Equivariant Topology and Derived Algebra

A conference in honor of John Greenlees' $60^{\rm th}$ birthday NTNU, August 2019

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The present volume is a collection of papers whose topics connect to several major themes from John Greenlees' vast mathematical career.

Conference

The catalyst for this proceedings was a week-long conference held at NTNU (Trondheim) between the 29^{th} of July and the 2^{nd} of August 2019. This conference, entitled *Equivariant Topology and Derived Algebra*, was held in honor of John Greenlees' 60^{th} birthday. The conference consisted of 15 invited talks, 11 contributed talks, and 13 shorter *gong show* style talks, and was attended by a diverse group of over 90 international participants. The mathematical content was enhanced by a customary hiking excursion and a hearty conference dinner with beautiful scenic views.

Summary of the papers

We briefly outline the papers in this proceedings, while also taking the opportunity to connect them to the work of John Greenlees, which at the date of writing spans more than 90 papers and four research monographs 32, 45, 57, 73.

Comparing Dualities in the K(n)-local Category P. G. Goerss and M. J. Hopkins

Duality is a recurring theme through the work of John Greenlees, the starting place is perhaps Spanier–Whitehead duality. In modern language, the Spanier–Whitehead dual of a spectrum X is the function spectrum $DX = F(X, \mathbb{S})$, which arises from the commutative monoidal structure of the stable homotopy category. A common calculation is to show that the dual of the Moore spectrum for $\mathbb{Z}/2$ is simply a shift of that spectrum. A detailed examination of functional duals and Moore spectra is the subject of Greenlees' first published work, [93]. It is natural to look for generalisations of (Spanier–Whitehead) duality, for example [86] and [88] consider duality in the equivariant stable homotopy category, while [31] and [41] look more generally at questions of duality.

The first paper of this proceedings takes up this theme and investigates duality in the K(n)-local stable homotopy category, giving a full and detailed proof of a result relating K(n)-local Spanier–Whitehead duality to the more computable notion of Brown–Comenetz duality.

Axiomatic Representation Theory of Finite Groups by way of Groupoids

I. Dell'Ambrogio

A second major theme in the work of John Greenlees is representation theory, and in particular, the use and study of Mackey functors. The most immediate way Mackey functors appear in the work of Greenlees is via equivariant cohomology theories. These are a generalisation of cohomology theories that have G-spaces as input, and take G-Mackey functors as coefficients. The category of Mackey functors is also a rich and interesting category in its own right, as demonstrated in [40, 60, [73], [82], [85]. Indeed, three papers of this volume consider Mackey functors at length.

This paper considers very general notions of Mackey functors and gives a common conceptual framework for several different versions. It provides relations between these different versions and connects the theory to 2-categories and bisets.

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Chromatic Fracture Cubes O. Antolín-Camarena and T. Barthel

A sizable portion of John Greenlees' work is dedicated to the development of algebraic models for rational G-spectra, where G is a compact Lie group. Algebraic models for several groups have been established, including all finite groups, the circle, tori of arbitrary dimension, O(2)and SO(3) see [73, 57, [16, [11], 59, 52]. Greenlees has conjectured that an algebraic model (satisfying a list of key properties) exists for every compact Lie group G. A key tool for this project is an isotropy separation of the sphere spectrum in rational G-spectra. This separation is a pullback square similar to the arithmetic pullback square or the chromatic fracture square. As the sphere spectrum is the monoidal unit, the isotropy separation extends to a decomposition of the (homotopy) category of rational G-spectra into simpler building blocks. Recent work of Greenlees abstracts this machinery to the setting of axiomatic stable homotopy theory [1].

This paper generalises the familiar chromatic fracture square in the E(n)local stable homotopy category to a chromatic fracture cube. This cube provides a combinatorial decomposition of the category into monochromatic pieces. The E(n)-local stable homotopy category can be reconstructed by taking a homotopy limit of these monochromatic pieces over a certain diagram of diagrams.

An Introduction to Algebraic Models for Rational G-Spectra D. Barnes and M. Kędziorek

As mentioned above, a major project of Greenlees is the development of algebraic models for rational *G*-spectra, where *G* is a compact Lie group, see [73], 57, 16, 11, 59, 52. The initial case is where *G* is a finite group, here the algebraic model for rational *G*-spectra is a finite product over conjugacy classes of subgroups $H \leq G$ of graded $\mathbb{Q}[W_G H]$ modules ($W_G H$ is the Weyl group of *H* in *G*). One of the ways to prove this result uses the idempotent splitting of rational *G*-Mackey functors, see Appendix A of [73]. There are many papers building upon that work, such as constructing an algebraic model for naive-commutative ring *G*-spectra [9].

This paper gives an introduction to rational Mackey functors and summarises the main techniques used to obtain algebraic models for rational G-spectra, concentrating on the case of a finite group G. It discusses the topological and algebraic parallels of using idempotents to

split the category of rational G-spectra and rational G-Mackey functors. It also briefly mentions the techniques to obtain algebraic models when G is not finite.

Monoidal Bousfield Localizations and Algebras over Operads D. White

Commutative monoidal structures appear throughout John Greenlees' work, occurring with algebraic origins [63], [72], [74], topological origins [4], [9] and bridging the divide between algebra and topology: [13], [36], [37], [75]. Moreover, the construction of algebraic models for rational G-spectra often depends upon making use of (commutative) monoidal structures in both topology and algebra. For example, the isotropy separation arguments require that certain localizations of the sphere spectrum are still commutative monoids. This property is not automatic, even under suitable cofibrancy conditions.

This paper characterizes those Bousfield localizations that respect (commutative) monoidal structures, and moreover proves that these localizations preserve algebras over cofibrant operads. This general machinery can be used to retrieve many classical results which have repeatedly been used in the work of Greenlees.

Stratification and Duality for Unipotent Finite Supergroup Schemes

D. Benson, S. B. Iyengar, H. Krause and J. Pevtsova

A recent direction in the work of Greenlees is the study of tensortriangulated categories, triangulated categories with compatible symmetric monoidal product and function object. A central example is the stable homotopy category, arising from homotopy (co)fibre sequences and the smash product and function spectrum. The equivariant stable homotopy category for a compact Lie group G is an even richer example, see [1], [3] and [8]. A localization of a tensor-triangulated category is (roughly speaking) a method to construct a quotient category by inverting certain maps in the original category (an idea closely related to Bousfield localisations) and is deeply related to questions of duality, as in [33].

The purpose of this paper is to give an outline of the theory of localisation and duality by applying it to the stable module category of a unipotent finite supergroup scheme.

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Bi-incomplete Tambara Functors

A. J. Blumberg and M. A. Hill

One of the recent themes in equivariant homotopy theory is to understand commutative ring objects in G-spaces and G-spectra. This is reflected in John Greenlees' work through the research on commutativity described above and more directly, as in [16] [9]. The subtlety and complications of equivariant commutativity can be described using certain class of G-operads, called N_{∞} operads. Algebras over an N_{∞} operad \mathcal{O} in G-topological spaces correspond, roughly speaking, to a G-spectrum with transfers determined by \mathcal{O} . Thus, one might think of \mathcal{O} as governing the additive structure of a G-spectrum. Algebras over an N_{∞} operad \mathcal{O} in G-spectra (as opposed to G-spaces) correspond, roughly speaking, to \mathcal{O} -commutative ring G-spectra, that is, ring G-spectra with norm maps on homotopy groups determined by \mathcal{O} . Thus, in this case one might think of \mathcal{O} as governing the multiplicative ring structure of a G-spectrum. The natural question is: how one can mix the various additive and multiplicative structures?

This paper investigates the compatibility conditions between incomplete additive transfers and incomplete multiplicative norms in the algebraic setting of G-Tambara functors and provides a full description of the possible interactions of these two classes of maps.

Homotopy Limits of Model Categories, Revisited J. E. Bergner

A key observation of the paper [1] is that the algebraic models for rational *G*-equivariant spectra can be described as homotopy limits of diagrams of model categories. This observation developed from homotopy pullback constructions in [1] based on isotropy separation, building on machinery of Greenlees–Shipley [20, [21], [25]. Homotopy limits also occur in the paper of Antolín-Camarena–Barthel (in the setting of $(\infty, 1)$ categories) further demonstrating their ubiquity.

The final paper of this proceedings provides a comprehensive outline of the machinery required for constructing homotopy limits of diagrams of Quillen model categories and left Quillen functors between them, collecting previous work of the author. Moreover the paper provides a wealth of important examples of this homotopy limit construction. The paper also provides some warning on working with diagrams which come with a mix of left and right Quillen functors.

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Monoidal Bousfield Localizations and Algebras over Operads David White

Abstract

We give conditions on a monoidal model category \mathcal{M} and on a set of maps \mathcal{C} so that the Bousfield localization of \mathcal{M} with respect to \mathcal{C} preserves the structure of algebras over various operads. This problem was motivated by an example that demonstrates that, for the model category of equivariant spectra, preservation does not come for free, even for cofibrant operads. We discuss this example in detail and provide a general theorem regarding when localization preserves P-algebra structure for an arbitrary operad P.

We characterize the localizations that respect monoidal structure and prove that all such localizations preserve algebras over cofibrant operads. As a special case we recover numerous classical theorems about preservation of algebraic structure under localization, in the context of spaces, spectra, chain complexes, and equivariant spectra. We also provide several new results in these settings, and we sharpen a recent result of Hill and Hopkins regarding preservation for equivariant spectra. To demonstrate our preservation result for non-cofibrant operads, we work out when localization preserves commutative monoids and the commutative monoid axiom, and again numerous examples are provided. Finally, we provide conditions so that localization preserves the monoid axiom.

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5.1 Introduction

Bousfield localization is a powerful tool in homotopical algebra, with classical applications to homology localization for spaces and spectra **11**, to cellularization and nullification 21, and to *p*-localization and completion 12. Hirschhorn generalized the machinery of Bousfield localization to the setting of model categories 32, inverting any class of morphisms generated by a set. This general framework has seen tremendous applications: it allows for the passage from levelwise model structures to stable model structures 38, it is used to set up motivic homotopy theory 35, and it allows for the study of combinatorial model categories via simplicial presheaves 16. The interplay between left Bousfield localization and monoidal structure has often proven fruitful, e.g. to put an E_{∞} -algebra structure on connective K-theory 19, for homotopy theoretic computations involving generalized Eilenberg-Maclane spaces 13, 15, 21, and, recently, to create an equivariant spectrum with a certain periodicity that is used to resolve the Kervaire invariant one problem 31. In this paper, we further the study of the interplay between left Bousfield localization and monoidal model categories, we provide conditions so that left Bousfield localization preserves algebras over operads (and several important model categorical axioms), and we apply our results to numerous classical and new examples of interest.

Structured ring spectra have had numerous applications in stable homotopy theory [19, 38, 43]. Nowadays, structured ring spectra are often thought of as algebras over operads acting in any of the monoidal model categories for spectra. It is therefore natural to ask the extent to which Bousfield localization preserves such algebraic structure. For Bousfield localizations at homology isomorphisms this question is answered in [19] and [43]. The case for spaces is subtle and is addressed in [13], [15], and [21]. More general Bousfield localizations are considered in [14].

The preservation question may also be asked in the context of equivariant and motivic spectra, and it turns out the answer is far more subtle. In Example 5.36, we discuss an example of a naturally occurring Bousfield localization of equivariant spectra that preserves the type of algebraic structure considered in [19] but fails to preserve the equivariant commutativity needed for the landmark results in [31]. We generalize this example in [5.37].

In order to understand this and related examples, we find conditions on a model category \mathcal{M} and on a class of maps \mathcal{C} so that the left Bousfield localization $L_{\mathcal{C}}$ with respect to \mathcal{C} preserves the structure of algebras over various operads. After a review of the pertinent terminology in Section 5.2 we give our general preservation result in Section 5.3, which we state here for the reader's convenience.

Theorem 1 Let \mathcal{M} be a monoidal model category and \mathcal{C} a class of morphisms such that the Bousfield localization $L_{\mathcal{C}}(\mathcal{M})$ exists and is a monoidal model category. Let P be an operad valued in \mathcal{M} . If the categories of P-algebras in \mathcal{M} and in $L_{\mathcal{C}}(\mathcal{M})$ inherit transferred semimodel structures from \mathcal{M} and $L_{\mathcal{C}}(\mathcal{M})$ (with weak equivalences and fibrations defined via the forgetful functor) then $L_{\mathcal{C}}$ preserves P-algebras.

In general, it is difficult to check that *P*-algebras in $L_{\mathcal{C}}(\mathcal{M})$ inherit a transferred semi-model structure. To make it easier to check this hypothesis, in Section 5.4 we characterize when $L_{\mathcal{C}}(\mathcal{M})$ is a monoidal model category, proving the following theorem.

Theorem 2 Suppose \mathcal{M} is a cofibrantly generated monoidal model category in which cofibrant objects are flat (i.e., for all weak equivalences f and all cofibrant K, $f \otimes id_K$ is a weak equivalence). Then $L_{\mathcal{C}}(\mathcal{M})$ is a monoidal model category with cofibrant objects flat if and only if every map of the form $f \otimes id_K$, where f is in \mathcal{C} and K is cofibrant, is a \mathcal{C} -local equivalence. If the domains of the generating cofibrations I are cofibrant, it suffices to consider K in the set of domains and codomains of the morphisms in I.

Most monoidal model categories encountered in nature satisfy the property that cofibrant objects are flat, as examples in Section 5.5 demonstrate. Furthermore, given a set of morphisms C, the condition that $f \otimes id_K$ be a C-local equivalence is easy to check in practice (for example, it is true for every localization of spaces and for every stable localization of spectra). In Section 5.5 we apply these theorems to numerous model categories and localizations of interest, obtaining preservation results for Σ -cofibrant operads such as A_{∞} and E_{∞} in model categories of spaces, spectra, chain complexes, and equivariant spectra. We recover several classical preservation results, and prove several new preservation results. We also provide counterexamples, such as Example 5.7 and Example 5.26 to show that the hypotheses of these theorems are really necessary.

In Section 5.5 we present a lattice of equivariant operads that interpolate between non-equivariant E_{∞} -algebra structure and equivariant E_{∞} -algebra structure. We apply our results to determine which localizations preserve the type of algebraic structure encoded by these operads. This new collection of operads is different from the N_{∞} -operads of 10

(that interpolate between equivariant E_{∞} -algebra structure and genuine commutative structure), and a common generalization of both types of operads is discussed in [27]. Example 5.36 demonstrates that it is possible to preserve equivariant E_{∞} -algebra structure, but fail to preserve genuine commutative structure. This motivates the latter half of the paper.

In Section 5.6 we turn to preservation of structure over non-cofibrant operads, specifically, preservation of commutative monoids. For categories of spectra the phenomenon known as rectification means that preservation of strict commutativity is equivalent to preservation of E_{∞} -structure, but for general model categories (including equivariant spectra) there can be Bousfield localizations that preserve the latter type of structure and not the former. In the companion paper 58 we introduced a condition on a monoidal model category called the *commutative monoid axiom*, that guarantees that the category of commutative monoids inherits a model structure. We build on this work in Section 5.6 by providing conditions on the maps in C so that Bousfield localization preserves the commutative monoid axiom, proving the following theorem.

Theorem 3 Assume \mathcal{M} is a cofibrantly generated monoidal model category satisfying the strong commutative monoid axiom and with domains of the generating cofibrations cofibrant. Suppose that $L_{\mathcal{C}}(\mathcal{M})$ is a monoidal Bousfield localization with generating trivial cofibrations $J_{\mathcal{C}}$. Then $L_{\mathcal{C}}(\mathcal{M})$ satisfies the strong commutative monoid axiom if and only if $\operatorname{Sym}^{n}(f)$ is a \mathcal{C} -local equivalence for all $n \in \mathbb{N}$ and for all $f \in J_{\mathcal{C}}$. This occurs if and only if $\operatorname{Sym}(-)$ preserves \mathcal{C} -local equivalences between cofibrant objects.

The hypotheses of this theorem are difficult to check, requiring complex arguments unraveling the symmetric group actions. However, in Section 5.7, we apply Theorems 1 and 3 to obtain preservation results for commutative monoids in spaces, spectra, chain complexes, and equivariant spectra. We recover classical preservation results, and several new preservation results, including Theorem 5.56, which sharpens and generalizes the main theorem of 30. This is the main application of the paper, and gives a concrete explanation of what goes wrong in Example 5.36 when a specific localization fails to preserve equivariant commutative monoid structure.

Finally, in Section 5.8 we provide conditions so that $L_{\mathcal{C}}(\mathcal{M})$ satisfies the monoid axiom when \mathcal{M} does, proving the following theorem (see Section 5.8 for definitions of the unfamiliar terms, from 5).

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Theorem 4 Suppose \mathcal{M} is a cofibrantly generated, *h*-monoidal, left proper model category such that the (co)domains of I are cofibrant and are finite relative to the *h*-cofibrations and cofibrant objects are flat. Then for any monoidal Bousfield localization $L_{\mathcal{C}}$, the model category $L_{\mathcal{C}}(\mathcal{M})$ satisfies the monoid axiom.

In general, it is difficult to check the hypotheses of this theorem. Fortunately, because Theorem $\boxed{1}$ only requires transferred semi-model structures, Theorem $\boxed{4}$ is not required for preservation, but is required in order to have a comprehensive study of the relationship between left Bousfield localization and monoidal structure, as the monoid axiom is often required for purposes other than transferring model structures. As always, we provide applications of Theorem $\boxed{4}$ to the examples of interest in this paper: spectra, spaces, chain complexes, and equivariant spectra. Roughly half of this paper consists of applications to these examples. In the setting of Theorem $\boxed{4}$, this requires some new results, including a verification that the commonly studied model structures on symmetric spectra are *h*-monoidal, and the introduction of new model structures on equivariant spectra that are combinatorial and *h*-monoidal.

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5.2 Preliminaries

We assume the reader is familiar with basic facts about model categories. Excellent introductions to the subject can be found in [18], [32], and [34]. Throughout the paper we will assume \mathcal{M} is a cofibrantly generated model

category [34], Section 2.1], with generating cofibrations I and generating trivial cofibrations J.

Let *I*-cell denote the class of transfinite compositions of pushouts of maps in *I*, and let *I*-cof denote retracts of such. In order to run the small object argument, we will assume the domains *K* of the maps in *I* (and *J*) are κ -small relative to *I*-cell (resp. *J*-cell), i.e. given a regular cardinal $\lambda \geq \kappa$ and any λ -sequence $X_0 \to X_1 \to \ldots$ formed of maps $X_\beta \to X_{\beta+1}$ in *I*-cell, then the map of sets $\lim_{\substack{\to\\ \beta < \lambda}} \mathcal{M}(K, X_\beta) \to \mathcal{M}(K, \lim_{\substack{\to\\ \beta < \lambda}} X_\beta)$ is a bijection. An object is small if there is some κ for which it is κ -small. See Chapter 10 of [32] for a more thorough treatment of this material. For any object *X* we have a cofibrant replacement $QX \to X$ and a fibrant replacement $X \to RX$.

We will at times also need the hypothesis that \mathcal{M} possesses sets of generating (trivial) cofibrations I and J with domains (hence codomains) cofibrant. This hypothesis is satisfied by all model categories of interest in this paper, but does not come for free, even for combinatorial model categories \mathcal{M} . An example, due to Carlos Simpson, is discussed in Remark 5.22. A method for finding sets I and J with cofibrant domains is given in Lemma 5.19.

