MODEL STRUCTURES ON COMMUTATIVE MONOIDS IN GENERAL
MODEL CATEGORIES

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ABSTRACT. We provide conditions on a monoidal model category \( M \) so that the subcategory of commutative monoids in \( M \) inherits a model structure from \( M \) in which a map is a weak equivalence or fibration if and only if it is so in \( M \). We then investigate properties of cofibrations of commutative monoids, functoriality of the passage from a commutative monoid \( R \) to the category of commutative \( R \)-algebras, rectification between \( E_\infty \)-algebras and commutative monoids, and the relationship between commutative monoids and monoidal Bousfield localization functors. In the final section we provide numerous examples of model categories satisfying our hypotheses.

1. Introduction

In recent years, the importance of monoidal model categories has been demonstrated by a number of striking results related to structured (equivariant) ring spectra, c.f. [9], [22], [17], [28], [41]. Commutative monoids played a key role in many of these applications, and it became important to have a model structure on objects with commutative structure, compatibly with the monoidal model structure on the underlying category \( M \).

The non-commutative case was treated in [36], where the authors introduced the monoid axiom. They prove that if \( M \) satisfies the monoid axiom then the category of monoids in \( M \) inherits a model structure from \( M \) with weak equivalences (resp. fibrations) maps that are weak equivalences (resp. fibrations) in \( M \). They then verify that the monoid axiom holds for all examples of interest.

In this paper we will take a similar approach and introduce the commutative monoid axiom, which guarantees us that commutative monoids in \( M \) inherit a model structure. In [36], the authors refer to the commutative situation as “intrinsically more complicated” and indeed there are several known cases where commutative monoids cannot inherit a model structure in the way above, e.g. commutative differential graded algebras over a field of nonzero characteristic, \( \Gamma \)-spaces, and non-positive model structures on symmetric or orthogonal spectra (due to an example of Gaunce Lewis in [24]). Side-stepping Lewis’s example required the introduction of positive variants on diagram spectra in [28], and the convenient model structure on symmetric spectra introduced in [38] (nowadays referred to as the positive flat model structure). We discuss these examples in Section 5.
One way to get around these obstacles is to work with $E_\infty$-algebras everywhere and never ask for strict commutativity. It is much easier to place a model structure on $E_\infty$-algebras because $E_\infty$ is a cofibrant operad, while $Com$ is not. We feel it is important to also be able to treat the strict commutative case, in particular because outside of categories of structured ring spectra one does not know that there is a Quillen equivalence between $E_\infty$-algebras and strictly commutative monoids (because $Com$ is not even $\Sigma$-cofibrant, one cannot use the general rectification results in [2]). The crucial hypothesis which allows such a Quillen equivalence in the case of structured ring spectra is that for all cofibrant $X$, the map $(E_{\Sigma n})_+ \wedge_{\Sigma_n} X^{\vee n} \to X^{\vee n}/\Sigma_n$ is a weak equivalence. It is important to note that this hypothesis is not necessary for strictly commutative monoids to inherit a model structure (in particular, it fails for simplicial sets). This hypothesis appears to be more related to the rectification question than to the question of existence of model structures. We address the point further in Section 4.2.

Due to the difficulties associated with passing model structures to categories of commutative monoids, several important papers have folded the existence of a model structure on commutative monoids into their hypotheses. This is done in Assumption 1.1.0.4 in [41] and in Hypothesis 5.5 in [37], among other places. The results in Section 3 provide checkable conditions on $\mathcal{M}$ so that those hypotheses are satisfied.

We remark that a different axiom on $\mathcal{M}$ which guarantees commutative monoids inherit a model structure has appeared as Proposition 4.3.21 in [25]. However, it is pointed out in [26] that this work contains some errors and as written does not apply to the positive model structure on symmetric spectra. Furthermore, we will demonstrate that it does not apply to topological spaces, though it does apply to chain complexes over a field of characteristic zero. Our commutative monoid axiom is more general, and does apply in these situations.

After a review of model category preliminaries in Section 2, we will proceed to state the commutative monoid axiom and prove our main result in Section 3 highlighting differences from the situation of [25] as we go. We additionally discuss when a cofibration of commutative monoids forgets to a cofibration in $\mathcal{M}$, and we introduce the strong commutative monoid axiom to guarantee this occurs. Following [36], we place the details of the proofs of these main results in Appendix B and we also prove in Appendix A that it is sufficient to check the strong commutative monoid axiom on the generating (trivial) cofibrations. Using this, we collect examples in Section 5. We include additional results regarding functoriality of the passage from $R$ to commutative $R$-algebras, regarding rectification between $Com$ and $E_\infty$, and remarks regarding the interplay between the strong commutative monoid axiom and Bousfield localization in Section 4.

