapping spaces, etc. Combinatorial semi is Quillen equiv. to combinatorial model.

Left properness

A model category $\mathcal{M}$ is left proper if in pushout on left below, f is a weak equivalence. It's like the gluing property in Top.



Upside 1 (right): a lift $QB \to X$ yields a lift $B \to X$.
Upside 2: pushout square with one leg a cofibration is homotopy pushout square.
Semi-model version: when A, B cofibrant, these results are automatic.

Semi-model Smith theorem

Theorem (Batanin-W.)

Suppose $\mathcal{M}$ is a locally presentable category with a class $\mathscr{W}$ of weak equivalences and a set of maps I satisfying

1. *$\mathscr{W}$ is κ-accessible, closed under retracts, two out of three.*
2. *Any morphism in inj(I) is a weak equivalence.*
3. *Within cof $I \cap \mathscr{W}$, maps with cofibrant domain are closed under pushouts to arbitrary cofibrant objects and under transfinite composition.*
4. *Domains of I and initial object are cofibrant.*

Then there is a cofibrantly generated semi-model structure on $\mathcal{M}$ with generating cofibrations I, generating trivial cofibrations J, cofibrations cof I, and fibrations defined by the right lifting property with respect to J. Furthermore, the generating trivial cofibrations J have cofibrant domains.

Semi-model Bousfield localization

Theorem (Bousfield localization without left properness)

Suppose that $\mathcal{M}$ is a combinatorial semi-model category whose generating cofibrations have cofibrant domain, and $\mathcal{C}$ is a set of morphisms of $\mathcal{M}$. Then there is a semi-model structure $L_{\mathcal{C}}(\mathcal{M})$ on $\mathcal{M}$, whose weak equivalences are the $\mathcal{C}$-local equivalences, whose cofibrations are the same as $\mathcal{M}$, and whose fibrant objects are the $\mathcal{C}$-local objects. Furthermore, $L_{\mathcal{C}}(\mathcal{M})$ satisfies the universal property that, for any any left Quillen functor of semi-model categories $F : \mathcal{M} \to \mathcal{N}$ taking $\mathcal{C}$ into the weak equivalences of N, then F is a left Quillen functor when viewed as $F : L_{\mathcal{C}}(\mathcal{M}) \to \mathcal{N}$.

Quillen for semi means *U* preserves (trivial) fibrations, and *F* preserves (trivial) cofibrations (between cofibrant objects).

Applications of localization w.o. left properness

1. Voevodsky's radditive functors ($L_C\mathcal{M}$ only a semi).
2. Inverting operations in operads and rings.
3. Parameterized spectra; C^*-algebras; Richter $C(ch^\Sigma)$
4. Localizing O-alg: GH, HZ, B
5. Bacard lax diagrams and enrichment.
6. Toen dgCat(k) and derived algebraic geometry
7. Costello-Gwilliam prefactorization algebras.
8. Functor calculus, esp. for *Cat* and *Graph* (Vicinsky).
9. Left localization after right localization; E_2-model str
10. More?

Localization for k-operads

Theorem (Batanin-W.)

If $\mathbb{V}$ is a symmetric monoidal combinatorial model category and (P, A) encodes k-operads or $SO(\mathbb{V})$ or $BO(\mathbb{V})$, then

1. *The projective semi-model structure on $Alg_P(\mathbb{V})$ exists;*
2. *For any proper fundamental localizer the local semi-model model structure $Alg_P^W(\mathbb{V})$ exists and its fibrant objects are exactly W-locally constant P-algebras;*
3. *The local model structure $Alg_P^W(\mathbb{V})$ coincides with the transferred semi-model structure from $U : Alg_P(\mathbb{V}) \to [Q_k^{op}, \mathbb{V}]^W_{proj}$.*

W_∞-locally constant k-operads are higher braided operads.

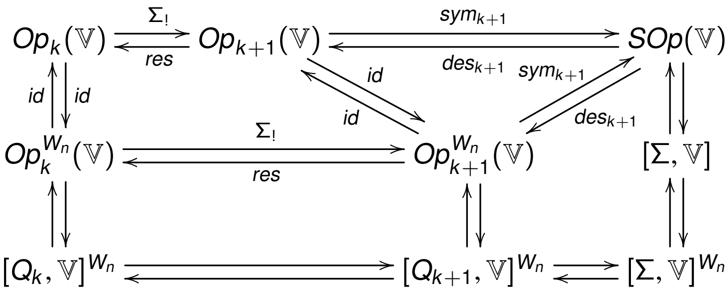
We lift old Batanin results from homotopy level to model.

Plan for proving Baez-Dolan Stabilization

Have: $U : Op_k(\mathbb{V}) \to [Q_k^{op}, \mathbb{V}]$ and left adjoint F.

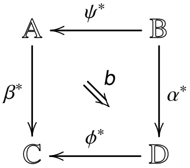
First: Transfer model structures, letting $\mathbb{V} = \Theta_n(Sp_0)$

Next: prove Quillen equivalences. We have (for $0 \leq n \leq \infty$):



Beck-Chevalley and Quillen equivalences

A square of right adjoints and a natural transformation



is called Beck-Chevalley if the natural transformation

$$\textbf{bc} : \phi_! \beta^* \to \alpha^* \psi_!$$

is an isomorphism. Upshot: if $(\phi_!, \phi^*)$ is an adjoint equivalence and β^*, α^* reflect isos then $(\phi_!, \phi^*)$ is adjoint equivalence.