Our model category \mathcal{M} will be a closed symmetric monoidal category with product \otimes and unit $S \in \mathcal{M}$. Additionally, we assume the following two axioms, which make \mathcal{M} a monoidal model category [34]. Chapter 4].

- 1 Unit Axiom: For any cofibrant X, the map $QS \otimes X \to S \otimes X \cong X$ is a weak equivalence.
- 2 Pushout Product Axiom: Given any $f: X_0 \to X_1$ and $g: Y_0 \to Y_1$ cofibrations, $f \Box g: X_0 \otimes Y_1 \coprod_{X_0 \otimes Y_0} X_1 \otimes Y_0 \to X_1 \otimes Y_1$ is a cofibration. Furthermore, if, in addition, f or g is a trivial cofibration, then $f \Box g$ is a trivial cofibration.

Note that the pushout product axiom is equivalent to the statement that $-\otimes$ – is a Quillen bifunctor. Furthermore, it is sufficient to check the pushout product axiom on the generating maps I and J, by Proposition 4.2.5 of [34]. When we need \mathcal{M} to be a simplicial model category, we require the SM7 axiom, which is analogous to the pushout product axiom. We refer the reader to Definition 4.2.18 in [34] for details.

We will at times also need to assume that *cofibrant objects are flat* in \mathcal{M} , i.e. that whenever X is cofibrant and f is a weak equivalence then $f \otimes X$ is a weak equivalence. When a monoidal model category satisfies this condition, it is called a *tensor model category* in [22] (Section 12).

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Finally, we remind the reader of the monoid axiom of Definition 3.3 in 50.

Given a class of maps C in \mathcal{M} , let $C \otimes \mathcal{M}$ denote the class of maps $f \otimes id_X$ where $f \in C$ and $X \in \mathcal{M}$. A model category is said to satisfy the *monoid axiom* if every map in (Trivial-Cofibrations $\otimes \mathcal{M}$)-cell is a weak equivalence.

We will be discussing preservation of algebraic structure as encoded by an operad. Let P be an operad valued in \mathcal{M} (for a general discussion of the interplay between operads and homotopy theory see $[\Omega]$). Let P-alg(\mathcal{M}) denote the category whose objects are P-algebras in \mathcal{M} (i.e. admit an action of P) and whose morphisms are P-algebra homomorphisms (i.e. respect the P-action). The free P-algebra functor from \mathcal{M} to P-alg(\mathcal{M}) is left adjoint to the forgetful functor. We will say that P-alg(\mathcal{M}) inherits a model structure from \mathcal{M} if the model structure is transferred across this adjunction, i.e. if a P-algebra homomorphism is a weak equivalence (resp. fibration) if and only if it is so in \mathcal{M} . In Section 4 of $[\Omega]$, an operad P is said to be admissible if P-alg(\mathcal{M}) inherits a model structure in this way.

Finally, we remind the reader about the process of Bousfield localization as discussed in [32]. This is a general machine that starts with a (nice) model category \mathcal{M} and a set of morphisms \mathcal{C} and produces a new model structure $L_{\mathcal{C}}(\mathcal{M})$ on the same category in which maps in \mathcal{C} are now weak equivalences. Furthermore, this is done in a universal way, introducing the smallest number of new weak equivalences as possible. When we say Bousfield localization we will always mean left Bousfield localization. So the cofibrations in $L_{\mathcal{C}}(\mathcal{M})$ will be the same as the cofibrations in \mathcal{M} .

Bousfield localization proceeds by first constructing the fibrant objects of $L_{\mathcal{C}}(\mathcal{M})$ and then constructing the weak equivalences. In both cases this is done via homotopy function complexes map(-, -). If \mathcal{M} is a simplicial or topological model category then one can use the hom-object in *sSet* or *Top*. Otherwise a framing is required to construct the homotopy function complex. We refer the reader to [34] or [32] for details on this process.

An object N is said to be *C*-local if it is fibrant in \mathcal{M} and if for all $g: X \to Y$ in \mathcal{C} , $\operatorname{map}(g, N) : \operatorname{map}(Y, N) \to \operatorname{map}(X, N)$ is a weak equivalence in *sSet*. These objects are precisely the fibrant objects in $L_{\mathcal{C}}(\mathcal{M})$. A map $f: A \to B$ is a *C*-local equivalence if for all N as above, $\operatorname{map}(f, N) : \operatorname{map}(B, N) \to \operatorname{map}(A, N)$ is a weak equivalence. These maps are precisely the weak equivalences in $L_{\mathcal{C}}(\mathcal{M})$.

It is often more convenient to work with left Bousfield localizations that invert a set of cofibrations (i.e. with left derived Bousfield local-

ization). This can always be guaranteed in the following way. For any map f let Qf denote the cofibrant replacement and let \tilde{f} denote the left factor in the cofibration-trivial fibration factorization of Qf. Then \tilde{f} is a cofibration between cofibrant objects and we may define $\tilde{\mathcal{C}} = \{\tilde{f} \mid f \in \mathcal{C}\}$. Localization with respect to $\tilde{\mathcal{C}}$ yields the same result as localization with respect to \mathcal{C} , so our assumption that the maps in \mathcal{C} are cofibrations between cofibrant objects loses no generality. We thus make the following convention.

Convention 5.1 Throughout this paper we assume C is a set of cofibrations between cofibrant objects, and that the model category $L_{\mathcal{C}}(\mathcal{M})$ exists.

The existence of $L_{\mathcal{C}}(\mathcal{M})$ can be guaranteed by assuming \mathcal{M} is left proper and either combinatorial (as discussed in [3]) or cellular (as discussed in [32]). A model category is *left proper* if pushouts of weak equivalences along cofibrations are again weak equivalences. We will make this a standing hypothesis on \mathcal{M} . However, as we have not needed the cellularity or combinatoriality assumptions for our work, outside of the existence of $L_{\mathcal{C}}(\mathcal{M})$, we have decided not to assume them. In this way if a Bousfield localization is known to exist for some reason other than the theory in [32] then our results will be applicable.

5.3 General Preservation Result

In this section we provide a general result regarding when Bousfield localization preserves P-algebras. Essentially, this means that (up to weak equivalence) the localization of a P-algebra is again a P-algebra and the localization morphism is a P-algebra homomorphism. We make this precise in Definition 5.2.

Throughout this section, let \mathcal{M} be a monoidal model category and let \mathcal{C} be a set of maps in \mathcal{M} such that Bousfield localization $L_{\mathcal{C}}(\mathcal{M})$ is a also monoidal model category. On the model category level the functor $L_{\mathcal{C}}$ is the identity. So when we write $L_{\mathcal{C}}$ as a functor we shall mean the composition of derived functors $\operatorname{Ho}(\mathcal{M}) \to \operatorname{Ho}(L_{\mathcal{C}}(\mathcal{M})) \to \operatorname{Ho}(\mathcal{M})$, i.e. $E \to L_{\mathcal{C}}(E)$ is the unit map of the adjunction $\operatorname{Ho}(\mathcal{M}) \leftrightarrows \operatorname{Ho}(L_{\mathcal{C}}(\mathcal{M}))$. In particular, for any E in \mathcal{M} , $L_{\mathcal{C}}(E)$ is weakly equivalent to $R_{\mathcal{C}}QE$ where $R_{\mathcal{C}}$ is a choice of fibrant replacement in $L_{\mathcal{C}}(\mathcal{M})$ and Q is a cofibrant replacement in \mathcal{M} .

Let P be an operad valued in \mathcal{M} . Because the objects of $L_{\mathcal{C}}(\mathcal{M})$

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are the same as the objects of \mathcal{M} , P is also valued in $L_{\mathcal{C}}(\mathcal{M})$. Thus, we may consider P-algebras in both categories and these classes of objects agree (because the P-algebra action is independent of the model structure). We denote the categories of P-algebras by P-alg (\mathcal{M}) and P-alg $(L_{\mathcal{C}}(\mathcal{M}))$. These are identical as categories, but in a moment they will receive different model structures. Inspired by [14], we make the following definition.

Definition 5.2 Assume that \mathcal{M} and $L_{\mathcal{C}}(\mathcal{M})$ are monoidal model categories, P is an operad valued in \mathcal{M} , and U is the forgetful functor from P-algebras to \mathcal{M} . Then $L_{\mathcal{C}}$ is said to *preserve* P-algebras if the following two properties are satisfied.

- 1 When E is a P-algebra there is some P-algebra \tilde{E} such that $U\tilde{E}$ is weakly equivalent in \mathcal{M} to the localization $L_{\mathcal{C}}(UE) := R_{\mathcal{C}}Q(UE)$, where $R_{\mathcal{C}}$ is fibrant replacement in $L_{\mathcal{C}}(\mathcal{M})$ and Q is cofibrant replacement in \mathcal{M} .
- 2 In addition, when E is a cofibrant P-algebra, then there is a choice of Ein P-alg (\mathcal{M}) with $U\widetilde{E}$ local in \mathcal{M} , there is a P-algebra homomorphism $r_E: E \to \widetilde{E}$, and there is a weak equivalence $\beta_E: L_{\mathcal{C}}(UE) \to U\widetilde{E}$ such that $\beta_E \circ l_{UE} \cong Ur_E$ in Ho (\mathcal{M}) .

This definition also appears in **[6]**, where it is compared with other notions of preservation, e.g., preservation in the homotopy category. The condition $\beta_E \circ l_{UE} \cong Ur_E$ means that r_E is a lift of the localization map $l_{UE} : UE \to L_{\mathcal{C}}(UE)$ to the category of algebras, at least up to homotopy.

We are ready to prove the main result of this section. Recall that when we say P-alg(\mathcal{M}) inherits a model structure from \mathcal{M} we mean that this model structure is transferred by the free-forgetful adjunction. In particular, a map of P-algebras f is a weak equivalence (resp. fibration) if and only if f is a weak equivalence (resp. fibration) in \mathcal{M} . A version of this result for semi-model categories will be proven as Corollary 5.5, after semi-model categories are defined. An alternative proof is given in Theorem 5.2 of 6.

Theorem 5.3 Let \mathcal{M} be a monoidal model category such that the Bousfield localization $L_{\mathcal{C}}(\mathcal{M})$ exists and is a monoidal model category. Let P be an operad valued in \mathcal{M} . If the categories of P-algebras in \mathcal{M} and in $L_{\mathcal{C}}(\mathcal{M})$ inherit model structures from \mathcal{M} and $L_{\mathcal{C}}(\mathcal{M})$ then $L_{\mathcal{C}}$ preserves P-algebras.

Proof Let $R_{\mathcal{C}}$ denote fibrant replacement in $L_{\mathcal{C}}(\mathcal{M})$, let $R_{\mathcal{C},P}$ denote fibrant replacement in P-alg $(L_{\mathcal{C}}(\mathcal{M}))$, let Q_P denote cofibrant replacement in P-alg (\mathcal{M}) , and let Q denote cofibrant replacement in \mathcal{M} . We will prove the first form of preservation and our method of proof will allow us to deduce the second form of preservation in the special case where E is a cofibrant P-algebra.

Let *E* be a *P*-algebra, and define $\widetilde{E} = R_{\mathcal{C},P}Q_P(E)$. First, *Q* is the left derived functor of the identity adjunction between \mathcal{M} and $L_{\mathcal{C}}(\mathcal{M})$, and $R_{\mathcal{C}}$ is the right derived functor of the identity, so $L_{\mathcal{C}}(UE) \simeq R_{\mathcal{C}}Q(UE)$. We must therefore show $R_{\mathcal{C}}Q(UE) \simeq UR_{\mathcal{C},P}Q_P(E)$.

The map $q: Q_P E \to E$ is a trivial fibration in P-alg (\mathcal{M}) , hence Uq is a trivial fibration in \mathcal{M} . The map $QUE \to UE$ is also a weak equivalence in \mathcal{M} . Consider the following lifting diagram in \mathcal{M} :

The lifting axiom gives the map $QUE \rightarrow UQ_PE$ and it is necessarily a weak equivalence in \mathcal{M} by the two out of three property.

Since $Q_P E$ is a *P*-algebra in \mathcal{M} it must also be a *P*-algebra in $\mathcal{L}_{\mathcal{C}}(\mathcal{M})$, since the monoidal structure of the two categories is the same. We may therefore apply $R_{\mathcal{C},P}$ to $Q_P E$. We next consider the following lift in $\mathcal{L}_{\mathcal{C}}(\mathcal{M})$, which exists because the left vertical map is a trivial cofibration in $\mathcal{L}_{\mathcal{C}}(\mathcal{M})$ and $UR_{\mathcal{C},P}Q_P E$ is fibrant in $\mathcal{L}_{\mathcal{C}}(\mathcal{M})$:

In this diagram the top horizontal map is U applied to a weak equivalence in P-alg $(L_{\mathcal{C}}(\mathcal{M}))$. Because the model category P-alg $(L_{\mathcal{C}}(\mathcal{M}))$ inherits weak equivalences from $L_{\mathcal{C}}(\mathcal{M})$, this map is a weak equivalence in $L_{\mathcal{C}}(\mathcal{M})$. The left vertical map is also a weak equivalence in $L_{\mathcal{C}}(\mathcal{M})$. Therefore, by the two out of three property, the lift is a weak equivalence in $L_{\mathcal{C}}(\mathcal{M})$. We make use of this map as the horizontal map in the lower right corner of the diagram below.

The top horizontal map $QUE \rightarrow UQ_PE$ in the following diagram is

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the first map we constructed, which was proven to be a weak equivalence in \mathcal{M} . The square in the diagram below is then obtained by applying $R_{\mathcal{C}}$ to that map. In particular, $R_{\mathcal{C}}QUE \rightarrow R_{\mathcal{C}}UQ_{P}E$ is a weak equivalence in $L_{\mathcal{C}}(\mathcal{M})$:

We have shown that both of the bottom horizontal maps are weak equivalences in $L_{\mathcal{C}}(\mathcal{M})$. Thus, by the two out of three property, their composite $R_{\mathcal{C}}QUE \to UR_{\mathcal{C},P}Q_{P}E$ is a weak equivalence in $L_{\mathcal{C}}(\mathcal{M})$. All the objects in the bottom row are fibrant in $L_{\mathcal{C}}(\mathcal{M})$, so these \mathcal{C} -local equivalences are actually weak equivalences in \mathcal{M} .

As E was a P-algebra and Q_P and $R_{\mathcal{C},P}$ are endofunctors on categories of P-algebras, it is clear that $R_{\mathcal{C},P}Q_PE$ is a P-algebra. We have just shown that $L_{\mathcal{C}}(UE)$ is weakly equivalent to this P-algebra, so we are done.

When E is assumed to be a cofibrant P-algebra. We have seen that there is an \mathcal{M} -weak equivalence $R_{\mathcal{C}}QUE \to UR_{\mathcal{C},P}Q_{P}E$, and above we took $R_{\mathcal{C},P}Q_{P}E$ in \mathcal{M} as our representative for $L_{\mathcal{C}}(UE)$ in $\operatorname{Ho}(\mathcal{M})$. Since E is a cofibrant P-algebra, there are weak equivalences $E \leftrightarrows Q_{P}(E)$ in P-alg $(L_{\mathcal{C}}(\mathcal{M}))$. This is because all cofibrant replacements of a given object are weakly equivalent, e.g. by diagram (5.1). So passage to $Q_{P}(E)$ is unnecessary when E is cofibrant, and we take $\tilde{E} = R_{\mathcal{C},P}E$ as our representative for $L_{\mathcal{C}}(E)$. Observe that $U\tilde{E}$ is local as the model structure on P-algebras is transferred. The P-algebra morphism $r_{E}: E \to \tilde{E}$ is just the fibrant replacement map $R_{\mathcal{C},P}$, and lifts the localization map $UE \to L_{\mathcal{C}}(UE)$ in $\operatorname{Ho}(\mathcal{M})$. The comparison β_{E} is the following lift in $L_{\mathcal{C}}(\mathcal{M})$:



The two out of three property guarantees that β_E is a weak equivalence (again using that the model structure on *P*-algebras is transferred), and the diagram above demonstrates that $\beta_E \circ l_{UE} \cong Ur_E$ in Ho(\mathcal{M}). \Box

This theorem alone would not be a satisfactory answer to the question of when $L_{\mathcal{C}}$ preserves *P*-algebras, because there is no clear way to check the hypotheses. For this reason, in the coming sections we will discuss conditions on \mathcal{M} and *P* so that *P*-algebras inherit model structures, and then we will discuss which localizations $L_{\mathcal{C}}$ preserve these conditions (so that P-alg($L_{\mathcal{C}}(\mathcal{M})$) inherits a model structure from $L_{\mathcal{C}}(\mathcal{M})$). One such condition on \mathcal{M} is the monoid axiom. In Section 5.8, we discuss which localizations $L_{\mathcal{C}}$ preserve the monoid axiom. However, it will turn out that the monoid axiom is not necessary in order for our preservation results to apply. This is because the work in 33 and 53 produces semi-model structures on *P*-algebras and these will be enough for our proof above to go through.

Observe that in the proof above we only used formal properties of fibrant and cofibrant replacement functors, and the fact that the model structures on P-algebras were inherited from \mathcal{M} and $L_{\mathcal{C}}(\mathcal{M})$. So the same proof works when P-algebras only form semi-model categories, a notion we define presently. The motivating example is when $\mathcal{D} = P$ -alg is obtained from \mathcal{M} via the general transfer principle for transferring a model structure across an adjunction (see Lemma 2.3 in [50] or Theorem 12.1.4 in [23]) when not all the conditions needed to get a full model structure are satisfied. The following definition is taken from [8] and [7], and is distilled from the definitions in [3] and [53]. Recall that, for a set of morphisms S, inj S refers to the class of morphisms having the right lifting property with respect to S.

Definition 5.4 A semi-model structure on a category \mathcal{D} consists of classes of weak equivalences \mathcal{W} , fibrations \mathcal{F} , and cofibrations \mathcal{Q} satisfying the following axioms:

- M1 Fibrations are closed under pullback.
- M2 The class \mathcal{W} is closed under the two out of three property.
- M3 $W, \mathcal{F}, \mathcal{Q}$ are all closed under retracts.
- M4 i Cofibrations have the left lifting property with respect to trivial fibrations.
- ii Trivial cofibrations whose domain is cofibrant have the left lifting property with respect to fibrations.
- M5 i Every map in \mathcal{D} can be functorially factored into a cofibration followed by a trivial fibration.
- ii Every map whose domain is cofibrant can be functorially factored into a trivial cofibration followed by a fibration.

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If, in addition, \mathcal{D} is bicomplete, then we call \mathcal{D} a semi-model category. \mathcal{D} is said to be cofibrantly generated if there are sets of morphisms I'and J' in \mathcal{D} such that inj I' is the class of trivial fibrations, inj J' is the class of fibrations in \mathcal{D} , the domains of I' are small relative to I'-cell, and the domains of J' are small relative to maps in J'-cell whose domain is cofibrant.

In practice, there is often an adjunction $F : \mathcal{M} \leftrightarrows \mathcal{D} : U$ where \mathcal{M} is a model category, U is a forgetful functor, the weak equivalences and fibrations in \mathcal{D} are maps that forget to weak equivalences and fibrations in \mathcal{M} , and the generating (trivial) cofibrations of \mathcal{D} are maps of the form F(I) and F(J) where $F : \mathcal{M} \to \mathcal{D}$ is the free algebra functor and I and J are the generating (trivial) cofibrations of \mathcal{M} .