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2. Preliminaries

We assume the reader is familiar with basic facts about model categories. Excellent introductions to the subject can be found in [8], [21], or [18]. Throughout the paper we will assume $M$ is a cofibrantly generated model category, i.e. there is a set $I$ of cofibrations and a set $J$ of trivial cofibrations which permit the small object argument (with respect to some cardinal $\kappa$), and a map is a (trivial) fibration if and only if it satisfies the right lifting property with respect to all maps in $J$ (resp. $I$). Let $I$-cell denote the class of transfinite compositions of pushouts of maps in $I$, and let $I$-cof denote retracts of such. In order to run the small object argument, we will assume the domains $K$ of the maps in $I$ (and $J$) are $\kappa$-small relative to $I$-cell (resp. $J$-cell), i.e. given a regular cardinal $\lambda \geq \kappa$, any $\lambda$-sequence $X_0 \to X_1 \to \ldots$ formed of maps $X_\beta \to X_{\beta+1}$ in $I$-cell, then the map of sets $\operatorname{colim}_{\beta < \lambda} M(K, X_\beta) \to M(K, \operatorname{colim}_{\beta < \lambda} X_\beta)$ is a bijection. An object is small if there is some $\kappa$ for which it is $\kappa$-small. See Chapter 10 of [18] for a more thorough treatment of this material. For any object $X$ we have a cofibrant replacement $QX \to X$ and a fibrant replacement $X \to RX$.

Our model category $M$ will also be a monoidal category with product $\otimes$ and unit $S \in M$. In order to ensure that the monoidal structure interacts nicely with the model structure (e.g. to guarantee it passes to a monoidal structure on the homotopy category $\text{Ho}(M)$ whose unit is given by $S$) we must assume

1. Unit Axiom: For any cofibrant $X$, the map $QS \otimes X \to S \otimes X \equiv X$ is a weak equivalence

2. Pushout Product Axiom: Given any $f : X_0 \to X_1$ and $g : Y_0 \to Y_1$ cofibrations, $f \Box g : X_0 \otimes Y_1 \coprod_{X_0 \otimes Y_0} X_1 \otimes Y_0 \to X_0 \otimes Y_0$ is a cofibration. Furthermore, if either $f$ or $g$ is trivial then $f \Box g$ is trivial.

If these hypotheses are satisfied then $M$ is called a monoidal model category. Note that the pushout product axiom forces $\otimes$ to be a Quillen bifunctor. Furthermore, it is sufficient to check the pushout product axiom on the generating maps $I$ and $J$, by Lemma 3.5 in [36].

We turn now to the problem of placing model structures on subcategories of algebras. Let $P$ be an operad valued in $M$. For this discussion it will be fine to think of $P$ as $\text{Ass}$ or $\text{Com}$. For a general discussion of operads see [2]. Let $P$-alg denote the subcategory of $M$ whose objects are $P$-algebras (i.e. admit an action of $P$) and whose morphisms are $P$-algebra homomorphisms (i.e. respect the $P$-action).
Let \( P : \mathcal{M} \to \mathcal{P}\text{-alg} \) be the free \( \mathcal{P}\text{-alg} \) functor and let \( U : \mathcal{P}\text{-alg} \to \mathcal{M} \) be the forgetful functor. Then \((P, U)\) is an adjoint pair. When \( \mathcal{P} = \text{Ass} \), the free monoid functor \( X \mapsto S \coprod X \coprod X \coprod \ldots \) has been known to topologists for years as the James construction. When \( \mathcal{P} = \text{Com} \), the free commutative monoid functor \( X \mapsto S \coprod X \coprod X \coprod / \Sigma_2 \coprod \ldots \) is sometimes called the \( SP^\infty \) functor, or the Dold-Thom functor.

In order for there to be a model structure on \( \mathcal{P}\text{-alg} \) which is compatible with the model structure on \( \mathcal{M} \), it must be the model structure which is transferred across the pair \((P, U)\) so that \((P, U)\) forms a Quillen pair. In particular, a weak equivalence or fibration of \( \mathcal{P}\text{-alg} \)s will be a map which is a weak equivalence or fibration in \( \mathcal{M} \).