Homotopical Beck-Chevalley

The square above is homotopy Beck-Chevalley if $\mathbb{L}\phi_!\mathbb{R}\beta^*(-) \to \mathbb{R}\alpha^*\mathbb{L}\psi_!(-)$ is an isomorphism in $\mathrm{Ho}(\mathbb{D})$. This occurs if α^* preserves weak equivalences and β^* preserves cofibrant objects.

Upshot: if $(\phi_!, \phi^*)$ is a Quillen equivalence and β^*, α^* reflect weak equivalences between fibrant objects, then $(\phi_!, \phi^*)$ is a Quillen equivalence.

Application: Quillen equivalences of categories of algebras over substitudes. Given $(f, g) : (P, A) \to (Q, B)$, if $(g_!, g^*)$ is Q.E. then so is $(f_!, f^*)$.

Lift Q.E. $[A, \mathbb{V}]^W_{proj} \leftrightarrows [B, \mathbb{V}]^W_{proj}$ to $\mathrm{Alg}^W_P(\mathbb{V}) \leftrightarrows \mathrm{Alg}^W_Q(\mathbb{V})$.

Stabilization for locally constant k-operads

Let $\mathbb{V}$ be a combinatorial symmetric monoidal model category with cofibrant unit.

Theorem (Batanin-W.)

For $k \geq 3$ and $2 \leq n+1 \leq k$, the symmetrisation functor $sym_k : Op_k^{W_n}(\mathbb{V}) \to SO(\mathbb{V})$ and the suspension functor $\Sigma_! : Op_k^{W_n}(\mathbb{V}) \to Op_m^{W_n}(\mathbb{V})$ (for $k < m \leq \infty$) are left Quillen equivalences.

For $k = 2$, use braided operads $BO(\mathbb{V})$, and for $1 \leq n \leq \infty$ $bsym_2 : Op_2^{W_n}(\mathbb{V}) \to BO(\mathbb{V})$ is a left Quillen equivalence.

Baez-Dolan stabilization follows from this, for $\Theta_n Sp_0$, $Seg^{n+k}(\mathcal{M})$, $nQcat$, $\Theta_n Sp$-Segal Cat, $sPSh(\Delta \times \Theta_n)_{BR}$, or Tamsamani's $PC^n(M)$.

n-truncated model categories

Let $\mathbb{V}$ be *n-truncated* i.e. all Map(X,Y) are W_n-local in sSet. Then $[A,\mathbb{V}]^{W_r}_{proj} \to [A,\mathbb{V}]^{W_\infty}_{proj}$ is a Q.E. for $r \geq n+1$, and:

Corollary (Stabilization for Higher Braided Operads)

For $n \geq 0$ and $3 \leq n+2 \leq k \leq \infty$, the symmetrisation functor $sym_k : Op_k^{W_\infty}(\mathbb{V}) \to SO(\mathbb{V})$ and the suspension functor $\Sigma_! : Op_k^{W_\infty}(\mathbb{V}) \to Op_m^{W_\infty}(\mathbb{V})$ (for $k < m \leq \infty$) are left Quillen equivalences.
For $k = 2$, use braided operads $BO(\mathbb{V})$, and for $1 \leq n \leq \infty$ $bsym_2 : Op_2^{W_\infty}(\mathbb{V}) \to BO(\mathbb{V})$ is a left Quillen equivalence.

Note: $\Theta_n Sp_0$ is n-truncated. Or: truncate Tamsamani, Simpson, Ara, or Bergner-Rezk models via $\tau_{\leq n}$ localization. Truncate from (∞, n)-categories to weak n-cat.

Baez-Dolan Stabilization; Batanin 2017

Let G_k = cof rep of $I \in Op_k$, and $B_k(\mathbb{V}) = G_k$-alg. Note $i : \Sigma_! G_k \to G_{k+1}$.

Theorem (Baez-Dolan Stabilization)

Let $0 \leq n$ and $\mathbb{V}$ a n-truncated monoidal combinatorial model category with cofibrant unit. Then $i_! : B_k(\mathbb{V}) \to B_{k+1}(\mathbb{V})$ and $(j_k)_! : B_k(\mathbb{V}) \to E_\infty(\mathbb{V})$ are left Quillen equivalences for $k \geq n+2$.

Apply this with Rezk's $\mathbb{V} = \Theta_n Sp_m$, $n+m$-truncated model for $(n+m, n)$-categories, where Sp_m is m-truncation on sSet (local objects are m-types). Or: $\tau_n Seg^{n+k}(\mathcal{M})$, $\tau_k PC^n(\mathcal{M})$, $\tau_k nQcat$, $\tau_k \Theta_n Sp$-Segal Cat (model on $[\Delta^{op}, \Theta_n Sp]$), or $\tau_k sPSh(\Delta \times \Theta_n)_{BR}$.

Baez-Dolan Stabilization

Corollary (Stabilisation for Rezk's $(n+m, n)$-categories)

The suspension functor induces the left Quillen equivalence

$$i_! : B_k(\Theta_n Sp_m) \to B_{k+1}(\Theta_n Sp_m)$$

for $k \geq m+n+2$ and, hence, an equivalence between homotopy categories of Rezk's k-tuply monoidal $(n+m, n)$-categories and Rezk's $(k+1)$-tuply monoidal $(n+m, n)$-categories. Baez-Dolan stabilization also holds for Tamsamani, Simpson, Ara, and Bergner-Rezk models of weak n-categories , and will hold for other models (e.g., n-relative categories, n-fold Segal spaces) if suitable monoidal products are discovered.

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