Note that the only difference between a semi-model structure and a model structure is that one of the lifting properties and one of the factorization properties requires the domain of the map in question to be cofibrant. Because fibrant and cofibrant replacements are constructed via factorization, (4) implies that every object has a cofibrant replacement and that cofibrant objects have fibrant replacements. So one could construct a fibrant replacement functor that first does cofibrant replacement and then does fibrant replacement. These functors behave as they would in the presence of a full model structure.

We are now prepared to state our preservation result in the presence of only a semi-model structure on P-algebras. When we say P-algebras inherit a semi-model structure we mean with weak equivalences and fibrations reflected and preserved by the forgetful functor. We state this as a corollary because its proof is so similar to that of Theorem 5.3

Corollary 5.5 Let \mathcal{M} be a monoidal model category such that the Bousfield localization $L_{\mathcal{C}}(\mathcal{M})$ exists and is a monoidal model category. Let P be an operad valued in \mathcal{M} . If the categories of P-algebras in \mathcal{M} and in $L_{\mathcal{C}}(\mathcal{M})$ inherit transferred semi-model structures from \mathcal{M} and $L_{\mathcal{C}}(\mathcal{M})$ then $L_{\mathcal{C}}$ preserves P-algebras.

Proof The proof proceeds exactly as the proof of Theorem 5.3 We highlight where care must be taken in the presence of semi-model categories. As remarked above, the cofibrant replacement Q_P in the semi-model category P-alg(\mathcal{M}) exists and the cofibrant replacement map $Q_P E \to E$ is a weak equivalence in P-alg(\mathcal{M}), hence in \mathcal{M} , because the semi-model structure is transferred. Diagram (5.1) is a lifting diagram in \mathcal{M} , so still yields a weak equivalence $QUE \to UQ_P E$.

Next, the fibrant replacement $R_{\mathcal{C}}UQ_{P}E$ is a replacement in the model category $L_{\mathcal{C}}(\mathcal{M})$. The fibrant replacement $Q_{P}E \to R_{\mathcal{C},P}Q_{P}E$ is a fibrant replacement in the semi-model category P-alg $(L_{\mathcal{C}}(\mathcal{M}))$, and exists because $Q_{P}E$ is cofibrant in P-alg $(L_{\mathcal{C}}(\mathcal{M}))$. The resulting object $R_{\mathcal{C},P}Q_{P}E$ is fibrant in P-alg $(L_{\mathcal{C}}(\mathcal{M}))$ hence in $L_{\mathcal{C}}(\mathcal{M})$, since the semimodel structure is transferred. The lift in (5.2) is a lift in $L_{\mathcal{C}}(\mathcal{M})$, and again by the two out of three property in $L_{\mathcal{C}}(\mathcal{M})$ the diagonal map $R_{\mathcal{C}}UQ_{P}E \to UR_{\mathcal{C},P}Q_{P}E$ is a \mathcal{C} -local equivalence.

Next, the map $R_{\mathcal{C}}QUE \to R_{\mathcal{C}}UQ_{P}E$ in (5.3) is a fibrant replacement of the map $QUE \to UQ_{P}E$ in the model category $L_{\mathcal{C}}(\mathcal{M})$, and so the argument that $R_{\mathcal{C}}QUE \to R_{\mathcal{C}}UQ_{P}E$ is a \mathcal{C} -local equivalence remains unchanged. The composite across the bottom of (5.3), $R_{\mathcal{C}}QUE \to$ $UR_{\mathcal{C},P}Q_{P}E$ is a weak equivalence between fibrant objects in $L_{\mathcal{C}}(\mathcal{M})$ and so is a weak equivalence in \mathcal{M} , as in the proof of Theorem 5.3

Finally, for the case of E cofibrant in the semi-model category Palg(\mathcal{M}), note that the localization map $E \to L_{\mathcal{C}}(E)$ is again fibrant replacement $E \to R_{\mathcal{C},P}E$ in P-alg($L_{\mathcal{C}}(\mathcal{M})$). This exists because the domain is cofibrant by assumption. By construction, this map is a Palgebra morphism, as desired. The lift defining β in (5.4) occurs in $L_{\mathcal{C}}(\mathcal{M})$, and the rest of the proof only uses that weak equivalences and fibrations in P-alg($L_{\mathcal{C}}(\mathcal{M})$) forget to weak equivalences and fibrations in $L_{\mathcal{C}}(\mathcal{M})$.

Remark 5.6 Corollary 5.5 has been generalized to algebras over colored operads in 62, and to right Bousfield localization in 59. It has been applied to localizations of Smith ideals in 61.

5.4 Monoidal Bousfield Localizations

In both Theorem 5.3 and Corollary 5.5 we assumed that $L_{\mathcal{C}}(\mathcal{M})$ is a monoidal model category. In this section we provide conditions on \mathcal{M} and \mathcal{C} so that this occurs. First, we provide an example demonstrating that the pushout product axiom can fail for $L_{\mathcal{C}}(\mathcal{M})$, even if it holds for \mathcal{M} . The author learned this example from Mark Hovey.

Example 5.7 It is not true that every Bousfield localization of a monoidal model category is a monoidal model category. Let $R = \mathbb{F}_2[\Sigma_3]$. An R module is simply an \mathbb{F}_2 vector space with an action of the symmetric group Σ_3 . Because R is a Frobenius ring, we may pass from R-mod to

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the stable module category StMod(R) by identifying any two morphisms whose difference factors through a projective module.

Section 2.2 of [34] introduces a model category \mathcal{M} of R-modules whose homotopy category is StMod(R), then proves \mathcal{M} is a finitely generated, combinatorial, stable model category in which all objects are cofibrant (hence, \mathcal{M} is also left proper). Proposition 4.2.15 of [34] proves that for $R = \mathbb{F}_2[\Sigma_3]$, this model category is a monoidal model category because R is a Hopf algebra over \mathbb{F}_2 . The monoidal product of two R-modules is $M \otimes_{\mathbb{F}_2} N$ where R acts via its diagonal $R \to R \otimes_{\mathbb{F}_2} R$.

We now check that cofibrant objects are flat in \mathcal{M} . By the pushout product axiom, $X \otimes -$ is left Quillen. Since all objects are cofibrant, all weak equivalences are weak equivalences between cofibrant objects. So Ken Brown's lemma implies $X \otimes -$ preserves weak equivalences.

Let $f: 0 \to \mathbb{F}_2$, where the codomain has the trivial Σ_3 action. We'll show that the Bousfield localization with respect to f cannot be a monoidal Bousfield localization. First observe that if an object is f-locally trivial then it has no Σ_3 -fixed points, i.e., fails to admit Σ_3 -equivariant maps from \mathbb{F}_2 (the non-identity element would need to be taken to a Σ_3 -fixed point because the Σ_3 -action on \mathbb{F}_2 is trivial).

If the pushout product axiom held in $L_f(\mathcal{M})$ then the pushout product of two *f*-locally trivial cofibrations *g*, *h* would have to be *f*-locally trivial. We will now demonstrate an *f*-locally trivial object *N* for which $N \otimes_{\mathbb{F}_2} N$ is not *f*-locally trivial, so $(\emptyset \to N) \square (\emptyset \to N)$ is not a trivial cofibration in $L_f(\mathcal{M})$.

Define $N \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ where the element (12) sends a = (1, 0) to b = (0, 1)and the element (123) sends a to b and b to c = a + b. The reader can check that (12)(123) acts the same as $(123)^2(12)$, so that this is a welldefined Σ_3 -action. This object N is f-locally trivial, since localization by f kills both factors of \mathbb{F}_2 . However, $N \otimes_{\mathbb{F}_2} N$ is not f-locally trivial because $N \otimes_{\mathbb{F}_2} N$ does admit any map from \mathbb{F}_2 taking the non-identity element of \mathbb{F}_2 to the Σ_3 -invariant element $a \otimes a + b \otimes b + c \otimes c$. Thus, $L_f(\mathcal{M})$ is not a monoidal model category.

There are two ways to get around examples such as the above. One can change the mapping space used to define the localization, e.g., using the derived internal hom rather than a homotopy function complex (as in [3], Definition 4.45]), or one can place hypotheses on the set of morphisms C that we are inverting, to ensure that the C-local equivalences play nicely with the monoidal product. These two approaches are, in fact, equivalent.

A similar program, which amounts to a condition on the set of mor-

phisms C, was conducted in 14, in order to guarantee that localizations of stable model categories commute with suspension. Similarly, a condition on a stable localization to ensure that it is additionally monoidal was given in Definition 6.2 of 2 and the same condition appeared in Theorem 4.46 of 3. This condition states that $C \Box I$ is contained in the C-local equivalences.

Remark 5.8 The counterexample above fails to satisfy the condition that $C \square I$ is contained in the *C*-local equivalences. If this condition were satisfied then I would be contained in the *f*-local equivalences and this would imply all cofibrant objects (hence all objects) are *f*-locally trivial. But $0 \rightarrow N \otimes_{\mathbb{F}_2} N$ is not *f*-locally trivial. Thus, this counterexample has no bearing on the work of [2] or [3].

Remark 5.9 The counterexample demonstrates a general principle that we now highlight. In any *G*-equivariant world, there are multiple spheres due to the different group actions. In the example above, one can suspend by representations of the symmetric group Σ_n , i.e. copies of \mathbb{F}_2 on which Σ_n acts. The 1-point compactification of such an object is a sphere S^n on which Σ_n acts. A localization that kills a representation sphere should not be expected to respect the monoidal structure, because not all acyclic cofibrant objects can be built from one of the representation spheres alone. In particular, $N \otimes N$ will not be in the smallest thick subcategory generated by \mathbb{F}_2 . The point is that the homotopy categories of stable model categories in an equivariant context are not monogenic axiomatic stable homotopy categories in the sense of 37.

Note that this example also demonstrates that the monoid axiom can fail on $L_{\mathcal{C}}(\mathcal{M})$. The author does not know an example of a model category satisfying the pushout product axiom but failing the monoid axiom.

In our applications we will need to know that $L_{\mathcal{C}}(\mathcal{M})$ satisfies the pushout product axiom, the unit axiom, and the axiom that cofibrant objects are flat. We therefore give a name to such localizations, and then we characterize them. The reader is advised to keep Convention 5.1 in mind.

Definition 5.10 A Bousfield localization $L_{\mathcal{C}}$ is said to be a *monoidal* Bousfield localization if $L_{\mathcal{C}}(\mathcal{M})$ satisfies the pushout product axiom, the unit axiom, and the axiom that cofibrant objects are flat.

Theorem 5.11 Suppose that \mathcal{M} is a cofibrantly generated monoidal model category in which cofibrant objects are flat and the domains of

the generating cofibrations are cofibrant. Let I denote the generating cofibrations of \mathcal{M} . Then $L_{\mathcal{C}}$ is a monoidal Bousfield localization if and only if every map of the form $f \otimes id_K$, where f is in C and K is a domain or codomain of a map in I, is a C-local equivalence.

Theorem 5.12 Suppose \mathcal{M} is a cofibrantly generated monoidal model category in which cofibrant objects are flat. Then $L_{\mathcal{C}}$ is a monoidal Bousfield localization if and only if every map of the form $f \otimes id_K$, where f is in \mathcal{C} and K is cofibrant, is a \mathcal{C} -local equivalence.

Note that the condition $\mathcal{C} \Box I \subset \mathcal{C}$ -local equivalences, from [2, 3], implies the condition from these theorems. In fact, one can prove it is equivalent to $L_{\mathcal{C}}(\mathcal{M})$ being a monoidal model category, because \mathcal{C} can be taken to be a set of \mathcal{C} -local trivial cofibrations. However, the condition stated in the theorems above is easier to check. We shall prove Theorem 5.11 in Subsection 5.4.1 and we shall prove Theorem 5.12 in Subsection 5.4.2. These theorems demonstrate precisely what must be done if one wishes to invert a given set of morphisms \mathcal{C} and ensure that the resulting model structure is a monoidal model structure.

Definition 5.13 Suppose \mathcal{M} is left proper, is either cellular or combinatorial, and that the domains of the generating cofibrations are cofibrant. The *smallest monoidal Bousfield localization* which inverts a given set of morphisms \mathcal{C} is the Bousfield localization with respect to the set $\mathcal{C}' = \{\mathcal{C} \otimes id_K\}$ where K runs through the domains and codomains of the generating cofibrations I.

This notion has already been used in 39. The reason for the hypothesis on the domains of the generating cofibrations is to ensure that C' is a set. Requiring left properness and either cellularity or combinatoriality ensures that $L_{C'}$ exists. The smallest Bousfield localization has a universal property, that we now highlight.

Proposition 5.14 Suppose C' is the smallest monoidal Bousfield localization inverting C, and let $j : \mathcal{M} \to L_{\mathcal{C}'}(\mathcal{M})$ be the left Quillen functor realizing the localization. Suppose \mathcal{N} is a monoidal model category with cofibrant objects flat. Suppose $F : \mathcal{M} \to \mathcal{N}$ is a monoidal left Quillen functor such that $\mathbb{L}F$ takes the images of C in $\operatorname{Ho}(\mathcal{M})$ to isomorphisms in $\operatorname{Ho}(\mathcal{N})$. Then there is a unique monoidal left Quillen functor $\delta : L_{\mathcal{C}'}\mathcal{M} \to \mathcal{N}$ such that $\delta j = F$.

Proof Suppose $F : \mathcal{M} \to \mathcal{N}$ is a monoidal left Quillen functor, that \mathcal{N} has cofibrant objects flat, and that $\mathbb{L}F$ takes the images of \mathcal{C} in $Ho(\mathcal{M})$

to isomorphisms in Ho(\mathcal{N}). Then F also takes the images of maps in \mathcal{C}' to isomorphisms in Ho(\mathcal{N}), because for any $f \in \mathcal{C}$ and any cofibrant K, $F(f \otimes K) \cong F(f) \otimes F(K)$ is a weak equivalence in \mathcal{N} . This is because F(K) is cofibrant in \mathcal{N} (as F is left Quillen), cofibrant objects are flat in \mathcal{N} , and F(f) is a weak equivalence in \mathcal{N} by hypothesis.

The universal property of the localization $L_{\mathcal{C}'}$ then provides a unique left Quillen functor $\delta : L_{\mathcal{C}'}\mathcal{M} \to \mathcal{N}$ that is the same as F on objects and morphisms (Theorem 3.3.18 and Theorem 3.3.19 in [32]). In particular, δ is a monoidal functor and $\delta q = Fq : F(QS) \to F(S)$ is a weak equivalence in \mathcal{N} because the cofibrant replacement $QS \to S$ is the same in $L_{\mathcal{C}'}(\mathcal{M})$ as in \mathcal{M} . So δ is a unique monoidal left Quillen functor as required, and the commutativity $\delta j = F$ follows immediately from the definition of δ .

5.4.1 Proof of Theorem 5.11

In this section we will prove Theorem 5.11. We first prove that under the hypotheses of Theorem 5.11, cofibrant objects are flat in $L_{\mathcal{C}}(\mathcal{M})$.

Proposition 5.15 Let \mathcal{M} be a cofibrantly generated monoidal model category in which cofibrant objects are flat and the domains of the generating cofibrations are cofibrant. Let I denote the generating cofibrations of \mathcal{M} . Suppose that every map of the form $f \otimes id_K$, where f is in \mathcal{C} and K is a domain or codomain of a map in I, is a \mathcal{C} -local equivalence. Then cofibrant objects are flat in $L_{\mathcal{C}}(\mathcal{M})$.

Proof We must prove that the class of maps $\{g \otimes X \mid g \text{ is a } C\text{-local} equivalence and X is a cofibrant object} is contained in the <math>C\text{-local}$ equivalences. Let X be a cofibrant object in $L_{\mathcal{C}}(\mathcal{M})$ (equivalently, in \mathcal{M}). Let $g: A \to B$ be a C-local equivalence. To prove $-\otimes X$ preserves C-local equivalences, it suffices to show that it takes $L_{\mathcal{C}}(\mathcal{M})$ trivial cofibrations between cofibrant objects to weak equivalences. This is because we can always do cofibrant replacement on g to get $Qg: QA \to QB$. While Qg need not be a cofibration in general, we can always factor it into $QA \hookrightarrow Z \xrightarrow{\simeq} QB$. By abuse of notation we will continue to use the symbol QB to denote Z, and we will rename the cofibration $QA \to Z$ as Qg since Z is cofibrant and maps via a trivial fibration to B. Smashing with

X gives:



If we prove that $Qg \otimes X$ is a C-local equivalence, then $g \otimes X$ must also be by the two out of three property, since the vertical maps are weak equivalences in \mathcal{M} due to X being cofibrant and cofibrant objects being flat in \mathcal{M} . So we may assume that g is an $L_{\mathcal{C}}(\mathcal{M})$ trivial cofibration between cofibrant objects. Since X is built as a transfinite composition of pushouts of maps in I, we proceed by transfinite induction. For the rest of the proof, let K, K_1 , and K_2 denote domains/codomains of maps in I. These objects are cofibrant in \mathcal{M} by hypothesis, so they are also cofibrant in $L_{\mathcal{C}}(\mathcal{M})$.

For the base case X = K we appeal to Theorem 3.3.18 in [32]. The composition $F = id \circ K \otimes - : \mathcal{M} \to \mathcal{M} \to L_{\mathcal{C}}(\mathcal{M})$ is left Quillen because K is cofibrant. F takes maps in \mathcal{C} to weak equivalences by hypothesis. So Theorem 3.3.18 implies F induces a left Quillen functor $K \otimes - : L_{\mathcal{C}}(\mathcal{M}) \to L_{\mathcal{C}}(\mathcal{M})$. Thus, $K \otimes -$ takes \mathcal{C} -local equivalences between cofibrant objects to \mathcal{C} -local equivalences and in particular takes Qg to a \mathcal{C} -local equivalence. Note that this is the key place in this proof where we use the hypothesis that $L_{\mathcal{C}}$ is a monoidal Bousfield localization. This theorem is the primary tool when one wishes to get from a statement about \mathcal{C} to a statement about all \mathcal{C} -local equivalences.

For the successor case, suppose X_{α} is built from K as above and is flat in $L_{\mathcal{C}}(\mathcal{M})$. Suppose $X_{\alpha+1}$ is built from X_{α} and a map in I via a pushout diagram:



We smash this diagram with $g: A \to B$ and note that smashing a pushout square with an object yields a pushout square.


Because g is a cofibration of cofibrant objects, A and B are cofibrant. Because pushouts of cofibrations are cofibrations, $X_{\alpha} \hookrightarrow X_{\alpha+1}$ for all α . Because X_0 is cofibrant, X_{α} is cofibrant for all α . So all objects above are cofibrant. Furthermore, $g \otimes K_i = g \Box (0 \hookrightarrow K_i)$. Thus, by the pushout product axiom on \mathcal{M} and the fact that cofibrations in \mathcal{M} match those in $L_{\mathcal{C}}(\mathcal{M})$, these maps are cofibrations.