If such a model structure on \( \mathcal{P}\text{-alg} \) exists, we say it is inherited from \( \mathcal{M} \). Proving the existence of this model structure comes down to Lemma 2.3 in [36]:

**Lemma 2.1.** Suppose \( \mathcal{M} \) is cofibrantly generated and \( T \) is a monad which commutes with filtered direct limits. If the domains of \( T(I) \) and \( T(J) \) are small relative to \( T(I)\)-cell and \( T(J)\)-cell respectively and either

1. \( T(J)\)-cell \( \subseteq \mathcal{W} \), or
2. All objects are fibrant and every \( T\)-algebra has a path object (factoring \( \delta : X \to X \otimes X \) into \( \hookrightarrow \to \to \))

then \( T\text{-alg} \) inherits a cofibrantly generated model structure with fibrations and weak equivalences created by the forgetful functor to \( \mathcal{M} \).

When the conditions of this lemma are satisfied, \( \mathcal{P}\text{-alg} \) inherits a cofibrantly generated model structure in which \( P(I) \) and \( P(J) \) are the generating (trivial) cofibrations. The case \( \mathcal{P} = \text{Ass} \) was treated in [36] and checking the first condition of the lemma led to the introduction of the following axiom on a model category:

**Definition 2.2.** Given a class of maps \( C \) in \( \mathcal{M} \), let \( C \otimes \mathcal{M} \) denote the class of maps \( f \otimes id_X \) where \( f \in C \) and \( X \in \mathcal{M} \). A model category is said to satisfy the monoid axiom if every map in \( (\text{Trivial-Cofibrations} \otimes \mathcal{M})\)-cell is a weak equivalence.

Let \( A \) be any monoid and let \( R \) be any commutative monoid. In [36] and the follow-up paper [20], it is proven that if \( \mathcal{M} \) satisfies the monoid axiom and if the domains of \( I \) (resp. \( J \)) are small relative to \( (\mathcal{M} \otimes I)\)-cell (resp. \( (\mathcal{M} \otimes J)\)-cell), then the categories of (left or right) \( A\)-modules and of \( R\)-algebras inherit model structures from \( \mathcal{M} \). We will require the same smallness hypothesis in Section 3. It is satisfied automatically if \( \mathcal{M} \) is a combinatorial model category.

In [36] it is proven that it is sufficient to check the monoid axiom on the generating trivial cofibrations and that many model categories of interest satisfy the monoid axiom. We will conduct a similar program for the strong commutative monoid axiom in Section 5 and in Appendix A.
3. A model structure on commutative monoids

We are now ready to prove the commutative analog of the work summarized above. We first introduce the commutative analog to the monoid axiom.

**Definition 3.1.** A monoidal model category $\mathcal{M}$ is said to satisfy the **commutative monoid axiom** if whenever $h$ is a trivial cofibration in $\mathcal{M}$ then $h^\otimes n/\Sigma_n$ is a trivial cofibration in $\mathcal{M}$ for all $n > 0$.

Under this hypothesis, we state our main theorem:

**Theorem 3.2.** Let $\mathcal{M}$ be a cofibrantly generated monoidal model category satisfying the commutative monoid axiom and the monoid axiom, and assume that the domains of the generating maps $I$ (resp. $J$) are small relative to $(I \otimes \mathcal{M})$-cell (resp. $(J \otimes \mathcal{M})$-cell). Let $R$ be a commutative monoid in $\mathcal{M}$. Then the category $\text{CAlg}(R)$ of commutative $R$-algebras is a cofibrantly generated model category in which a map is a weak equivalence or fibration if and only if it is so in $\mathcal{M}$. In particular, when $R = S$ this gives a model structure on commutative monoids in $\mathcal{M}$.

It is clear from this description of $\text{CAlg}(R)$ that if $\mathcal{M}$ is simplicial then $\text{CAlg}(R)$ is simplicial. Simply use that (trivial) fibrations are created in $\mathcal{M}$ and use the pullback formulation of the SM7 axiom. Because $\text{CAlg}(R)$ is a subcategory of $\mathcal{M}$, we can also see that if $\mathcal{M}$ is combinatorial then $\text{CAlg}(R)$ is combinatorial.