Finally, the maps $g \otimes K_i$ are weak equivalences in $L_{\mathcal{C}}(\mathcal{M})$ by the base case above, while $g \otimes X_{\alpha}$ is a weak equivalence in $L_{\mathcal{C}}(\mathcal{M})$ by the inductive hypothesis. Thus, by the Cube Lemma (Lemma 5.2.6 in [34]), the map $g \otimes X_{\alpha+1}$ is a weak equivalence in $L_{\mathcal{C}}(\mathcal{M})$.

For the limit case, suppose $X = \lim_{\substack{\to \\ \alpha < \beta}} X_{\alpha}$ where each X_{α} is cofibrant and flat in $L_{\mathcal{C}}(\mathcal{M})$. Because each X_{α} is cofibrant, $g \otimes X_{\alpha} = g \Box (0 \hookrightarrow X_{\alpha})$ is still a cofibration. By the inductive hypothesis, each $g \otimes X_{\alpha}$ is also a \mathcal{C} -local equivalence, hence a trivial cofibration in $L_{\mathcal{C}}(\mathcal{M})$. Since trivial cofibrations are always closed under transfinite composition, $g \otimes X =$ $g \otimes \lim_{\rightarrow} X_{\alpha} = \lim_{\rightarrow} (g \otimes X_{\alpha})$ is also a trivial cofibration in $L_{\mathcal{C}}(\mathcal{M})$. \Box

We now pause for a moment to extract the key point in the proof above, where we applied the universal property of Bousfield localization. This is a reformulation Theorem 3.3.18 in 32 that we will need below.

Lemma 5.16 A left Quillen functor $F : \mathcal{M} \to \mathcal{M}$ induces a left Quillen functor $L_{\mathcal{C}}F : L_{\mathcal{C}}(\mathcal{M}) \to L_{\mathcal{C}}(\mathcal{M})$ if and only if for all $f \in \mathcal{C}$, F(f) is \mathcal{C} -local equivalence.

We turn now to the unit axiom.

Proposition 5.17 If \mathcal{M} satisfies the unit axiom then any Bousfield localization $L_{\mathcal{C}}(\mathcal{M})$ satisfies the unit axiom. If cofibrant objects are flat in

 \mathcal{M} then the map $QS \otimes Y \to Y$, induced by cofibrant replacement $QS \to S$, is a weak equivalence for all Y, not just cofibrant Y. Furthermore, for any weak equivalence $f : K \to L$ between cofibrant objects, $f \otimes Y$ is a weak equivalence.

Proof Since $L_{\mathcal{C}}(\mathcal{M})$ has the same cofibrations as \mathcal{M} , it must also have the same trivial fibrations. Thus, it has the same cofibrant replacement functor and the same cofibrant objects. Thus, the unit axiom on $L_{\mathcal{C}}(\mathcal{M})$ follows directly from the unit axiom on \mathcal{M} , because a weak equivalence in \mathcal{M} is in particular a \mathcal{C} -local equivalence.

We now assume cofibrant objects are flat and that Y is an object of \mathcal{M} . Consider the following diagram:



The top map is a weak equivalence by the unit axiom for the cofibrant object QY. The left vertical map is a weak equivalence because cofibrant objects are flat and QS is cofibrant. The right vertical is a weak equivalence by definition of QY. Thus, the bottom arrow is a weak equivalence by the two out of three property.

For the final statement we again apply cofibrant replacement to Y and we get

$$\begin{array}{c} K \otimes QY \longrightarrow L \otimes QY \\ & \downarrow \\ & \downarrow \\ & K \otimes Y \longrightarrow L \otimes Y \end{array}$$

Again the top horizontal map and the vertical maps are weak equivalences because cofibrant objects are flat (for the first use that QX is cofibrant, for the second use that K and L are cofibrant).

We turn now to proving Theorem 5.11 As mentioned in the proof of Proposition 5.15 if h and g are $L_{\mathcal{C}}(\mathcal{M})$ -cofibrations then they are cofibrations in \mathcal{M} and so $h \Box g$ is a cofibration in \mathcal{M} (hence in $L_{\mathcal{C}}(\mathcal{M})$) by the pushout product axiom on \mathcal{M} . To verify the rest of the pushout product axiom on $L_{\mathcal{C}}(\mathcal{M})$ we must prove that if h is a trivial cofibration in $L_{\mathcal{C}}(\mathcal{M})$ and g is a cofibration in $L_{\mathcal{C}}(\mathcal{M})$ then $h \Box g$ is a weak equivalence in $L_{\mathcal{C}}(\mathcal{M})$.

Proposition 5.18 Let \mathcal{M} be a cofibrantly generated monoidal model category in which cofibrant objects are flat and the domains of the generating cofibrations are cofibrant. Let I denote the generating cofibrations of \mathcal{M} . Suppose that every map of the form $f \otimes id_K$, where f is in \mathcal{C} and K is a domain or codomain of a map in I, is a \mathcal{C} -local equivalence. Then $L_{\mathcal{C}}(\mathcal{M})$ satisfies the pushout product axiom.

Proof We have already remarked that the cofibration part of the pushout product axiom on $L_{\mathcal{C}}(\mathcal{M})$ follows from the pushout product axiom on \mathcal{M} , since the two model categories have the same cofibrations. By Proposition 4.2.5 of **34** it is sufficient to check the pushout product axiom on generating (trivial) cofibrations. So suppose $h : X \to Y$ is an $L_{\mathcal{C}}(\mathcal{M})$ trivial cofibration and $g : K \to L$ is a generating cofibration in $L_{\mathcal{C}}(\mathcal{M})$ (equivalently, in \mathcal{M}). Then we must show $h \square g$ is an $L_{\mathcal{C}}(\mathcal{M})$ trivial cofibration

By hypothesis on \mathcal{M} , K and L are cofibrant. Because h is a cofibration, $K \otimes h$ and $L \otimes h$ are cofibrations by the pushout product axiom on \mathcal{M} (because $K \otimes h = (\emptyset \hookrightarrow K) \Box h$). By Proposition 5.15 cofibrant objects are flat in $L_{\mathcal{C}}(\mathcal{M})$. So $K \otimes h$ and $L \otimes h$ are also weak equivalences. In particular, $K \otimes -$ and $L \otimes -$ are left Quillen functors. Consider the following diagram:



The map $L \otimes X \to (K \otimes Y) \coprod_{K \otimes X} (L \otimes X)$ is a trivial cofibration because it is the pushout of a trivial cofibration. Thus, by the two out of three property for the lower triangle, $h \square g$ is a weak equivalence. Since we already knew it was a cofibration (because it is so in \mathcal{M}), this means it is a trivial cofibration. \square

We are now ready to complete the proof of Theorem 5.11.

Proof of Theorem 5.11 We begin with the forwards direction. Suppose $L_{\mathcal{C}}(\mathcal{M})$ satisfies the pushout product axiom and has cofibrant objects flat. Let f be any map in \mathcal{C} . Note that in particular, f is a \mathcal{C} -local equivalence.

Because cofibrant objects are flat, the map $f \otimes K$ is a C-local equivalence for any cofibrant K. So the collection $C \otimes K$ is contained in the C-local equivalences, where K runs through the class of cofibrant objects, i.e. L_C is a monoidal Bousfield localization.

For the converse, we apply our three previous propositions. That cofibrant objects are flat in $L_{\mathcal{C}}(\mathcal{M})$ is the content of Proposition 5.15. The unit axiom on $L_{\mathcal{C}}(\mathcal{M})$ follows from Proposition 5.17 applied to $L_{\mathcal{C}}(\mathcal{M})$. That the pushout product axiom holds on $L_{\mathcal{C}}(\mathcal{M})$ is Proposition 5.18

5.4.2 Proof of Theorem 5.12

We will now prove Theorem 5.12 following the outline above. The proof that cofibrant objects are flat in $L_{\mathcal{C}}(\mathcal{M})$ will proceed just as it did in Proposition 5.15 Proposition 5.17 again implies the unit axiom in $L_{\mathcal{C}}(\mathcal{M})$. Deducing the pushout product axiom on $L_{\mathcal{C}}(\mathcal{M})$ will be more complicated without the assumption on the domains of I. For this reason, we need the following lemma. First, let I' be obtained from the generating cofibrations I by applying any cofibrant replacement Q to all $i \in I$ and then taking the left factor in the cofibration-trivial fibration factorization of Qi. So I' consists of cofibrations between cofibrant objects.

Lemma 5.19 Suppose \mathcal{M} is a left proper model category cofibrantly generated by sets I and J in which the domains of maps in J are small relative to I-cell. Then the sets $I' \cup J$ and J cofibrantly generate \mathcal{M} .

Proof We verify the conditions given in Definition 11.1.2 of [32]. We have not changed J, so the fibrations are still precisely the maps satisfying the right lifting property with respect to J and the maps in J still permit the small object argument because the domains are small relative to J-cell.

Any map that has the right lifting property with respect to all maps in I is a trivial fibration, so will in particular have the right lifting property with respect to all cofibrations, hence with respect to maps in $I' \cup J$. Conversely, suppose p has the right lifting property with respect to all maps in $I' \cup J$. We are faced with the following lifting problem:



Because p has lifting with respect to $I' \cup J$, it has the right lifting property with respect to J. This guarantees us that p is a fibration. Now because \mathcal{M} is left proper, Proposition 13.2.1 in [32] applies to solve the lifting diagram above. In particular, because p has the right lifting property with respect to I', p must have the right lifting property with respect to I. Thus, p is a trivial fibration as desired.

We now turn to smallness. Any domain of a map in J is small relative to J-cell, but in general this would not imply smallness relative to I-cell. We have assumed the domains of maps in J are small relative to I-cell, so they are small relative to $(J \cup I')$ -cell because $J \cup I'$ is contained in I-cell.

Any domain of a map in I' is of the form QA for A a domain of a map in I. We will show QA is small relative to I-cell. As $J \cup I'$ is contained in I-cell this will show QA is small relative to $J \cup I'$. Consider the construction of QA as the left factor in



The map $\emptyset \to QA$ is in *I*-cell, so QA is a colimit of cells (let us say κ_A many), each of which is κ -small where κ is the regular cardinal associated to *I* by Proposition 11.2.5 of [32]. So for any λ greater than the cofinality of max(κ, κ_A), a map from QA to a λ -filtered colimit of maps in *I*-cell must factor through some stage of the colimit because all the cells making up QA will factor in this way. One can find a uniform λ for all objects QA by an appeal to Lemma 10.4.6 of [32].

Remark 5.20 In a combinatorial model category no smallness hypothesis needs to be made because all objects are small. In a cellular model category, the assumption that the domains of J are small relative to cofibrations is included. As these hypotheses are standard when working with left Bousfield localization, we shall say no more about the additional smallness hypothesis placed on J above.

Corollary 5.21 Suppose \mathcal{M} is a left proper model category cofibrantly generated by sets I and J in which the domains of maps in J are small relative to I-cell and are cofibrant. Then there exist a set of generating cofibrations I' with cofibrant domains.

Remark 5.22 Note that this corollary does not say that any left proper,

cofibrantly generated model category has generating sets I and J with cofibrant domains. There is an example due to Carlos Simpson (found on page 199 of 52) of a left proper, combinatorial model category that has no such sets I and J. In this example the cofibrations and trivial cofibrations are the same, so cannot be leveraged against one another in the way we have done above.

We are now prepared to prove Theorem 5.12.

Proof of Theorem 5.12 First, if $L_{\mathcal{C}}$ is a monoidal Bousfield localization then every map of the form $f \otimes id_K$, where $f \in \mathcal{C}$ and K is cofibrant, is a \mathcal{C} -local equivalence. This is because f is a \mathcal{C} -local equivalence and cofibrant objects are flat in $L_{\mathcal{C}}(\mathcal{M})$. We turn now to the converse.

Assume every map of the form $f \otimes id_K$, where $f \in \mathcal{C}$ and K is cofibrant, is a \mathcal{C} -local equivalence. Then cofibrant objects are flat in $L_{\mathcal{C}}(\mathcal{M})$. To see this, let X be cofibrant and define $F(-) = X \otimes -$. Then Lemma 5.16 implies F is left Quillen when viewed as a functor from $L_{\mathcal{C}}(\mathcal{M})$ to $L_{\mathcal{C}}(\mathcal{M})$. So F takes \mathcal{C} -local equivalences between cofibrant objects to \mathcal{C} -local equivalences. By the reduction at the beginning of the proof of Proposition 5.15, this implies F takes all \mathcal{C} -local equivalences to \mathcal{C} -local equivalences.

Next, the unit axiom on $L_{\mathcal{C}}(\mathcal{M})$ follows from the unit axiom on \mathcal{M} , by Proposition 5.17 Finally, we must prove the pushout product axiom holds on $L_{\mathcal{C}}(\mathcal{M})$. As in the proof of Proposition 5.18 Proposition 4.2.5 of 34 reduces the problem to checking the pushout product axiom on a set of generating (trivial) cofibrations. We apply Lemma 5.19 to \mathcal{M} and check the pushout product axiom with respect to this set of generating maps.

As in the case of Theorem 5.11, let $h: X \to Y$ be a trivial cofibration in $L_{\mathcal{C}}(\mathcal{M})$ and let $g: K \to L$ be a generating cofibration. By the lemma, the map g is either a cofibration between cofibrant objects or a trivial cofibration in \mathcal{M} . If the former, then the proof of Proposition 5.18 goes through verbatim and proves that $h \square g$ is an $L_{\mathcal{C}}(\mathcal{M})$ -trivial cofibration, since cofibrant objects are flat in $L_{\mathcal{C}}(\mathcal{M})$. If the latter, then because g is a trivial cofibration in \mathcal{M} and h is a cofibration in \mathcal{M} we may apply the pushout product axiom on \mathcal{M} to see that $h \square g$ is a trivial cofibration in \mathcal{M} (hence in $L_{\mathcal{C}}(\mathcal{M})$ too). This completes the proof of the pushout product axiom on $L_{\mathcal{C}}(\mathcal{M})$.

Remark 5.23 The use of the lemma demonstrates that this proposition proves something slightly more general. Namely, if \mathcal{M} is cofibrantly

generated, left proper, has cofibrant objects flat, and the class of cofibrations is closed under pushout product then \mathcal{M} satisfies the pushout product axiom.

Additionally, one could also prove the forwards direction in the theorem using only that $L_{\mathcal{C}}(\mathcal{M})$ satisfies the pushout product axiom. For any cofibrant K we have a cofibration $\phi_K : \emptyset \hookrightarrow K$. Note that for any $f \in \mathcal{C}$, $f \otimes K = f \Box \phi_K \subset \mathcal{C}$ -local equivalences, because f is a trivial cofibration in $L_{\mathcal{C}}(\mathcal{M})$.

We record this remark because in the future we hope to better understand the connection between monoidal Bousfield localizations and the closed localizations that appeared in 14, and this remark may be useful.

5.5 Preservation of algebras over Σ -cofibrant operads

In this section we will provide several applications of the results in the previous section. We remind the reader that for operads valued in \mathcal{M} , a map of operads $A \to B$ is said to be a trivial fibration if $A_n \to B_n$ is a trivial fibration in \mathcal{M} for all n. An operad P is said to be *cofibrant* if the map from the initial operad into P has the left lifting property in the category of operads with respect to all trivial fibrations of operads. An operad P is said to be Σ -cofibrant if it has this left lifting property only in the category of symmetric sequences. The E_{∞} -operads considered in [44] are Σ -cofibrant precisely because the n^{th} space is assumed to be an $E\Sigma_n$ space, where Σ_n is the symmetric group.

We begin with a theorem due to Markus Spitzweck, proven as Theorem 5 in 53 and as Theorem A.8 in 26, that makes it clear that the hypotheses of Corollary 5.5 are satisfied when $L_{\mathcal{C}}$ is a monoidal Bousfield localization and P is a cofibrant operad.

Theorem 5.24 Suppose P is a Σ -cofibrant operad and \mathcal{M} is a monoidal model category. Then P-alg is a semi-model category.

This theorem, applied to both \mathcal{M} and $L_{\mathcal{C}}(\mathcal{M})$ (if the localization is monoidal), endows the categories of P-algebras in \mathcal{M} and $L_{\mathcal{C}}(\mathcal{M})$ with inherited semi-model structures. By Corollary 5.5, monoidal Bousfield localizations preserve algebras over Σ -cofibrant operads. In particular, monoidal localizations preserve A_{∞} and E_{∞} -algebras in \mathcal{M} , since these algebras are encoded by A_{∞} and E_{∞} -operads P that are Σ -cofibrant (and

weakly equivalent to Ass and Com respectively in the category $Coll(\mathcal{M})$). A definition of A_{∞} and E_{∞} operads over general model categories \mathcal{M} can be found in [53], Section 8], among other places. When \mathcal{M} is a category of spectra we are free to work with operads valued in spaces because the Σ^{∞} functor will take a (Σ -cofibrant) space-valued operad to a (Σ -cofibrant) spectrum-valued operad with the same algebras.

5.5.1 Spaces and Spectra

We now provide examples demonstrating the power of Theorem 5.3. For topological spaces the situation is especially nice. We will always work in the context of pointed spaces, with the Quillen model structure.

Proposition 5.25 Let \mathcal{M} be the model category of (pointed) simplicial sets or k-spaces. Every Bousfield localization of \mathcal{M} is a monoidal Bousfield localization.

Proof For a review of the monoidal model structures on spaces and simplicial sets see Chapter 4 of **34**. Both are cellular, left proper, monoidal model categories with cofibrant objects flat and the domains of the generating cofibrations cofibrant.

For $\mathcal{M} = sSet$ or $sSet_*$, we can simply rely on Theorem 4.1.1 of [32], which guarantees that $L_{\mathcal{C}}(\mathcal{M})$ is a simplicial model category. The pushout product axiom is equivalent to the SM7 axiom for sSet, so this proves $L_{\mathcal{C}}(\mathcal{M})$ is a monoidal model category and hence that $L_{\mathcal{C}}$ is monoidal. There is also an elementary proof of this fact, obtained from the proof below by replacing F(-, -) everywhere by map(-, -).

We turn now to $\mathcal{M} = Top_*$. By definition, any Bousfield localization $L_{\mathcal{C}}$ will be a monoidal Bousfield localization as soon as we show $\mathcal{C} \wedge S^n_+$ is contained in the \mathcal{C} -local equivalences (the codomains of the generating cofibrations are contractible, so do not matter). As remarked in the discussion below Definition 4.1 in [39], for topological model categories Bousfield localization with respect to a set of cofibrations can be defined using topological mapping spaces rather than simplicial mapping spaces (at least when all maps in \mathcal{C} are cofibrations). Let F(X, Y) denote the space of based maps $X \to Y$.

We will make use of Proposition 3.2 in [35], a version of which states that because Top is left proper and cofibrantly generated, a map f is a weak local equivalence if and only if F(T, f) is a weak equivalence of topological spaces for all T in the (co)domains of the generating cofibrations I in Top_* .

Now consider the following equivalent statements, where T runs through the domains and codomains of generating cofibrations.

f is a C-local equivalence iff F(f, Z) is a weak equivalence for all C-local Z iff F(T, F(f, Z)) is a weak equivalence for all C-local Z and all T iff $F(T \wedge f, Z)$ is a weak equivalence for all C-local Z (by adjointness) iff $T \wedge f$ is a C-local equivalence

This proves that the class of C-local equivalences is closed under smashing with a domain or codomain of a generating cofibration, so $L_{\mathcal{C}}$ is a monoidal Bousfield localization. An analogous proof works for $\mathcal{M} = Top$.