Recall that a model category $\mathcal{M}$ is said to be **tractable** if the domains of the generating (trivial) cofibrations are cofibrant in $\mathcal{M}$. As the generating (trivial) cofibrations of $\text{CAlg}(R)$ are of the form $R \otimes \text{Sym}(I)$ (resp. $R \otimes \text{Sym}(J)$), these are cofibrant in $\text{CAlg}(R)$ if $\mathcal{M}$ is tractable and satisfies the commutative monoid axiom. Hence, tractability also passes from $\mathcal{M}$ to $\text{CAlg}(R)$.

**Proof sketch.** We will focus first on the case where $R$ is the monoidal unit $S$, and discuss general $R$ at the end. As commutative $S$-algebras are simply commutative monoids, we denote the category of commutative monoids $\text{CMon}(\mathcal{M})$ rather than $\text{CAlg}(S)$. We will verify condition (1) of Lemma 2.1 for the monad coming from the $(\text{Sym}, U)$ adjunction between $\mathcal{M}$ and $\text{CMon}(\mathcal{M})$. Let $J$ denote the generating trivial cofibrations of $\mathcal{M}$. We must prove that maps in $\text{Sym}(J)$-cell are weak equivalences. Given a trivial cofibration $h : K \to L$ in $\mathcal{M}$, we form the following pushout square in $\text{CMon}(\mathcal{M})$ and must prove that the bottom map is a map of the sort considered by the monoid axiom, so that transfinite compositions of such maps are weak equivalences in $\mathcal{M}$ (hence weak equivalences of commutative monoids):

$$
\begin{array}{ccc}
\text{Sym}(K) & \longrightarrow & \text{Sym}(L) \\
\downarrow & & \downarrow \\
X & \otimes & P \\
\end{array}
$$

Of course, in $\text{CMon}(\mathcal{M})$, the pushout is simply the tensor product, so $P \equiv X \otimes \text{Sym}(K)$ $\text{Sym}(L)$, but we will not make use of this fact. Following [36], we construct a
filtration of the map of commutative monoids $X \to P$ as a composition $P_n \to P_{n+1}$ of maps formed by pushout diagrams in $\mathcal{M}$. Doing so requires the decomposition of $\text{Sym}(K) = \bigsqcup_n \text{Sym}^n(K)$ where $\text{Sym}^n(K) = K^\otimes_n / \Sigma_n$.

Thinking of $P$ as formal products of elements from $X$ and from $L$ with relations in $K$ leads to a consideration of $n$-dimensional cubes to build products of length $n$ from the letters $X$, $K$, $L$. Because the map $\text{Sym}(K) \to X$ is adjoint to a map $K \to X$, we will in fact only need to consider $n$-dimensional cubes whose vertices are length $n$ words in the letters $K$ and $L$. Formally, for any subset $D$ of $[n] = \{1, 2, \ldots, n\}$ we obtain a vertex $C_1 \otimes \cdots \otimes C_n$ with $C_i = K$ if $i \notin D$ and $C_j = L$ if $j \in D$. The punctured cube is the cube with the vertex $L^\otimes_n$ removed. The map $h_\square^n$ is the induced map from the colimit $Q_n$ of the punctured cube to $L^\otimes_n$.

There is an action of $\Sigma_n$ on the cube which permutes the letters in the words (equivalently, which permutes the vertices in the cube in a way coherent with respect to the edges of the cube). Explicitly, the action is defined as follows. Any $\sigma \in \Sigma_n$ sends the vertex defined above to the vertex corresponding to $\sigma(D) \subset [n]$ using the action of $\Sigma_{|D|}$ on the $X$'s and $\Sigma_{n-|D|}$ on the $Y$'s. This action yields a $\Sigma_n$-action on $h_\square^n : Q_n \to L^\otimes_n$, and in a moment we will pass to $\Sigma_n$-coinvariants.

We now show how to obtain $P_n$ (which in this analogy is to be thought of as formal products of length $n$) from the cubes we have just described. The steps in the filtration of $X \to P$ are formed by pushouts of the maps $id_X \otimes h_\square^n / \Sigma_n$:

$$
\begin{array}{ccc}
X \otimes Q_n / \Sigma_n & \longrightarrow & X \otimes L^\otimes_n / \Sigma_n \\
\downarrow & & \downarrow \\
P_{n-1} & \longrightarrow & P_n
\end{array}
$$

The proof that the $P_n$ provide a filtration of $X \to P$ is delayed until Appendix [\ref{appendix}]. Assuming the commutative monoid axiom, the maps $h_\square^n / \Sigma_n$ are trivial cofibrations. Thus, the map $X \to P$ is a transfinite composite of pushouts of maps in $\mathcal{M} \otimes \{\text{trivial cofibrations}\}$. Hence, by the monoid axiom, $X \to P$ is a weak equivalence. Similarly, for any transfinite composition $f$ of pushouts of maps of the form $\text{Sym}(K) \to \text{Sym}(L)$, we may realize $f$ as a transfinite composition of maps $X \to P$ of the form above. As a transfinite composition of transfinite compositions is still a transfinite composition, the monoid axiom applies again and proves $f$ is a weak equivalence. Lemma [\ref{monoid-lemma}] now applies to produce the required model structure on commutative monoids.

To handle the case of commutative $R$-algebras, note that there is an equivalence of categories between $\text{CAlg}(R)$ and $(R \downarrow \text{CMon}(\mathcal{M}))$, the category of commutative monoids under $R$. So we may apply the remark after Proposition 1.1.8 of [\cite{21}] to conclude that this is a model category with cofibrations, fibrations, and weak equivalences inherited from $\text{CMon}(\mathcal{M})$. Note that this is a different approach from the one provided in [\cite{36}] because we do not pass through $R$-modules en route to commutative $R$-algebras. That $\text{CAlg}(R)$ is cofibrantly generated follows from [\cite{19}],...
where it is also shown that the generating cofibrations are given by the set $I_R$ of maps in $(R \downarrow \text{CMon}(M))$ where the map in $\text{CMon}(M)$ is in $I$. Under the equivalence of categories between $\text{CAlg}(R)$ and $(R \downarrow \text{CMon}(M))$, such maps are sent to maps in $R \otimes \text{Sym}(J)$. We can similarly identify the generating trivial cofibrations as $R \otimes \text{Sym}(J)$.

\[ \square \]

Remark 3.3. Notice that the proof in fact requires less than the full strength of the hypotheses. Rather than assuming the commutative monoid axiom and the monoid axiom separately, we could have assumed that transfinite compositions of pushouts of maps in $\{M \otimes h^{\Sigma_n} \mid h \text{ is a trivial cofibration}\}$ are contained in the weak equivalences. We will refer to this property as the weak commutative monoid axiom. Certain model categories discussed in Section 5 only satisfy this axiom and not the commutative monoid axiom. However, for reasons which will become clear in Corollary 3.8, we have chosen the commutative monoid axiom as the appropriate axiom for our applications.

The full proof in Appendix [3] will in fact prove more than just the theorem. It will also prove the commutative analog to Lemma 6.2 of [36], from which one can deduce the proposition below regarding when cofibrations of commutative monoids forget to cofibrations in $\mathcal{M}$. It is well-known to experts that obtaining the correct behavior of cofibrations under the forgetful functor is subtle in the commutative setting. Indeed, this was the motivation behind the convenient model structures introduced in [38] and [40]. In order to guarantee the desired behavior we must strengthen the commutative monoid axiom.

Definition 3.4. A monoidal model category $\mathcal{M}$ is said to satisfy the strong commutative monoid axiom if whenever $h$ is a (trivial) cofibration in $\mathcal{M}$ then $h^{\Sigma_n} \otimes \Sigma_n$ is a (trivial) cofibration in $\mathcal{M}$ for all $n > 0$. In particular, we are now assuming that cofibrations are closed under the operation $(-)^{\Sigma_n}$.

Proposition 3.5. Suppose $\mathcal{M}$ satisfies the strong commutative monoid axiom. Then for any commutative monoid $R$, a cofibration in $\text{CAlg}(R)$ with source cofibrant in $\mathcal{M}$ is a cofibration in $\mathcal{M}$.

Corollary 3.6. Suppose $\mathcal{M}$ satisfies the strong commutative monoid axiom and that $S$ is cofibrant in $\mathcal{M}$. Then any cofibrant commutative monoid is cofibrant in $\mathcal{M}$. If in addition $R$ is cofibrant in $\mathcal{M}$ then any cofibrant commutative $R$-algebra is cofibrant in $\mathcal{M}$.

See Appendix [3] for proof of this proposition.