The reader may wonder whether all Bousfield localizations preserve algebras over cofibrant operads in general model categories \mathcal{M} , i.e. whether all Bousfield localizations are monoidal. This is false, as demonstrated by the following example, from Section 6 in 14.

Example 5.26 Let \mathcal{M} be symmetric spectra, S-modules or orthogonal spectra. Recall that in pointed spaces, the n^{th} Postnikov section functor P_n is the Bousfield localization L_f corresponding to $f: S^{n+1} \to *$, for $n \geq 0$. Applying Σ^{∞} gives a map of spectra and we again denote by P_n the Bousfield localization with respect to this map. The Bousfield localization P_{-1} on \mathcal{M} does not preserve A_{∞} -algebras. If R is a non-connective A_{∞} -algebra then the unit map $\nu: S \to P_{-1}R$ is null because $\pi_0(P_{-1}R) = 0$. Thus, $P_{-1}R$ cannot admit a ring spectrum structure (not even up to homotopy) because $S \wedge P_{-1}R \to P_{-1}R \wedge P_{-1}R \to P_{-1}R$ is not a homotopy equivalence as it would have to be for $P_{-1}R$ to be a homotopy ring. It follows that the model category $P_{-1}\mathcal{M}$ fails the pushout product axiom, because if $P_{-1}\mathcal{M}$ satisfied the pushout product axiom, because if $P_{-1}\mathcal{M}$ such that the pushout product axiom, here 5.24, and Corollary 5.5 would imply that P_{-1} preserves A_{∞} -algebras.

In [14], examples of the sort above are prohibited by assuming that L-equivalences are closed under the monoidal product. It is then shown in Theorem 6.5 that for symmetric spectra this property is implied if the localization is *stable*, i.e. $L \circ \Sigma \simeq \Sigma \circ L$. We now compare our requirement that $L_{\mathcal{C}}$ be a monoidal Bousfield localization to existing results regarding preservation of monoidal structure.

Proposition 5.27 Let \mathcal{M} be a stable model category. Then every

monoidal Bousfield localization is stable. In a monogenic setting such as spectra, every stable localization is monoidal.

This is clear, since suspending is the same as smashing with the suspension of the unit sphere. The Postnikov section is clearly not stable, and indeed the counterexample above hinges on the fact that the section has truncated the spectrum by making trivial the degree in which the unit must live. Stable localizations preserve cofiber sequences, but P_{-1} does not. Under the hypothesis that localization respects the monoidal product, Theorem 6.1 of 14 proves that cofibrant algebras over a cofibrant colored operad valued in $sSet_*$ or Top_* are preserved. Theorem 5.3 recovers this result in the case of operads, and improves on it by extending the class of operads so that they do not need to be valued in $sSet_*$ or Top_* , by discussing preservation of non-cofibrant algebras, by weakening the cofibrancy required of the operad to Σ -cofibrancy (using Theorem 5.24 above), and by potentially weakening the hypothesis on the localization. A different generalization of 14 has been given in 26.

Proposition 5.28 Every Bousfield localization for which the local equivalences are closed under \otimes is a monoidal Bousfield localization, but the converse fails.

Proof To see why this is true, consider the maps id_K as *L*-equivalences when testing whether or not $id_K \otimes C$ is a *C*-local equivalence. To see that the converse fails, take *C* to be the generating trivial cofibrations of any cofibrantly generated model category in which the weak equivalences are not closed under \otimes .

Thus, our hypothesis on a monoidal Bousfield localization is strictly weaker than requiring *L*-equivalences to be closed under \otimes . Theorems 5.11 and 5.12 demonstrate that the hypothesis that $\mathcal{C} \otimes id_K$ is contained in the \mathcal{C} -local equivalences is best-possible, since $L_{\mathcal{C}}$ is a monoidal Bousfield localization if and only if this property holds, and without the pushout product axiom on $L_{\mathcal{C}}(\mathcal{M})$ the question of preservation of algebras under localization is not even well-posed. Note that cofibrant objects are flat for symmetric spectra by 5.3.10 in 38.

Remark 5.29 In light of the Postnikov Section example, the argument of Proposition 5.25 must break down for spectra. The precise place where the argument fails is the passage through map(T, map(f, Z)). In spectra, this expression has no meaning, because T is a spectrum but map(f, Z)is a space. So the argument of Proposition 5.25 relies precisely on the

fact that $\mathcal{M} = sSet$ (or $\mathcal{M} = Top$ in the topological case), so that the SM7 axiom for \mathcal{M} is precisely the same as the pushout product axiom.

Theorem 5.3 and Theorem 5.24 combine to prove that any monoidal Bousfield localization of spectra preserves A_{∞} and E_{∞} -algebras. In particular, A_{∞} and E_{∞} -algebras are preserved by stable Bousfield localizations such as L_E where E is a homology theory. So our results recover Theorems VIII.2.1 and VIII.2.2 of 19.

5.5.2 Equivariant Spectra

In order to specialize Corollary 5.5 to the case of *G*-equivariant spectra, where G is a compact Lie group, we must first understand the generating cofibrations. For Top^G , the (co)domains of maps in I take the form $((G/H) \times S^{n-1})_+$ and $((G/H) \times D^n)_+$ for H a closed subgroup of G, by Definition 1.1 in 42. For G-spectra, the situation is more complicated. We first fix a G-universe, i.e., a set \mathcal{U} of orthogonal G-representations closed under direct sums and summands, containing the trivial one-dimensional representation. For any finite dimensional orthogonal G-representation Wthere is an evaluation functor $Ev_W : S^G \to Top^G$. This functor has a left adjoint F_W (see Proposition 3.1 in 39 for more details). The (co)domains of maps in I take the form $F_W((G/H)_+ \wedge S^{n-1}_+)$ and $F_W((G/H)_+ \wedge D^n_+)$ by Definition 1.11 in 42, where W runs through our fixed G-universe. The latter are contractible, and so smashing with them does not make a difference. Observe that the domains of the generating cofibrations are cofibrant, since on the space level they are G-CW complexes, and on the spectra level they are created by the left Quillen functors F_W .

Left Bousfield localization yields the stable model structure, which we denote S^G . That S^G is a monoidal model category with cofibrant objects flat is verified in Proposition III.7.3 of [42], and may also be deduced from Corollary 4.4 in [39]. We could also work with the positive stable model structure S^G_+ , which has the same weak equivalences as S^G , but cofibrations defined by functors F_W where $W^G \neq 0$. The proof that these model structures are left proper and cellular can be found in the appendix of [27]. For $\mathcal{M} = S^G$ or S^G_+ , our preservation result (Corollary 5.5) together with Theorem [5.11] becomes:

Theorem 5.30 Let G be a compact Lie group. In S^G (resp. S^G_+), a Bousfield localizations L_C is monoidal if and only if each morphism in the set $C \wedge F_W((G/H)_+ \wedge S^{n-1}_+)$ is a C-local equivalence for all closed subgroups H of G, for all W in the universe (resp. all W such that $W^G \neq 0$), and

for all n (resp. n > 0). Furthermore, such localizations preserve P-algebra structures for any Σ -cofibrant P, including any equivariant E_{∞} -operad P.

Here a G-operad P is called equivariant E_{∞} if it is Σ -free, the spaces P(n) are G-CW complexes, and $P(n)^H \simeq *$ for all closed subgroups H of G. These operads are Σ -cofibrant with respect to the model structure on G-operads transferred from the model structure on G-collections $\prod_{n\geq 0} (Top^G)^{\Sigma_n}$. Here Top^G has the usual G-equivariant model structure, and $(Top^G)^{\Sigma_n}$ has the projective model structure, where Σ_n is the sym-

and $(Top^{\circ})^{\mathbb{Z}_n}$ has the projective model structure, where Σ_n is the symmetric group. Hence, a morphism $f = (f_n)$ is a weak equivalence (resp. fibration) if f_n^H is a weak equivalence (resp. fibration) in Top for every closed subgroup H of G and every n. Note that these operads do not encode genuine equivariant commutativity. To do that, subgroups of $G \times \Sigma_n$ would need to be considered. In particular, there is not a Quillen equivalence between algebras over an E_{∞} -operad and commutative equivariant ring spectra. The N_{∞} -operads of Π were introduced to encode genuine commutativity in a homotopy coherent way (relative to a choice of a collection of families of subgroups of $G \times \Sigma_n$ for $n \geq 0$) and were constructed in [27] as cofibrant replacements of the operad Com in various model structures on the category of G-operads, corresponding to the choice of a collection of families of subgroups of $G \times \Sigma_n$.

Ignoring suspensions, Theorem 5.30 demonstrates that monoidal Bousfield localizations are precisely the ones for which $L_{\mathcal{C}}$ respects smashing with $(G/H)_+$ for all subgroups H. In this light, Theorem 5.30 can be seen as a generalization of Proposition 5.27, saying that if $L_{\mathcal{C}}$ respects stabilization with respect to all the objects $F_W((G/H)_+ \wedge S^{n-1}_+)$ then $L_{\mathcal{C}}$ is monoidal. We think of these monoidal localizations as the ones that can 'see' the information of all subgroups. A natural question is: what if $L_{\mathcal{C}}$ can only 'see' the information of some subgroups H? To answer this question, we must consider the following model structures, from Theorem 6.3 in [42] (on spectra either the stable or positive stable model structure can be used):

Definition 5.31 Let \mathcal{F} be a family of closed subgroups of G, i.e. a nonempty set of subgroups closed under conjugation and taking subgroups. Then the \mathcal{F} -fixed point model structure on pointed G-spaces is a cofibrantly generated model structure in which a map f is a weak equivalence (resp. fibration) if and only if f^H is a weak equivalence (resp. fibration) in Top for all $H \in \mathcal{F}$. We will denote this model structure by $Top^{\mathcal{F}}$. The

generating (trivial) cofibrations are $(G/H \times g)_+$, where g is a generating (trivial) cofibration of topological spaces, and $H \in \mathcal{F}$.

The corresponding cofibrantly generated model structure on G-spectra will be denoted $\mathcal{S}^{\mathcal{F}}$. Again, weak equivalences (resp. fibrations) are maps fsuch that f^H is a weak equivalence (resp. fibration) of orthogonal spectra for all $H \in \mathcal{F}$. The generating (trivial) cofibrations are $F_W((G/H)_+ \wedge g)$ as H runs through \mathcal{F} , g runs through the generating (trivial) cofibrations of spaces, and W runs through our fixed G-universe \mathcal{U} .

With the generating cofibrations in hand, Theorem 5.11 implies that monoidal Bousfield localizations in $S^{\mathcal{F}}$ are characterized by the property that $\mathcal{C} \wedge (G/H)_+$ is a \mathcal{C} -local equivalence for all $H \in \mathcal{F}$ (again, ignoring suspensions). One can also define \mathcal{F} -fixed point semi-model structures $Oper^{\mathcal{F}}$ on the category of G-operads by applying the general machinery of Theorem 12.2.A in [23]. Indeed, [27] demonstrates how to define full model structures on these categories of operads, for even more general families of subgroups.

Definition 5.32 Let $E_{\infty}^{\mathcal{F}}$ be the cofibrant replacement for the operad *Com* in the \mathcal{F} -fixed point semi-model structure on *G*-operads.

These operads form a lattice (ordered by family inclusion) interpolating between non-equivariant E_{∞} (corresponding to the family $\mathcal{F} = \{e\}$) and equivariant E_{∞} (corresponding to the family $\mathcal{F} = \{All\}$ and denoted E_{∞}^{G}). To understand the algebraic structure encoded by $E_{\infty}^{\mathcal{F}}$, we pause to introduce some new terminology.

Given a G-space X and a closed subgroup H, one may restrict the G action to H and obtain an H-space denoted $res_H(X)$. This association is functorial and lifts to a functor $res_H : S^G \to S^H$. This restriction functor has a left adjoint $G_+ \wedge_H (-)$, the induction functor. We refer the reader to Section 2.2.4 of [31] for more details. If one shifts focus to commutative monoids $Comm_G$ in S^G (equivalently to genuine E_{∞} -algebras) then there is again a restriction functor $res_H : Comm_G \to Comm_H$ and it again has a left adjoint functor $N_H^G(-)$ called the norm. This functor is discussed in Section 2.3.2 of [31].

An $E_{\infty}^{\mathcal{F}}$ -algebra X has a multiplicative structure on $res_H(X)$ (compatible with the transfers) for every $H \in \mathcal{F}$. However, $N_H^G(res_H(X))$ need not have a multiplicative structure. This is related to Example 5.37 below. These operads $E_{\infty}^{\mathcal{F}}$ have been generalized and further studied in [27], which includes a comparison between these operads and the N_{∞} -operads of [10], results about transferred model structures, and rec-

tification results. For now we will focus on how $E_{\infty}^{\mathcal{F}}$ -algebra structure interacts with Bousfield localization. First, observe that both $Oper^{\mathcal{F}}$ and $\mathcal{S}^{\mathcal{F}}$ are $Top^{\mathcal{F}}$ -model structures (in the sense of Definition 4.2.18 in [34]) and the cofibrancy of $E_{\infty}^{\mathcal{F}}$ is relative to the \mathcal{F} -model structure. Thus, from a model category theoretic standpoint, $E_{\infty}^{\mathcal{F}}$ -algebras are best viewed in $\mathcal{S}^{\mathcal{F}}$. The following two theorems also have formulations for the positive stable model structure, in analogy with Theorem [5.30], that we leave to the reader.

Theorem 5.33 Let $\mathcal{M} = \mathbb{S}^G$ and let \mathfrak{F} be a family of closed subgroups of G. Assume $F_W((G/H)_+ \wedge S^{n-1}_+) \wedge \mathcal{C}$ is contained in the \mathcal{C} -local equivalences for all $H \in \mathfrak{F}$, for all n, and for all W in the universe. Then $L_{\mathcal{C}}$ preserves $E_{\mathfrak{F}}^{\mathfrak{S}}$ -structure.

Localizations of the form above are \mathcal{F} -monoidal but not necessarily G-monoidal. For this reason, when $X \in E^G_{\infty}$ -alg, $L_{\mathcal{C}}(X)$ has $E^{\mathcal{F}}_{\infty}$ -algebra structure but may not have E^G_{∞} -algebra structure. More generally, we have the following result, encoding the fact that if we work in $\mathcal{S}^{\mathcal{K}}$ rather than \mathcal{S}^G , then localizations are compatible with both \mathcal{K} and \mathcal{F} . Because there are now two families involved, the localization will preserve algebraic structure corresponding to the meet of these two families in the lattice of families.

Theorem 5.34 Let \mathcal{M} be the \mathcal{K} -fixed point model structure on the category of G-spectra and let \mathcal{K}' be a subfamily of \mathcal{K} . Assume the morphisms $F_W((G/H)_+ \wedge S^{n-1}_+) \wedge \mathcal{C}$ are contained in the \mathcal{C} -local equivalences for all $H \in \mathcal{K}'$, for all n, and for all W in the universe. Then $L_{\mathcal{C}}$ takes any $E_{\infty}^{\mathcal{F}}$ -algebra to an $E_{\infty}^{\mathcal{F}\cap\mathcal{K}'}$ -algebra.

Proof In order to apply Corollary 5.5 first forget to the model structure $S^{\mathcal{F}\cap\mathcal{K}'}$ and observe that any $E^{\mathcal{F}}_{\infty}$ -algebra is sent to a $E^{\mathcal{F}\cap\mathcal{K}'}_{\infty}$ -algebra. The hypothesis on $L_{\mathcal{C}}$ guarantees that $L_{\mathcal{C}}$ is a monoidal Bousfield localization with respect to the $\mathcal{F}\cap\mathcal{K}'$ model structure, and so $E^{\mathcal{F}\cap\mathcal{K}'}_{\infty}$ is preserved. \Box

Remark 5.35 It is easy to produce examples of localizations $L_{\mathcal{C}}$ that reduce $E_{\infty}^{\mathcal{F}}$ -algebra structure to $E_{\infty}^{\mathcal{F}'}$ -algebra structure for any families $\mathcal{F}' \subsetneq \mathcal{F}$ of closed subgroups of G, by generalizing the Postnikov section 5.26 For every closed subgroup $H \in \mathcal{F} \setminus \mathcal{F}'$, consider a truncation of the spectrum $F_W((G/H)_+ \wedge S_+^{n-1})$. Localizing with respect to the wedge of these truncation maps will take $E_{\infty}^{\mathcal{F}}$ -algebras to $E_{\infty}^{\mathcal{F}'}$ -algebras, by Theorem 5.34

Together, Theorem 5.34 and Remark 5.35 resolve the preservation

question for operads in the lattice $E_{\infty}^{\mathcal{F}}$. As expected, preservation of lesser algebraic structure comes down to requiring a less stringent condition on the Bousfield localization. The least stringent condition is for $\mathcal{F} = \{e\}$ and recovers the notion of a stable localization (i.e. one which is monoidal on the category of spectra after forgetting the *G*-action). However, because none of the operads $E_{\infty}^{\mathcal{F}}$ rectify with respect to the *Com* operad, we do not have preservation results for commutative equivariant ring spectra. For the remainder of the section, we discuss localizations that preserve $E_{\infty}^{\mathcal{F}}$ -algebra structure but fail to preserve commutative structure. We begin with the example that motivated this paper, which the author learned from a talk given by Mike Hill at Oberwolfach (the proceedings can be found in [29]). A similar example appeared in [45]. This example will be generalized in Example [5.37] below.

Example 5.36 There are localizations that destroy genuine commutative structure but that preserve equivariant E_{∞} -algebra structure. For this example, let G be a (non-trivial) finite group. Consider the reduced real regular representation $\overline{\rho}$ obtained by taking the quotient of the real regular representation ρ by the trivial representation. We write $\overline{\rho}_G = \rho_G - 1$ where 1 means the trivial representation $\mathbb{R}[e]$. Taking the one-point compactification of this representation yields a representation sphere $S^{\overline{\rho}}$. There is a natural inclusion $a_{\overline{\rho}} : S^0 \to S^{\overline{\rho}}$ induced by the inclusion of the trivial representation into $\overline{\rho}$. Consider the spectrum $E = \mathbb{S}[a_{\overline{\rho}}^{-1}]$ obtained from the unit \mathbb{S} (certainly a commutative algebra in \mathbb{S}^G) by localization with respect to $a_{\overline{\rho}}$. We will show that this spectrum cannot be commutative.

First, for any proper H < G, the restriction $\rho_G|_H$ is $[G:H]\rho_H$, so $\overline{\rho}_G|_H = [G:H]\overline{\rho}_H + ([G:H]1-1)$. Let k = [G:H] - 1. Observe that $res_H S^{\overline{\rho}_G} = (S^{\overline{\rho}_H})^{\#[G:H]} \wedge S^k$. It follows that $res_H(E)$ is contractible, because k > 0.

If E were a commutative equivariant ring spectrum, then the counit of the norm-restriction adjunction would provide a ring homomorphism $N_H^G res_H(E) \rightarrow E$. But the domain is contractible for every proper subgroup H because $res_H(E)$ is contractible. This cannot be a ring map unless E to be contractible, and we know E is not contractible, because its G-fixed points are not contractible.

Justin Noel has pointed out that the localization in Example 5.36 is smashing, hence monoidal (this is clear from the reformulation in Example 5.37) but was not clear to the author from the formulation above). It follows that the localization preserves E_{∞} -algebra structure,

by Theorem 5.30, and hence takes commutative monoids to E_{∞} -algebras. Because *any* such *H* will lead to a failure of $L(\mathbb{S}) = E$ to be commutative, Example 5.36 is in some sense maximally bad. We now leverage this observation to generalize Example 5.36.

Recall from Definition 5.7 of [27] that an \mathcal{F} - N_{∞} -operad P is a generalization of an N_{∞} -operad, where P-algebras have multiplicative norms for all $H \in \mathcal{F}$. The formulation of Example 5.37 matches the presentation from [29] for the case when G is finite, \mathcal{F} is the family of all subgroups of G, and \mathcal{P} the family of proper subgroups.

Example 5.37 Let G be a compact Lie group and \mathcal{F} a family of closed subgroups of G. If X is an algebra over an \mathcal{F} - N_{∞} -operad then there is a localization L sending X to an $E_{\infty}^{\mathcal{F}}$ -algebra. Consider the cofiber sequence $E\mathcal{P}_+ \to S^0 \to \widetilde{E}\mathcal{P}$ for any family \mathcal{P} properly contained in \mathcal{F} . Recall the fixed-point property of the space $E\mathcal{P}$ (discussed very nicely in Section 7 of [49]) and deduce:

$$(E\mathcal{P}_+)^H \simeq \begin{cases} *_+ = S^0 & \text{if } H \in \mathcal{P} \\ \varnothing_+ = * & \text{if } H \notin \mathcal{P} \end{cases}$$

For all H, the H-fixed points of S^0 are S^0 , so that the cofiber obtained by mapping this space into S^0 satisfies the following fixed-point property

$$(\widetilde{E}\mathfrak{P})^H \simeq \begin{cases} * & \text{if } H \in \mathfrak{P} \\ S^0 & \text{if } H \notin \mathfrak{P} \end{cases}$$

Now apply Σ^{∞}_+ to the map $S^0 \to \widetilde{E}\mathcal{P}$. If G is a finite group, and \mathcal{F} is the family of all subgroups of G, then the resulting map $\mathbb{S} \to E$ is the same localization map considered in Example 5.36 (see Section 7 of 49). Returning to the general case, note that E is not contractible because $E\mathcal{P}_+$ is not homotopy equivalent to S^0 (since \mathcal{P} is properly contained in \mathcal{F}), though $res_H(E\mathcal{P}_+)$ is homotopy equivalent to $res_H(S^0)$ for any $H \in \mathcal{P}$. In this formulation it is clear that the map $\mathbb{S} \to E$ is a nullification that kills all maps out of the induced cells $G_+ \wedge_H (H/K)_+ \cong (G/H)_+$ for all $H \in \mathcal{P}$.

This localization is monoidal with respect to the \mathcal{F} -model structure, so E is still an $E_{\infty}^{\mathcal{F}}$ -algebra by Theorem 5.34. When G is finite, Example 5.37 makes it clear that the localization is simply inverting a homotopy element (namely: the Euler class $a_{\overline{\rho}}$ discussed in Section 2.6.3 of [31]). The presentation in Example 5.37 has several benefits of its own: it generalizes to compact Lie groups G, it demonstrates that a smaller localization

than Example 5.36 is needed to destroy \mathcal{F} - N_{∞} -algebra structure rather than N_{∞} -algebra structure, and it shows how localization can reduce one's place in the lattice of \mathcal{F} - N_{∞} -algebras without reducing it all the way down to $E_{\infty}^{\mathcal{F}}$. To see this, observe that the localization E can still admit some multiplicative norms, for subgroups $H \in \mathcal{F} \setminus \mathcal{P}$. Hence, Ecan still be a \mathcal{K} - N_{∞} -algebra if the family \mathcal{K} only intersects \mathcal{P} in the trivial subgroup (using the fact that \mathbb{S} is an N_{∞} -algebra for all choices of families of subgroups). Examples such as that of II.2.3 [40] can be used for this purpose.

Motivated by Example 5.37, we devote the next two sections to determining when a left Bousfield localization must preserve commutative structure. We will see that the key compatibility condition required is that the maps in C respect the free commutative monoid functor. In the case of equivariant spectra, this will imply that C respects the multiplicative norm functors.

5.6 Bousfield Localization and Commutative Monoids

In this section we turn to the interplay between monoidal Bousfield localizations and commutative monoids, i.e. algebras over the (non-cofibrant) operad *Com*. In 58, the following theory is developed as Definition 3.1, Theorem 3.2, and Corollary 3.8. Here Σ_n is the symmetric group.

Definition 5.38 A monoidal model category \mathcal{M} is said to satisfy the commutative monoid axiom if whenever h is a trivial cofibration in \mathcal{M} then $h^{\Box n}/\Sigma_n$ is a trivial cofibration in \mathcal{M} for all n > 0.

If, in addition, the class of cofibrations is closed under the operation $(-)^{\Box n}/\Sigma_n$ then \mathcal{M} is said to satisfy the *strong commutative monoid axiom*.

Theorem 5.39 Let \mathcal{M} be a cofibrantly generated monoidal model category satisfying the commutative monoid axiom and the monoid axiom, and assume that the domains of the generating maps I (resp. J) are small relative to $(I \otimes \mathcal{M})$ -cell (resp. $(J \otimes \mathcal{M})$ -cell). Let R be a commutative monoid in \mathcal{M} . Then the category CAlg(R) of commutative R-algebras is a cofibrantly generated model category in which a map is a weak equivalence or fibration if and only if it is so in \mathcal{M} . In particular, when R = Sthis gives a model structure on commutative monoids in \mathcal{M} .

Corollary 5.40 Let \mathcal{M} be a cofibrantly generated monoidal model category satisfying the commutative monoid axiom, and assume that the domains of the generating maps I (resp. J) are small relative to $(I \otimes \mathcal{M})$ -cell (resp. $(J \otimes \mathcal{M})$ -cell). Then for any commutative monoid R, the category of commutative R-algebras is a cofibrantly generated semi-model category in which a map is a weak equivalence or fibration if and only if it is so in \mathcal{M} .

While these results only make use of the commutative monoid axiom, in practice we usually desire the strong commutative monoid axiom so that in the category of commutative R-algebras cofibrations with cofibrant domains forget to cofibrations in \mathcal{M} . This is discussed further in 58 and numerous examples of model categories satisfying these axioms are given.

In order to apply the corollary above to verify the hypotheses of Corollary 5.5 we must give conditions on the maps C so that if \mathcal{M} satisfies the commutative monoid axiom then so does $L_{\mathcal{C}}(\mathcal{M})$. As for the pushout product axiom, our method will be to apply Lemma 5.16, which is just the universal property of Bousfield localization. However, $(-)^{\Box n}/\Sigma_n$ is not a functor on \mathcal{M} , but rather on $\operatorname{Arr}(\mathcal{M})$. The following lemma lets us instead work with the functor $\operatorname{Sym}^n : \mathcal{M} \to \mathcal{M}$ defined by $\operatorname{Sym}^n(X) = X^{\otimes n}/\Sigma_n$. This lemma is proved in Appendix A of 58, and appears in 25.

Lemma 5.41 Assume that for every $g \in I$, $g^{\Box n}/\Sigma_n$ is a cofibration. Suppose f is a trivial cofibration between cofibrant objects and $f^{\Box n}/\Sigma_n$ is a cofibration for all n. Then $f^{\Box n}/\Sigma_n$ is a trivial cofibration for all n if and only if $\operatorname{Sym}^n(f)$ is a trivial cofibration for all n.

With this lemma in hand, we are ready to prove the main result of this section, regarding preservation of the commutative monoid axiom by Bousfield localization.

Theorem 5.42 Assume \mathcal{M} is a cofibrantly generated monoidal model category satisfying the strong commutative monoid axiom and with domains of the generating cofibrations cofibrant. Suppose that $L_{\mathcal{C}}(\mathcal{M})$ is a monoidal Bousfield localization with generating trivial cofibrations $J_{\mathcal{C}}$. If $\operatorname{Sym}^{n}(f)$ is a \mathcal{C} -local equivalence for all $n \in \mathbb{N}$ and for all $f \in J_{\mathcal{C}}$, then $L_{\mathcal{C}}(\mathcal{M})$ satisfies the strong commutative monoid axiom. In particular, the category of commutative monoids inherits a transferred semi-model structure from $L_{\mathcal{C}}(\mathcal{M})$.

Remark 5.43 The condition of Theorem 5.42, that for all $n \in \mathbb{N}$ and for all $f \in J_{\mathcal{C}}$, Symⁿ(f) is a \mathcal{C} -local equivalence is equivalent to the condition that Sym(-) preserves C-local equivalences between cofibrant objects. The latter condition implies the former because $\text{Sym}^n(f)$ is a retract of Sym(f), and the maps in $J_{\mathcal{C}}$ may be assumed to have cofibrant domains, as shown in [36], Proposition 4.3]. That the former implies the latter follows from Theorem 5.6 of 6, which shows that the existence of a transferred semi-model structure on commutative monoids in $L_{\mathcal{C}}(\mathcal{M})$ implies $\operatorname{Sym}(-)$ preserves C-local equivalences between cofibrant objects. This result, together with Theorem 5.42, implies that both conditions are equivalent to existence of a transferred semi-model structure on commutative monoids in $L_{\mathcal{C}}(\mathcal{M})$. Furthermore, the existence of this semi-model structure is equivalent to $L_{\mathcal{C}}(\mathcal{M})$ satisfying the strong commutative monoid axiom, since the existence of the semi-model structure implies Sym(-) preserves \mathcal{C} -local equivalences, which implies the strong commutative monoid axiom by Theorem 5.42. We have refrained from stating Theorem 5.42 as an 'if and only if' to match the discussion in 6 where the converse was first noticed.

We turn now to the proof of Theorem 5.42 and to several related results. All of these results are meant to find the easiest possible condition to check on \mathcal{C} so that $L_{\mathcal{C}}(\mathcal{M})$ satisfies the commutative monoid axiom. Theorem 5.42 reduces the problem from having to check the class of all \mathcal{C} -local equivalences to only having to check the set $J_{\mathcal{C}}$ (which, unfortunately, is often mysterious in practice). It is tempting to try to prove Theorem 5.42 using Lemma 5.16 as we did in Theorem 5.11 since this would reduce the problem to checking the set $J \cup \mathcal{C}$ (which is much less mysterious than $J_{\mathcal{C}}$). However, Symⁿ is not a left adjoint. One could attempt to get around this by applying Lemma 5.16 with the functor Sym : $\mathcal{M} \to \text{CMon}(\mathcal{M})$, but this would require the existence of a model structure on $\text{CMon}(\mathcal{M})$ in which the weak equivalences are \mathcal{C} -local equivalences. As this is what we're trying to prove by obtaining the commutative monoid axiom on $L_{\mathcal{C}}(\mathcal{M})$, this approach is doomed to fail. Instead, we opt for a more technical argument, following the techniques of 58.

Proof of Theorem 5.42 By Appendix A of 58, if $(-)^{\Box n}/\Sigma_n$ takes generating (trivial) cofibrations to (trivial) cofibrations, then it takes all (trivial) cofibrations to (trivial) cofibrations. The generating cofibrations of $L_{\mathcal{C}}(\mathcal{M})$ are the same as those in \mathcal{M} and \mathcal{M} satisfies the strong commutative monoid axiom, so the class of cofibrations of $L_{\mathcal{C}}(\mathcal{M})$ is closed under the operation $(-)^{\Box n}/\Sigma_n$.

Suppose, for every generating trivial cofibration $f: X \to Y$ of $L_{\mathcal{C}}(\mathcal{M})$, that $\operatorname{Sym}^{n}(f)$ is a \mathcal{C} -local equivalence. Because the domains of the generating cofibrations in \mathcal{M} are cofibrant, the same is true in $L_{\mathcal{C}}(\mathcal{M})$ (see Proposition 4.3 in [36]), so we may assume f has cofibrant domain and codomain. In particular, the proof of Lemma [5.41] implies $\operatorname{Sym}^{n}(f)$ is a cofibration, because $f^{\Box k}/\Sigma_{k}$ is a cofibration for all k and the domain Xof f is cofibrant.

By hypothesis, $\operatorname{Sym}^n(f)$ is a trivial cofibration of $L_{\mathcal{C}}(\mathcal{M})$ for all n. We are therefore in the situation of Lemma 5.41 and may conclude that $f^{\Box n}/\Sigma_n$ is a trivial cofibration for all n. We now apply the result from Appendix A of 58 to conclude that all trivial cofibrations of $L_{\mathcal{C}}(\mathcal{M})$ are closed under the operation $(-)^{\Box n}/\Sigma_n$. The main theorem of 58 produces the resulting transferred semi-model structure on commutative monoids.

If we know more about \mathcal{M} in the statement of Theorem 5.42 then we can in fact get a sharper condition regarding the generating trivial cofibrations $J_{\mathcal{C}}$. One way to better understand the trivial cofibrations in $L_{\mathcal{C}}(\mathcal{M})$ is via the theory of framings. Briefly, the idea here is to cofibrantly replace morphisms in \mathcal{C} in an appropriate simplicial model category. One starts by shifting focus from \mathcal{M} to the category of cosimplicial objects \mathcal{M}^{Δ} , i.e., functors from the simplex category Δ to \mathcal{M} . For any model category \mathcal{M} , \mathcal{M}^{Δ} carries the Reedy model structure [34]. Theorem 5.2.5]. For an object X of \mathcal{M} , a cosimplicial resolution is a cofibrant replacement, in the Reedy model structure, of the constant cosimplicial object valued at X. As cofibrant replacement is functorial, this also defines the notion of a cosimplicial resolution of a morphism f in \mathcal{M} . For a class of morphisms \mathcal{C} in \mathcal{M} , Definition 4.2.1 of [32] defines the full class of horns on \mathcal{C} to be the class

$$\Lambda(\mathcal{C}) = \{ f \square i_n \mid f \in \mathcal{C}, n \ge 0 \}$$

where $i_n : \partial \Delta[n] \to \Delta[n]$ is the natural inclusion, and $\tilde{f} : \tilde{A} \to \tilde{B}$ is a Reedy cofibration between cosimplicial resolutions, obtained by factoring the cosimplicial resolution of f into a Reedy cofibration followed by a Reedy trivial fibration. In the case where C is a set and \mathcal{M} is cofibrantly generated, Definition 4.2.2 of [32] defines an *augmented set of C-horns* to be $\overline{\Lambda(C)} = \Lambda(C) \cup J$. Finally, 4.2.5 of [32] defines a set $\widehat{\Lambda(C)}$ to be a set of relative *I*-cell complexes with cofibrant domains obtained from $\overline{\Lambda(C)}$ via cofibrant replacement. Note that, according to the erratum to [32],

we do not know that the domains of maps in $\Lambda(\mathcal{C})$ are cofibrant, but we do know that they are small relative to I.

We now advertise the surprising and powerful Theorem 4.11 in [2]. This result states that if \mathcal{M} is proper and stable, if the \mathcal{C} -local objects are closed under Σ (such $L_{\mathcal{C}}$ are called *stable*), and if \mathcal{C} consists of cofibrations between cofibrant objects then $J_{\mathcal{C}}$ is $J \cup \Lambda(\mathcal{C})$. The last hypothesis is a standing assumption for this paper. The key input to [2], Theorem 4.11] is the observation that for such \mathcal{M} , a map is a \mathcal{C} -fibration if and only if its fiber is \mathcal{C} -fibrant.

Corollary 5.44 Suppose \mathcal{M} is a stable, proper, simplicial model category satisfying the strong commutative monoid axiom. Suppose that $L_{\mathcal{C}}$ is a stable and monoidal Bousfield localization such that for all $n \in \mathbb{N}$ and $f \in \mathcal{C}$, $\operatorname{Sym}^{n}(f)$ is a \mathcal{C} -local equivalence. Then $L_{\mathcal{C}}(\mathcal{M})$ satisfies the strong commutative monoid axiom. In particular, the category of commutative monoids inherits a transferred semi-model structure from $L_{\mathcal{C}}(\mathcal{M})$.

Proof By Theorem 5.42 we must only check that Sym^n takes maps in $J_{\mathcal{C}} = J \cup \Lambda(\mathcal{C})$ to \mathcal{C} -local equivalences. By the commutative monoid axiom on \mathcal{M} , maps in J are taken to weak equivalences, so we must only consider maps in $\Lambda(\mathcal{C})$.

The reason for the hypothesis that \mathcal{M} is simplicial is Remark 5.2.10 in [34], which states that the functor $\widetilde{A}^m = A \otimes \Delta[m]$ is a cosimplicial resolution of A (at least, when A is cofibrant), in both \mathcal{M} and $L_{\mathcal{C}}(\mathcal{M})$. Thus, we may take our map in $\Lambda(\mathcal{C})$ to be of the form $(f \otimes \Delta[m]) \Box i_n$ where $f : A \to B$ is in \mathcal{C} , since $\widetilde{f}^m = f \otimes \Delta[m]$.

The map $(f \otimes \Delta[m]) \Box i_n$ can be realized as the corner map in the diagram

If we can prove that $(g \otimes K)^{\Box n} / \Sigma_n$ is a C-local trivial cofibration for any C-local trivial cofibration g between cofibrant objects then we can apply the same reasoning from the proof of Proposition 5.18 to deduce that the corner map becomes a C-local trivial cofibration after applying $(-)^{\Box n}/\Sigma_n$. This reasoning goes by proving that after applying $(-)^{\Box n}/\Sigma_n$ the lower curved map and the top horizontal map are C-local trivial cofibrations, so the bottom horizontal map is as well (because it is a pushout), and hence the corner map is a weak equivalence by the two out of three property. This reasoning works because whenever f is a pushout of g then $f^{\Box n}/\Sigma_n$ is a pushout of $g^{\Box n}/\Sigma_n$ as shown in Appendix A of 58.

Because $g \otimes K$ is a C-local trivial cofibration between cofibrant objects, we may apply Lemma 5.41 to reduce the final step to checking that if $\operatorname{Sym}^n(g)$ is a C-local trivial cofibration for all n then so is $\operatorname{Sym}^n(g \otimes K)$. This is proven in 25.

When the hypotheses of stability and properness are dropped one can no longer easily write down the set $J_{\mathcal{C}}$. However, Theorem 4.1.1 (and its proof, notably 4.3.1) in [32] demonstrate that the class of maps $X \to L_{\mathcal{C}}(X)$ are contained in $\widehat{\Lambda(\mathcal{C})}$ -cell. Given a \mathcal{C} -local trivial cofibration $g: X_1 \to X_2$ between cofibrant objects, applying fibrant replacement $L_{\mathcal{C}}$ results in a map $L_{\mathcal{C}}(g)$, that is a weak equivalence between cofibrant objects. An appeal to Ken Brown's lemma on the functor Symⁿ and to the two out of three property reduces the verification that $(-)^{\Box n}/\Sigma_n$ takes gto a \mathcal{C} -local equivalence to verifying that $(-)^{\Box n}/\Sigma_n$ takes $X_i \to L_{\mathcal{C}}(X_i)$ to \mathcal{C} -local equivalences.

Since such maps are in $\Lambda(\mathcal{C})$ -cell, by Appendix A of 58 one must only show that maps in $\widetilde{\Lambda(\mathcal{C})}$ are taken to \mathcal{C} -local equivalences by $(-)^{\Box n}/\Sigma_n$ (that they are taken to cofibrations is immediate by the strong commutative monoid axiom on \mathcal{M}). This observation leads to the following result, which we have recently learned was independently discovered in 25.