Corollary 3.7. Assume $S$ is cofibrant in $\mathcal{M}$ and that $\mathcal{M}$ satisfies the strong commutative monoid axiom. If $f$ is a cofibration between cofibrant objects then $\text{Sym}(f)$ is a cofibration in $\mathcal{M}$. In particular, if $X$ is cofibrant in $\mathcal{M}$ then $\text{Sym}(X)$ is cofibrant in $\mathcal{M}$.
Proof. Because the model structure on $\text{CMon}(\mathcal{M})$ is transferred from that of $\mathcal{M}$, the functor $\text{Sym}(-)$ is left Quillen, and hence preserves cofibrations. So $\text{Sym}(f)$ is a cofibration of commutative monoids because $f$ is a cofibration in $\mathcal{M}$. If the source $K$ of $f$ is cofibrant then the source of $\text{Sym}(f)$ is a cofibrant commutative monoid, by applying $\text{Sym}(-)$ to the cofibration $\varnothing \hookrightarrow K$. By Corollary 3.6 the source of $\text{Sym}(f)$ is cofibrant in $\mathcal{M}$. By Proposition 3.5 $\text{Sym}(f)$ is a cofibration in $\mathcal{M}$.

Recall that the point of positive model structures on diagram spectra (e.g. symmetric spectra or orthogonal spectra) is to break the cofibrancy of $S$ and so avoid Lewis’s obstruction [24] to having a model structure on commutative ring spectra. Thus, these corollaries do not apply to positive model categories of spectra. In [38], a variant on the positive model structure is introduced in which cofibrant commutative ring spectra are cofibrant as spectra. This model structure was known in that paper as the convenient model structure, and later as the positive flat model structure. We do not know how to obtain this ‘convenient’ property for general model categories. We suspect it has something to do with forcing the cofibrations to contain the monomorphisms.

The proof of Theorem 3.2 makes clear precisely where the monoid axiom is being used, and hence why the smallness hypotheses are needed. If $\mathcal{M}$ does not satisfy the monoid axiom, then we can make this step work by assuming $X$ is a cofibrant commutative monoid. In this case, [20] and [39] make it clear that a semi-model structure can be obtained. We summarize this as a corollary, so that we may reference it in [42].

Corollary 3.8. Let $\mathcal{M}$ be a cofibrantly generated monoidal model category satisfying the commutative monoid axiom, and assume that the domains of the generating maps $I$ (resp. $J$) are small relative to $(I \otimes \mathcal{M})$-cell (resp. $(J \otimes \mathcal{M})$-cell). Then for any commutative monoid $R$, the category of commutative $R$-algebras is a cofibrantly generated semi-model category in which a map is a weak equivalence or fibration if and only if it is so in $\mathcal{M}$.

Proof. We begin with the case where $R = S$, so that we are building a semi-model structure on $\text{CMon}(\mathcal{M})$. Consider the proof of Theorem 3.2. All of the model category axioms are purely formal except for factorization of an arbitrary map into a trivial cofibration followed by a fibration, and except for lifting of trivial cofibrations against fibrations (which follows from factorization and the retract property). The monoid axiom is not used until we have already proven that the pushout of commutative monoids

$$
\begin{array}{ccc}
\text{Sym}(K) & \longrightarrow & \text{Sym}(L) \\
\downarrow & & \downarrow \\
X & \longrightarrow & P
\end{array}
$$
can be factored into \( X = P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P \) where each \( P_{n-1} \rightarrow P_n \) is a pushout of \( X \otimes f_{\Sigma_n}^{\text{fin}} \). By the commutative monoid axiom, \( f_{\Sigma_n}^{\text{fin}} \) is a trivial cofibration. Without the monoid axiom it is not clear how to proceed unless \( X \) is cofibrant. If \( X \) is cofibrant, then this map has the form \( (\otimes \hookrightarrow X) \Box f_{\Sigma_n}^{\text{fin}} \) and hence is a trivial cofibration by the pushout product axiom. Thus, the pushout \( P_{n-1} \rightarrow P_n \) must also be a trivial cofibration, and the composite \( X \rightarrow P \) is a composite of trivial cofibrations and hence a trivial cofibration.

From this argument, we may conclude that factorization into a trivial cofibration followed by a fibration works for maps with cofibrant domain. Similarly, lifting of trivial cofibrations \( g \) against fibrations is satisfied if \( g \) has cofibrant domain. In particular, notice that if all objects are cofibrant in \( \mathcal{M} \) then a semi-model structure is the same as a model structure, because adding the assumption that \( X \) is cofibrant changes nothing.

For the case of a general commutative monoid \( R \), observe that the semi-model structure on \( \text{CMon}(\mathcal{M}) \) yields a semi-model structure on \( \text{CAlg}(R) = R \downarrow \text{CMon}(\mathcal{M}) \) purely formally, since all of the model category axioms are satisfied if and only if they are satisfied in \( \text{CMon}(\mathcal{M}) \). To prove the resulting semi-model structure is cofibrantly generated, we again reference [19]. The semi-model structure on \( \text{CAlg}(R) \) is transferred from the structure on \( \text{CMon}(\mathcal{M}) \) along the adjunction \( R \otimes - : \text{CMon}(\mathcal{M}) \rightleftarrows \text{CAlg}(R) : U \), and hence along the adjunction \( \mathcal{M} \rightleftarrows \text{CAlg}(R) \) obtained from composing with the adjunction \( \text{Sym}(\cdot) : \mathcal{M} \rightleftarrows \text{CMon}(\mathcal{M}) : U \). This yields sets of maps \( FI \) and \( FJ \) which cofibrantly generate a semi-model structure on \( \text{CAlg}(R) \).

That the resulting semi-model structure on \( \text{CAlg}(R) \) matches the one inherited from \( \text{CMon}(\mathcal{M}) \) follows precisely as in [19] applied to the adjunction \( R \otimes - : \text{CMon}(\mathcal{M}) \rightleftarrows \text{CAlg}(R) : U \). Weak equivalences are clearly the same. In order to prove that fibrations are the same, one must use the fact that in the cofibrantly generated semi-model category \( \text{CMon}(\mathcal{M}) \), a map is a fibration if and only if it has the right lifting property with respect to the generating trivial cofibrations. Thankfully, this is part of the definition of a cofibrantly generated semi-model category given in [39]. Lastly, if two semi-model structures have the same fibrations and weak equivalences then they have the same cofibrations, because a map is a cofibration if and only if it satisfies the left lifting property with respect to trivial fibrations (see [39]).

Observe that if one wishes to obtain on \( \text{CAlg}(R) \) a semi-model structure over \( \mathcal{M} \) in the terminology of [39] then one must also assume that \( S \) and \( R \) are cofibrant so that the initial object in \( \text{CAlg}(R) \) forgets to a cofibrant object in \( \mathcal{M} \).

As the filtration given in Appendix [8] is related to Harper’s filtration for general operads from [16], we pause for a moment to compare these two approaches.
Remark 3.9. Harper’s general machinery describes the map $P_{n-1} \to P_n$ as a pushout

$$\begin{array}{ccc}
Com_X[n] \otimes_{\Sigma_n} Q_n & \longrightarrow & Com_X[n] \otimes_{\Sigma_n} L^{\Sigma_n} \\
\downarrow & & \downarrow \\
P_{n-1} & \longrightarrow & P_n
\end{array}$$

where $Com_X$ is the enveloping operad. One may use Proposition 7.6 in [16] to write $Com_X[n] = X$ with the trivial $\Sigma_n$ action. Thus, $P_{n-1} \to P_n$ can be written as the pushout of $X \otimes f^{Com}/\Sigma_n$ and Harper’s filtration makes it clear that the commutative monoid axiom is precisely the right hypothesis.

In a similar way, $Ass_X[n] = X^{\otimes n+1} \cdot \Sigma_n$, i.e. the coproduct of $n!$ copies of $X^{\otimes n+1}$ with the free $\Sigma_n$ action. So in that case the $(\cdot \otimes \Sigma_n)(\cdot)$ provides a cancellation and Harper’s filtration reduces to a pushout of $X^{\otimes n+1} \otimes f^{Com}$. We see immediately why the monoid axiom is necessary.

Finally, one could realize commutative $R$-algebras as algebras over the operad $Com_R$ and in this case Harper’s filtration would be a pushout of a map of the form $(Com_R)_A[n] \otimes_{\Sigma_n} f^{Com}$ where $A$ is a commutative $R$-algebra. In this case, the formula in Proposition 7.6 yields $(Com_R)_A[n] = R \otimes A$ and so the maps $P_{n-1} \to P_n$ are pushouts of $(R \otimes A) \otimes f^{Com}/\Sigma_n$. In this way we see that in the presence of the commutative monoid axiom but in the absence of the the monoid axiom we need both $R$ and $A$ to be cofibrant in order to ensure that this map is a trivial cofibration, i.e. to obtain on $CAlg(R)$ a semi-model structure over $M$. This is the commutative analog of Theorem 3.3 in [20], in which cofibrancy of $R$ was required to achieve a semi-model structure on $R$-algebras. There the formula $(Ass_R)_A[n] = R \otimes A \cdot \Sigma_n$ means that the relevant pushout takes the form $R \otimes A \otimes f^{Com}$ and this makes clear why both $R$ and $A$ must be cofibrant in the absence of the monoid axiom.