Theorem 5.45 Suppose \mathcal{M} is a cofibrantly generated, simplicial model category satisfying the strong commutative monoid axiom and with domains of the generating cofibrations cofibrant. Suppose that for all $n \in \mathbb{N}$ and $f \in \mathcal{C}$, $\operatorname{Sym}^n(f)$ is a \mathcal{C} -local equivalence. Then $L_{\mathcal{C}}(\mathcal{M})$ satisfies the strong commutative monoid axiom. In particular, the category of commutative monoids inherits a transferred semi-model structure from $L_{\mathcal{C}}(\mathcal{M})$.

As the proof of this Theorem appears in [25], we will content ourselves with the sketch of the proof given above and we refer the interested reader to [25] for details. With a careful analysis of $\widehat{\Lambda(\mathcal{C})}$ the author believes one could remove the need for \mathcal{M} to be simplicial. However,

lacking equations of the sort found in Remark 5.2.10 of 34, he does not know how to proceed.

Remark 5.46 The commutative monoid axiom has a natural generalization to an arbitrary operad P. The proof of Proposition 7.6 in [28] demonstrates a precise hypothesis on \mathcal{M} so that P-algebras inherit a model structure, namely that for all $A \in P$ -alg and for all n, $P_A[n] \otimes_{\Sigma_n} (-)^{\Box n}$ preserves trivial cofibrations (where P_A is the enveloping operad). If these hypotheses are only satisfied for cofibrant A then P-alg inherits a semi-model structure. We hope in the future to study the types of localizations that preserve these axioms, so that Corollary 5.5 may be applied to deduce preservation results for arbitrary operads P. We conjecture that the correct condition on a localization is that for all $f \in C$, for all $A \in P$ -alg, and for all n, then $P_A[n] \otimes_{\Sigma_n} f^{\Box n}$ is contained in the C-local equivalences. Assuming a P-algebra analogue of Lemma 5.41, the proof of Corollary 5.44 will go through, if we assume $P_A[n] \otimes_{\Sigma_n} f^{\Box n}$ is contained in the C-local equivalences, for all f of the form $g \otimes K$ where $g \in C$ and K is a simplicial set.

Remark 5.47 Theorem 5.42 also has a converse, that the author discovered in joint work with Michael Batanin 6 (Theorem 5.6 and Example 5.9). For nicely behaved model categories, including all examples considered in this paper, the following are equivalent:

- 1 $L_{\mathcal{C}}$ preserves *P*-algebras,
- 2 P-alg $(L_{\mathcal{C}}(\mathcal{M}))$ admits a transferred semi-model structure from $L_{\mathcal{C}}(\mathcal{M})$,
- 3 $L_{\mathcal{C}}$ lifts to a localization of *P*-algebras (inverting the maps $P(\mathcal{C})$),
- 4 U preserves local equivalences,

and any of these statements implies P(-) preserves C-local equivalences between cofibrant objects. A dual result, for the situation of right Bousfield localization, appears in [60]. It follows that, for any of the situations from Theorem [5.42], [5.44] or [5.45] L_C preserves commutative monoids if and only if Sym(-) preserves C-local equivalences between cofibrant objects. Note that the condition that the objects be cofibrant is no obstacle, in any model category satisfying the strong commutative monoid axiom, since C can be taken to be a set of cofibrations between cofibrant objects, and Symⁿ(X) is cofibrant whenever X is cofibrant, if the commutative monoid axiom is satisfied. This follows from the filtration on Symⁿ(\emptyset) \rightarrow Symⁿ(X) from Lemma A.3 of [58]. Although the positive stable model structure only satisfies the (weak) commutative monoid axiom, one can use the positive flat stable model structure to prove all statements needed for the positive stable model structure, as demonstrated in 58.

5.7 Preservation of Commutative Monoids

We turn now to the question of preservation under Bousfield localization of commutative monoids. We will be applying Theorem 5.42 and Corollary 5.5 for this purpose in a moment, but we first remark on a simpler case where the hypotheses of Theorem 5.42 are not necessary.

5.7.1 Spectra

Preservation of commutative monoids by monoidal Bousfield localizations is easy in certain categories of spectra, because of the property that for all cofibrant X in \mathcal{M} , the map $(E\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \to X^{\wedge n}/\Sigma_n$ is a weak equivalence, where the symmetric group Σ_n acts by permuting copies of X. This property was first noticed in [19], and we will now discuss it more generally.

Recall that, given a morphism of operads $\alpha: O \to P$, the two operads O and P are said to satisfy *rectification* if the induced adjunction between P-alg and O-alg is a Quillen equivalence. In [58], we introduced the *rectification axiom*, which states that if $Q_{\Sigma_n}S \to S$ is cofibrant replacement for the unit S in the projective model structure \mathcal{M}^{Σ_n} of Σ_n -objects in \mathcal{M} , then for all cofibrant X in \mathcal{M} , the map $Q_{\Sigma_n}S \otimes_{\Sigma_n} X^{\otimes n} \to X^{\otimes n}/\Sigma_n$ is a weak equivalence (this is the natural generalization of the property from [19] mentioned above, and was further generalized in [63]). Observe that this property automatically holds on $L_{\mathcal{C}}(\mathcal{M})$ if it holds on \mathcal{M} , because the cofibrant objects are the same and the weak equivalences are contained in the \mathcal{C} -local equivalences. We now prove that in the presence of the rectification axiom, preservation results for commutative monoids are particularly nice.

Theorem 5.48 Let QCom denote a Σ -cofibrant replacement of Comin \mathcal{M} . Let \mathcal{M} be a monoidal model category satisfying the conditions of Theorem 4.6 of [58], so that the rectification axiom implies that QComand Com rectify. Let $L_{\mathcal{C}}$ be a monoidal Bousfield localization. Then $L_{\mathcal{C}}$ preserves commutative monoids. In particular:

For positive (flat) symmetric spectra, positive (flat) orthogonal spectra,

or S-modules, QCom is E_{∞} and any monoidal Bousfield localization preserves strict commutative ring spectra.

For positive (flat) G-equivariant orthogonal spectra, QCom is E_{∞}^{G} and any monoidal Bousfield localization preserves strict commutative equivariant ring spectra.

Proof Let E be a commutative monoid, so in particular E is a QCom algebra via the map $QCom \rightarrow Com$. Because QCom is Σ -cofibrant, QCom-algebras in both \mathcal{M} and $L_{\mathcal{C}}(\mathcal{M})$ inherit semi-model structures. Corollary 5.5 implies $L_{\mathcal{C}}(E)$ is weakly equivalent to some QCom-algebra E_Q . The rectification axiom in $L_{\mathcal{C}}(\mathcal{M})$ now implies E_Q is weakly equivalent to a commutative monoid \widehat{E} .

Currently, this result is only known to apply to the categories of spectra listed in the statement of the theorem. We conjectured in **58** that the rectification axiom implies rectification between QCom and Com for general \mathcal{M} . If this conjecture is proven then the theorem will apply to all \mathcal{M} satisfying the rectification axiom. Even if the conjecture is false, the following proposition demonstrates that when \mathcal{M} satisfies the rectification axiom then the conditions of Theorem **5.42** are satisfied and so any monoidal localization preserves commutative monoids.

Proposition 5.49 Suppose \mathbb{N} is a monoidal model category satisfying the rectification axiom. Then $\operatorname{Sym}^n(-)$ takes trivial cofibrations between cofibrant objects to weak equivalences.

In particular, if $L_{\mathcal{C}}(\mathcal{M})$ is a monoidal Bousfield localization and \mathcal{M} satisfies the rectification axiom, then $L_{\mathcal{C}}$ preserves commutative monoids.

Proof The first part is proven as Proposition 4.6 in [58], and we refer the reader there for a proof. For the second part, we apply the first part with $\mathcal{N} = L_{\mathcal{C}}(\mathcal{M})$, using our observation that the rectification axiom holds on $L_{\mathcal{C}}(\mathcal{M})$ whenever it holds on \mathcal{M} . Thus, $\operatorname{Sym}^n : L_{\mathcal{C}}(\mathcal{M}) \to L_{\mathcal{C}}(\mathcal{M})$ takes \mathcal{C} -local trivial cofibrations between cofibrant objects to \mathcal{C} -local equivalences. In particular, the hypotheses of Theorem 5.42 are satisfied and we may deduce from Corollary 5.5 that $L_{\mathcal{C}}$ preserves commutative monoids.

5.7.2 Spaces

We turn our attention now to simplicial sets and topological spaces. Rectification is known to fail (see Example 4.4 in [58]), so even though all localizations are monoidal we may not apply the result above. For

spaces the path connected commutative monoids are weakly equivalent to generalized Eilenberg-Mac Lane spaces, i.e. products of Eilenberg-Mac Lane spaces. Preservation of commutative monoids has been proven for pointed CW complexes as Theorem 1.4 in 15.

Theorem 5.50 Let \mathcal{M} be the category of pointed CW complexes. Let \mathcal{C} be any set of maps. Then Sym(-) preserves \mathcal{C} -local equivalences and $L_{\mathcal{C}}$ sends GEMs to GEMs.

As a special case of this theorem, we recover classical results of Bousfield, e.g. parts of Theorem 5.1 and Lemma 9.8 from 13. The proof of Theorem 5.50 is based on work of Dror Farjoun, Chapter 4 of 21, in the setting of $\mathcal{M} = sSet$. That work is generalized in 58 to hold for the category of k-spaces. So we may extend the theorem above to k-spaces as well. Observe that the theorem above implies both sSet and k-spaces satisfy the conditions of Theorem 5.42 because Sym^n is a retract of Sym.

Theorem 5.51 Let \mathcal{M} be either simplicial sets or k-spaces. Then every Bousfield localization preserves GEMs.

Thus, we have extended the result above and Theorem 4.B.4 in [21] to a wider class of topological spaces than spaces having the homotopy type of a CW complex.

5.7.3 Chain Complexes

When k is a field of characteristic zero, there are model structures on $Ch(k)_{\geq 0}$ and Ch(k) where the weak equivalences are quasi-isomorphisms, the fibrations are morphisms that are degreewise surjections in positive degree, and the cofibrations are the degreewise split monomorphisms [34, Theorem 2.3.11]. All operads are Σ -cofibrant, hence all operad-algebras are preserved by any monoidal Bousfield localization. That cofibrant objects are flat is an easy exercise, using the observation that, for every cofibrant A, the map $A \otimes QX \to A \otimes X$ induced by cofibrant replacement (i.e. projective resolution) is a quasi-isomorphism.

This model structure is cofibrantly generated by morphisms of the form $I := \{i_n : S^{n-1} \to D^n \mid n \in \mathbb{N}\}$ and $J := \{j_n : 0 \to D^n \mid n \in \mathbb{N}\}$ where the *sphere* S^{n-1} is the chain complex that is R in degree n-1 (and 0 elsewhere), the *disk* D^n is the chain complex that is R in degrees n-1 and n (and 0 elsewhere) with the only non-zero differential being the identity on R, i_n is the natural inclusion, and j_n is the unique morphism from the zero module.

Proposition 5.52 Let k be a field of characteristic zero. The only Bousfield localizations of $Ch(k)_{\geq 0}$ are truncations.

Proof Since k is a field, all modules are free. Hence, all chain complexes are wedges of spheres, and morphisms between them are generated by zero maps and identities on copies of R. Inverting an identity map changes nothing. Inverting a zero map has the effect of killing some object, also known as a nullification.

Furthermore, killing the wedge sum k^2 in degree n is the same as killing k in degree n, and this also kills copies of k in higher degrees, because differentials are identity maps. Thus, the localization is completely determined by the lowest dimension in which the first nullification occurs. The localization is therefore equivalent to $0 \rightarrow V$ where V is the sphere on k in that dimension.

Corollary 5.53 All Bousfield localizations of $Ch(k)_{\geq 0}$ are monoidal and hence preserve algebras over any operad P.

Remark 5.54 For unbounded chain complexes, truncations need not preserve algebraic structure. For example, if $f: S^{-2} \to D^{-3}$ gets inverted then just as with the Postnikov Section, an algebra will be taken to an object with no unit.

Quillen proved in Proposition 2.1 of Appendix B of 47 that bounded chain complexes over a field of characteristic zero satisfies the commutative monoid axiom. The proof that all quasi-isomorphisms are closed under Symⁿ goes via the cofiber and the 5-lemma on homology groups. The key observation is that Symⁿ(-) commutes with homology. The same proof demonstrates that Symⁿ preserves C-local equivalences for all $L_{\mathcal{C}}$ as above. Hence, all Bousfield localizations of $Ch(k)_{\geq 0}$ preserve commutative differential graded algebras. Of course, this can also be seen directly from the description of $L_{\mathcal{C}}$ as a truncation.

5.7.4 Equivariant Spectra

We conclude this section by returning to our motivating example, Example 5.36. Throughout this section, G is a finite group, since otherwise we do not know how to transfer a semi-model structure to commutative equivariant ring spectra. Note that when it was discovered, Example 5.36 represented a potential gap in the proof of the Kervaire Invariant One Theorem (because the spectrum $\Omega = D^{-1}MU^{(4)}$ needed to be

commutative for the computations in 31). Thankfully, the following theorem from 30 demonstrates that Ω was indeed commutative.

Theorem 5.55 Let G be a finite group. Let L be a localization of equivariant spectra. If for all L-acyclics Z and for all subgroups H, $N_H^G Z$ is L-acyclic, then for all commutative G-ring spectra R, L(R) is a commutative G-ring spectrum.

The hypothesis in this theorem is designed so that the proof in [19] regarding preservation of E_{∞} -structure under localization (i.e. via the skeletal filtration) may go through. We now specialize our preservation result to the context of *G*-spectra, by combining Theorem [5.3]. Theorem [5.45] and Remark [5.47] Recall from Proposition [5.27] that monoidal localizations are stable. In order to have a transferred semi-model structure on commutative monoids, we need to work with either the positive stable model structure (of Theorem 14.2 of [43]), the positive flat stable model structure (of Proposition B.4.1 of [31]). The proof that these model structures are left proper and cellular can be found in the appendix of [27].

Theorem 5.56 Let \mathcal{M} denote any of the positive model structure on G-spectra discussed above. Suppose $L_{\mathcal{C}}$ is a monoidal left Bousfield localization. Then the following are equivalent:

- 1 $L_{\mathcal{C}}$ preserves commutative equivariant ring spectra,
- 2 Symⁿ(-) preserves local equivalences between cofibrant objects, for all n,
- 3 Symⁿ(-) takes maps in C to local equivalences, and
- 4 Symⁿ(-) preserves L-acyclicity for all n.

That preservation of *L*-acyclics is the same as preservation of *L*-local equivalences as can be seen via the rectification axiom and the property that cofibrant objects are flat, but it is easier to observe that this equivalence holds for any stable localization in any stable model category (because consideration of cofibers allows one to reduce to the study of nullifications). In [30], several equivalent conditions are given in order for a localization to preserve commutative structure. Condition (4) above is equivalent to the condition that, for all *L*-acyclics *Z* and for all subgroups H, $N_H^G Z$ is *L*-acyclic. Hence, Theorem 5.56 sharpens Theorem 5.55 to make it an 'if and only if' result.

Another equivalent formulation states that preservation occurs whenever the functors $(E_G \Sigma_n)_+ \wedge_{\Sigma_n} (-)^{\wedge n}$ preserve *L*-acyclicity. This condition can be verified via the skeletal filtration of $E_G \Sigma_n$ into a homotopy colimit of Σ_n -free $G \times \Sigma_n$ sets *T* of the form $(G \times \Sigma_n)/\Gamma$ where Γ is the graph of a subgroup. This formulation of what is required for *L* to preserve commutativity is at the heart of the arguments in [31] and [10] and allows for the preservation machinery to be extended to N_∞ -operads in [27]. The condition is analogous to the non-equivariant condition that functors $(E\Sigma_n)_+ \wedge_{\Sigma_n} (-)^{\wedge n}$ preserve *L*-acyclicity.

By Theorem 6.3 of [27], for any complete N_{∞} -operad P whose spaces have the homotopy type of $G \times \Sigma_n$ -CW complexes, P-algebras are Quillen equivalent to commutative monoids. Hence, Theorem 5.48 implies the same conditions from Theorem 5.56 are equivalent to preservation of P-algebras for any (hence all) complete N_{∞} -operad P whose spaces have the homotopy type of $G \times \Sigma_n$ -CW complexes. Preservation results for non-complete N_{∞} -operads and for \mathcal{F} - N_{∞} -operads can be found in Section 7 of [27].

5.8 Bousfield localization and the monoid axiom

Recall that the monoid axiom is required to transfer a full model structure to the category of monoids in a monoidal model category [50]. However, Theorem 5.24 demonstrates that there is a transferred semi-model structure even if the monoid axiom is not satisfied. It follows from Corollary [5.5], that our preservation results do not require $L_{\mathcal{C}}(\mathcal{M})$ to satisfy the monoid axiom. However, the monoid axiom is an important part of the study of monoidal model categories, with many applications beyond the ability to transfer a model structure to monoids, and in this section we provide a result that guarantees it holds on $L_{\mathcal{C}}(\mathcal{M})$.

We remark that Proposition 3.8 of \square proves that $L_{\mathcal{C}}(\mathcal{M})$ inherits the monoid axiom from \mathcal{M} if $L_{\mathcal{C}}$ takes a special form similar to localization at a homology theory. In contrast, our result will place no hypothesis on the maps in \mathcal{C} at all, beyond our standing hypothesis that these maps are cofibrations. We additionally remark that [46] has independently considered the question of when Bousfield localization preserves the monoid axiom, towards the goal of rectification results in general categories of spectra.

In order to understand when Bousfield localization will preserve the monoid axiom we must introduce a definition, taken from 5. Note

that this is a different usage of the term *h*-cofibration than the usage in [19] where it means 'Hurewicz cofibration.' The meaning here is for 'homotopical cofibration' for reasons which will become clear.

Definition 5.57 A map $f: X \to Y$ is called an *h*-cofibration if the functor $f_!: X/\mathcal{M} \to Y/\mathcal{M}$ given by cobase change along f preserves weak equivalences. Formally, this means that in any diagram as below, in which both squares are pushout squares and w is weak equivalence, then w' is also a weak equivalence:



It is clear that any trivial cofibration is an *h*-cofibration, by the two out of three property. If \mathcal{M} is left proper then any cofibration is an *h*-cofibration (because $A \to A'$ is automatically a cofibration if f is). In fact, the converse holds as well [5, Lemma 1.2] and [5, Lemma 1.3] proves that *h*-cofibrations are closed under composition, pushout, and finite coproduct.

If \mathcal{M} is left proper. Proposition 1.5 in **5** proves that an *h*-cofibration is the same as a map f such that every pushout along f is a homotopy pushout (this version of the definition above was independently discovered in **56**) and also that f is an *h*-cofibration if and only if there is a factorization of f into a cofibration followed by a *cofiber equivalence* $w: W \to Y$. This means, for any map $g: W \to K$, the right-hand vertical map in the following pushout diagram is a weak equivalence:

$$\begin{array}{c|c} W \longrightarrow K \\ & & \\ w \\ & & \\ & \\ & \\ X \longrightarrow T \end{array}$$

We will make use of these various properties of h-cofibrations in this section. The purpose for introducing h-cofibrations is to make the following definition, which should be thought of as saying that the cofibrations in \mathcal{M} behave like inclusions of closed neighborhood deformation retracts of topological spaces.

Definition 5.58 \mathcal{M} is said to be *h*-monoidal if for each (trivial) cofibration f and each object Z, $f \otimes Z$ is a (trivial) *h*-cofibration.

We will find conditions so that Bousfield localization preserves the property of being *h*-monoidal, and we will then use this to deduce when Bousfield localization preserves the monoid axiom. In [5], *h*-monoidality is verified for the model categories of topological spaces, simplicial sets, equivariant spaces, chain complexes over a field (with the projective model structure), symmetric spectra (with the stable projective model structure), and several other model categories not considered in this paper. We now verify *h*-monoidality for the remaining model structures of interest in this paper. We remind the reader that an *injective model structure* has weak equivalences and cofibrations defined levelwise, and fibrations defined by the right lifting property.

Proposition 5.59 The following eight model structures on symmetric spectra are h-monoidal (4 stable and 4 unstable model structures):

- 1 The levelwise projective (stable) model structure (of Theorem 5.1.2 in 38, see also Proposition 1.14 of 5).
- 2 The positive (stable) model structure (of Theorem 14.1 and 14.2 in [43]).
- 3 The flat (stable) model structure (of Proposition 2.2 and Theorem 2.4 in [51], there called the S-model structure).
- 4 The positive flat (stable) model structure (obtained by redefining the cofibrations from the model structure above to be isomorphisms in level 0, see Proposition 3.1 in [51]).

Proof We appeal to Proposition 1.9 in [5], and make use of the injective (or injective stable for (5)-(8)) model structure on symmetric spectra, introduced in Definition 5.1.1 (resp. after Definition 5.3.6) of [38]. The references above prove that all eight of the model structures above are monoidal and that both injective model structures are left proper (e.g. because all objects are cofibrant). The final condition in Proposition 1.9 is that for any (trivial) cofibration f and any object X, the map $f \otimes X$ is a (trivial) cofibration in the corresponding injective model structure. The cofibration part of this is Proposition 4.15(i) in version 3 of Stefan Schwede's book project [48], since for all eight of the model structures above the cofibrations are contained in the flat cofibration. The trivial cofibration part is Proposition 4.15(iv) in [48], which includes statements for both levelwise and stable weak equivalences.

We turn now to orthogonal and equivariant orthogonal spectra. To mimic constructions from the theory of symmetric spectra built on

simplicial sets, we must build orthogonal spectra on the category Top_{Δ} of Δ -generated spaces (pointed, of course), which we now briefly discuss. The category of Δ -generated spaces is a locally presentable category of topological spaces that admits a combinatorial model structure [17, 20]. Essentially, these are spaces that are colimits of simplices, and the category of Δ -generated spaces is the final closure of the subcategory Δ of spaces consisting of the topological simplices Δ^n . The construction of the category of Δ -generated spaces is analogous to Vogt's construction of the category of compactly generated spaces as the colimit-closure of the category of compact Hausdorff spaces [55], and so proofs set in compactly generated spaces have analogues in Δ -generated spaces.

In both settings, care must be taken, as certain common constructions can take you out of the subcategory. For example, when computing limits, one needs Vogt's k-ification functor to get back into the category of compactly generated spaces, and there is an analogous "delta-fication" functor to get back into the category of Δ -generated spaces, used when taking limits or passing to subspaces takes you out of the category of Δ -generated spaces. For examples where this kind of care is taken, see [4, [17, 24].

For the two proofs below, these kinds of issues did not arise, for several reasons. First, [38] works at the level of simplices, and thus translates immediately to the setting of Δ -generated spaces. Secondly, Δ -generated spaces are closed under the smash product, fixed points commute with pushouts and filtered colimits along closed inclusions, and fixed points commute with the smash product. We note that in general, closed subspaces of Δ -generated spaces need not be Δ -generated, though this can be arranged by enlarging Δ . Nevertheless, the *H*-fixed points of a space in Top_{Δ}^G (or in Top_{Δ}^H) is a Δ -generated space, since the *H*-fixed points functor is a right adjoint (by the special adjoint functor theorem) to the functor $Top_{\Delta} \to Top_{\Delta}^H$ taking any space to the *H*-space with trivial *H*-action.

Let Sp_{Δ}^{O} denote orthogonal spectra built on Δ -generated spaces, i.e. where each space in the spectrum is a Δ -generated space. Let G be a compact Lie group and let GSp_{Δ}^{O} denote G-equivariant orthogonal spectra built on Δ -generated spaces. We first need a lemma regarding the existence of injective model structures.

Lemma 5.60 The following model structures exist and are left proper and combinatorial: the levelwise injective model structure on Sp_{Δ}^{O} , the

stable injective model structure on Sp_{Δ}^{O} , the levelwise injective model structure on GSp_{Δ}^{O} , and the stable injective model structure on GSp_{Δ}^{O} .

Proof Left properness will be inherited from Δ -generated spaces, where it is verified just as for topological spaces [17]. For the existence of these model structures, we proceed as in Theorem 5.1.2 and Lemma 5.1.4 of [38], starting with the level model structures. One chooses the set of generating cofibrations C to consist of one representative from each isomorphism class of cofibrations $i: X \to Y$ where Y is a cell spectrum built with countably many cells. This size restriction is required to ensure that C is a *set*. One similarly defines a set tC of generating trivial cofibrations.

As usual for injective model structures, it is difficult to precisely describe the morphisms in C or tC, but one can prove that morphisms with the right lifting property with respect to C are injective fibrations and level equivalences, just as in [38] Lemma 5.1.4]. The idea is to construct a lift bit-by-bit (that is, cell-by-cell), and use of Zorn's Lemma to construct a full lift. In the case of symmetric spectra, this involves looking at individual simplicies not yet covered by the partial lift, and generating a countable subspectrum from such a simplex.

In the Δ -generated case, one does the same thing with individual cells, using that the category of Δ -generated spaces is the final closure of Δ , and hence that arguments at the level of simplices translate directly to arguments about the corresponding Δ -generated spaces. The rest of Lemma 5.1.4 goes through mutatis mutandis, using formal properties of model categories, using properties of topological fibrations, and using Lemma 12.2 in [43] when checking that injective cofibrations are closed under smashing with an arbitrary object.

Together with the fact that a category of spectra built on a locally presentable category is again locally presentable, this proves the level injective model structures are combinatorial. The stable injective structures are obtained by Bousfield localization in the usual way, which exists because the levelwise structures are left proper and combinatorial. \Box

Proposition 5.61 Let G be a compact Lie group and fix a universe \mathcal{U} , that we take to mean a G-universe when working equivariantly. Assume all spectra are built on Δ -generated spaces. The following eight model structures (4 stable and 4 unstable) are h-monoidal:

1 The levelwise projective (stable) model structure on G-equivariant orthogonal spectra (of Theorem III.2.4 and III.4.2 in [42]).

- 2 The positive (stable) model structure on G-equivariant orthogonal spectra (of Theorem III.2.10 and III.5.3 in [42]).
- 3 The flat (stable) model structure on G-equivariant orthogonal spectra (of Theorem 2.3.13 of in [54]).
- 4 The positive flat (stable) model structure on G-equivariant orthogonal spectra (obtained by redefining the cofibrations from the model structure above to be isomorphisms in level 0, of Theorem 2.3.27 in [54]).

Taking G to be the trivial group yields eight model structures on orthogonal spectra, that this proposition proves are h-monoidal. They are the levelwise projective (stable) model structure (of Theorem 6.5 and Theorem 9.2 in [43]), the positive (stable) model structure on orthogonal spectra (of Theorem 14.1 and 14.2 in [43]), the flat (stable) model structure on orthogonal spectra (of Proposition 1.3.5 and 2.3.27 in [54]), and the positive flat (stable) model structure on orthogonal spectra (of Proposition 1.3.10 and Theorem 2.3.27 in [54]).

Proof of Proposition 5.61 The proof proceeds just as it does for Proposition 5.59, i.e. by comparison to the injective (stable) model structures in each of these settings. For the statement that for any cofibration fand any object X, the map $f \otimes X$ is a cofibration in the corresponding injective model structure, we appeal to Lemma 12.2 of 43 (which works equally well in the equivariant context). Finally, we turn to the statement that for any trivial cofibration f and any object X, the map $f \otimes X$ is a weak equivalence in the corresponding injective model structure. For the levelwise model structures above this property is inherited from spaces, e.g. by Lemma 12.2 in 43. For the stable model structures we appeal to the monoid axiom on all of the model structures in the theorem and to the fact that projective (stable) equivalences are the same as injective (stable) equivalences. The monoid axiom has been verified in 54 for all these model structures by Theorems 1.2.54 and 1.2.57 (both originally proven in (43), 1.3.10, 2.2.46 and 2.2.50 (both originally from (42)), and 2.3.27.

As previously mentioned, for the two proofs above, no constructions appeared that could take us out of the category of Δ -generated spaces. However, to prove other properties of Top_{Δ}^{G} and spectra built on them, such constructions do appear. The author intends to write a separate manuscript gathering together the required properties needed to work with (*G*-equivariant) Δ -generated spaces, in much the same way that May [44] and Vogt [55] did for compactly generated and compactly generated

weak Hausdorff spaces, and to prove further properties of spectra built on them.

We return now to the question of the monoid axiom. It is proven in Proposition 2.5 of [5] that if \mathcal{M} is left proper, *h*-monoidal, and the weak equivalences in $(\mathcal{M} \otimes I)$ -cell are closed under transfinite composition, then \mathcal{M} satisfies the monoid axiom. We will use this to find conditions on \mathcal{M} so that $L_{\mathcal{C}}(\mathcal{M})$ satisfies the monoid axiom. First, we improve Proposition 2.5 from [5] by replacing the third condition with the hypothesis that the (co)domains of I are finite relative to the class of *h*-cofibrations (in the sense of Section 7.4 of [34]).

Proposition 5.62 Suppose \mathcal{M} is cofibrantly generated, left proper, h-monoidal, and the (co)domains of I are finite relative to the class of h-cofibrations. Then \mathcal{M} satisfies the monoid axiom.

Proof We follow the proof of Proposition 2.5 in **5**. Consider the class $\{f \otimes Z \mid Z \in \mathcal{M}, f \in J\}$. As \mathcal{M} is *h*-monoidal, this is a class of trivial *h*-cofibrations. By Lemma 1.3 in **5**, *h*-cofibrations are closed under pushout. By Lemma 1.6 in **5**, because \mathcal{M} is left proper, trivial *h*-cofibrations are closed under pushouts (e.g. because weak equivalences are closed under homotopy pushout). In order to prove $\{f \otimes Z \mid Z \in \mathcal{M}, f \in J\}$ -cell is contained in the weak equivalences of \mathcal{M} we must only prove that transfinite compositions of trivial *h*-cofibrations are weak equivalences.

Consider a λ -sequence $A_0 \to A_1 \to \cdots \to A_{\lambda}$ of trivial *h*-cofibrations. Let j_{β} denote the map $A_{\beta} \to A_{\beta+1}$ in this λ -sequence. As in Proposition 17.9.4 of 32 we may construct a diagram



in which each A'_{β} is cofibrant, all the maps $A'_{\beta} \to A_{\beta}$ are trivial fibrations, and all the maps $A'_{\beta} \to A'_{\beta+1}$ are trivial cofibrations. Construction of this diagram proceeds by applying the cofibration-trivial fibration factorization iteratively to every composition $j_{\beta} \circ q_{\beta} : A'_{\beta} \to A_{\beta} \to A_{\beta+1}$ in order to construct $A'_{\beta+1}$. As j_{β} and q_{β} are both weak equivalences, so is their composite and so the cofibration $A'_{\beta} \to A'_{\beta+1}$ produced by the cofibration-trivial fibration factorization is a weak equivalence by the two out of three property.

We now show that the map $q_{\lambda} : A'_{\lambda} \to A_{\lambda}$ is a weak equivalence,

following the approach of Lemma 7.4.1 in 34. Consider the lifting problem



Where f is in the set I of generating cofibrations. Because the domains and codomains of maps in I are finitely presented we know that the map $X \to A'_{\lambda}$ factors through some finite stage A'_n . Similarly, $Y \to A_{\lambda}$ factors through some finite stage A_m . Let $k = \max(n, m)$. The map $A'_k \to A_k$ is a trivial fibration so there is a lift $g: Y \to A'_k$. Define $h: Y \to A'_{\lambda}$ as the composite with $A'_k \to A'_{\lambda}$.



Both triangles in the left-hand square commute by definition of lift. The triangle featuring g and h commutes because it is a composition. So the triangle featuring f and h commutes. The right-hand square commutes by construction of A'_{λ} and A_{λ} , so the trapezoid containing g and q_{λ} commutes. Thus, the triangle featuring h and q_{λ} commutes.

The existence of this lift h for all $f \in I$ proves that $A'_{\lambda} \to A_{\lambda}$ is a trivial fibration. Now consider that transfinite compositions of trivial cofibrations are always trivial cofibrations, so $A'_0 \to A'_{\lambda}$ is a weak equivalence. Furthermore, the vertical maps $q_0 : A'_0 \to A_0$ and $q_{\lambda} : A'_{\lambda} \to A_{\lambda}$ are trivial fibrations. So by the two out of three property, the map $A_0 \to A_{\lambda}$ is a weak equivalence as required.

It is shown in [5] that the compactness hypothesis of the proposition is satisfied for topological spaces, simplicial sets, equivariant and motivic spaces, and chain complexes. Similarly, it holds for all our categories of structured spectra because the sphere spectrum is \aleph_0 -compact as a spectrum. Lastly, it holds for all the stable analogues of these structures because the compactness hypothesis is automatically preserved by any Bousfield localization (the set of generating cofibrations of $L_{\mathcal{C}}(\mathcal{M})$ is simply I again).

Remark 5.63 The proof above only uses the fact that the maps j_β were *h*-cofibrations in order to factor $Y \to A_\lambda$ through some finite stage. So if
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the (co)domains of I are finite relative to the class of weak equivalences then the proof above demonstrates that weak equivalences are preserved under transfinite composition [34, Corollary 7.4.2]. This property was already known classically for sSet and Ch(k), but need not hold for a general model category.

Proposition 5.64 Suppose \mathcal{M} a cofibrantly generated, h-monoidal, left proper, and that the (co)domains of I are cofibrant and are finite relative to the class of h-cofibrations. Suppose cofibrant objects are flat. Let $L_{\mathcal{C}}$ be a monoidal Bousfield localization. Then $L_{\mathcal{C}}(\mathcal{M})$ is h-monoidal.

Proof Suppose $f : A \to B$ is a cofibration in $L_{\mathcal{C}}(\mathcal{M})$ and Z is any object of $L_{\mathcal{C}}(\mathcal{M})$. We must show $f \otimes Z$ is an h-cofibration in $L_{\mathcal{C}}(\mathcal{M})$. Because $L_{\mathcal{C}}(\mathcal{M})$ is left proper, Proposition 1.5 in **5** reduces us to proving that there is a factorization of $f \otimes Z$ into a cofibration followed by a cofiber equivalence $w : X \to B \otimes Z$, i.e. for any map $g : X \to K$ the right-hand vertical map in the following pushout diagram is a \mathcal{C} -local equivalence:



Because f is a cofibration in \mathcal{M} , the *h*-monoidality of \mathcal{M} guarantees us that $f \otimes Z$ is an *h*-cofibration in \mathcal{M} . Apply the cofibration-trivial fibration factorization in \mathcal{M} . Note that this is also a cofibration-trivial fibration factorization of $f \otimes Z$ in $L_{\mathcal{C}}(\mathcal{M})$ because cofibrations and trivial fibrations in the two model categories agree. The resulting $w : X \to B \otimes Z$ is a trivial fibration in either model structure. Because \mathcal{M} is left proper we know that the map w is a cofiber equivalence in \mathcal{M} by Proposition 1.5 in [5] applied to the *h*-cofibration $f \otimes Z$. So in any pushout diagram as above the map $K \to T$ is a weak equivalence in \mathcal{M} , hence in $L_{\mathcal{C}}(\mathcal{M})$. Thus, w is a cofiber equivalence in $L_{\mathcal{C}}(\mathcal{M})$ and its existence proves $f \otimes Z$ is an *h*-cofibration in $L_{\mathcal{C}}(\mathcal{M})$.

Now suppose f were a trivial cofibration in $L_{\mathcal{C}}(\mathcal{M})$ to start. We must show that $f \otimes Z$ is a \mathcal{C} -local equivalence. We do this first in the case where f is a generating trivial cofibration. By hypothesis, A and B are

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cofibrant. Apply cofibrant replacement to Z:

The fact that cofibrant objects are flat in $L_{\mathcal{C}}(\mathcal{M})$ implies the vertical maps are \mathcal{C} -local equivalences (because A and B are cofibrant) and that the top horizontal map is a \mathcal{C} -local equivalence (because QZ is cofibrant). By the two out of three property the bottom horizontal map is a \mathcal{C} -local equivalence.

By Lemma 1.3 in [5], the class of *h*-cofibrations is closed under cobase change and retracts. By Lemma 1.6, the class of trivial *h*-cofibrations is closed under cobase change (because $L_{\mathcal{C}}(\mathcal{M})$ is left proper). Weak equivalences are always closed under retract. Finally, by Proposition [5.62 the class of trivial *h*-cofibrations is closed under transfinite composition by our compactness hypothesis on \mathcal{M} (equivalently, on $L_{\mathcal{C}}(\mathcal{M})$). So for a general *f* in the trivial cofibrations of $L_{\mathcal{C}}(\mathcal{M})$, realize *f* as a retract of $g \in J_{\mathcal{C}}$ -cell, so that $g \otimes Z$ is a transfinite composite of pushouts of maps of the form $j \otimes Z$ for $j \in J_{\mathcal{C}}$. We have just proven that all $j \otimes Z$ are trivial *h*-cofibrations and closure properties imply $g \otimes Z$ and hence $f \otimes Z$ are trivial *h*-cofibrations as well. \Box

Theorem 5.65 Suppose \mathcal{M} is a cofibrantly generated, left proper, hmonoidal model category such that the (co)domains of I are cofibrant and are finite relative to the h-cofibrations and cofibrant objects are flat. Then for any monoidal Bousfield localization $L_{\mathcal{C}}$, the model category $L_{\mathcal{C}}(\mathcal{M})$ satisfies the monoid axiom.

Proof Apply Proposition 5.62 to the category $L_{\mathcal{C}}(\mathcal{M})$. By Proposition 5.64, $L_{\mathcal{C}}(\mathcal{M})$ is *h*-monoidal. It is left proper because \mathcal{M} is left proper. To verify the monoid axiom, consider a λ -sequence of maps that are pushouts of maps in $\{f \otimes Z \mid f \text{ is a trivial cofibration in } L_{\mathcal{C}}(\mathcal{M})\}$. Such maps are *h*-cofibrations in \mathcal{M} because \mathcal{M} is *h*-monoidal, *f* is a cofibration in \mathcal{M} , and *h*-cofibrations are closed under pushout. Thus, the hypothesis that the (co)domains of *I* are finite relative to the *h*-cofibrations in \mathcal{M} is sufficient to construct the lift in Proposition 5.62 and to prove the transfinite composition part of the proof of the monoid axiom. \Box

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