We conclude this section with a remark comparing our approach and results with the approach outlined by Lurie in [25], in which he proved:

**Theorem 3.10.** Let $M$ be a left proper, combinatorial, tractable, monoidal model category satisfying the monoid axiom and with a cofibrant unit. Assume further that

(*) If $h$ is a cofibration then $h^{Com}$ is a cofibration in the projective model structure on $M^{\Sigma_n}$ for all $n$. Such maps $h$ are called power cofibrations.

Then $CMon(M)$ has a model category structure with weak equivalences and fibrations inherited from $M$.

The difference between this result and Theorem 3.2 is that in Theorem 3.2 we do not require $M$ to be left proper, we do not require the unit to be cofibrant, we do not require the model structure to be tractable, we weaken combinatoriality to a much lesser smallness hypothesis, and we weaken (*) to the commutative monoid axiom. We have also discussed how to remove the monoid axiom. Note that Lurie
also assumes $\mathcal{M}$ is simplicial, but never uses this assumption. The assumption that the unit is cofibrant is part of what Lurie requires of a monoidal model category. However, the unit is not cofibrant in the positive and positive flat model structures on categories of spectra. For this reason, Theorem 3.10 cannot apply to such examples as stated, but elements of the proof have been made to apply to the positive flat stable model structure in [31].

We refer to condition (*) as Lurie’s hypothesis. It implies the strong commutative monoid axiom as shown in Lemma 4.3.28 of [25]. The key observation is that $(-)/\Sigma_n : \mathcal{M}^{\Sigma_n} \to \mathcal{M}$ is the left adjoint of a Quillen pair where the right adjoint is the constant diagram functor (i.e. endows an object with the trivial $\Sigma_n$ action). Thus, if (*) is satisfied and we apply this map to the projective cofibration $f_{\Sigma n}$ we obtain the strong commutative monoid axiom. However, (*) assumes strictly more than the strong commutative monoid axiom, as evidenced in Section 5 where we show that simplicial sets and topological spaces satisfy the latter but not the former.

Note that Lurie’s Proposition 4.3.21 is slightly more general than what we’ve stated above in that it only requires that there is some combinatorial model structure $\mathcal{M}_V$ on the relative category $\mathcal{M}$, and that $\mathcal{M}_V$ has cofibrations $V$ generated by cofibrations between cofibrant objects and satisfying (*). In this case $\mathcal{M}$ is said to be freely powered by $V$. We could also do our work in that level of generality, but choose not to because it seems unnatural to place a hypothesis on a model category which references the existence of some other model category. The point is that this extra generality does not buy us anything because $\mathcal{M}$ and $\mathcal{M}_V$ will be Quillen equivalent by Lurie’s Remark 4.3.20.

Lurie does not prove that it is sufficient to check hypothesis (*) on the generating (trivial) cofibrations, but this has been done in [31].

4. Additional Results

4.1. Functoriality and Homotopy Invariance. We turn now to the question of whether or not the passage from $R$ to $\text{CAlg}(R)$ is functorial and has good homotopy theoretic properties. Following [26], we provide a condition so that the homotopy theory of commutative $R$-algebras only depends on the weak equivalence type of $R$. Recall that a monoidal model category $\mathcal{M}$ is said to satisfy the property that cofibrant objects are flat if for all cofibrant $X$ and all weak equivalences $f$, the map $X \otimes f$ is a weak equivalence. This property can be viewed as a global version of the unit axiom (which is the same statement restricted to the cofibrant replacement map $f : QS \to S$).

**Theorem 4.1.** Suppose $\mathcal{M}$ satisfies the conditions of Theorem 3.2

1. The passage from $R$ to $\text{CAlg}(R)$ is functorial: given a ring homomorphism $f : R \to T$, restriction and extension of scalars provides a Quillen adjunction between $\text{CAlg}(R)$ and $\text{CAlg}(T)$. 

(2) Suppose that for any cofibrant left $R$-module $N$, the functor $N \otimes_R -$ preserves weak equivalences. Let $f : R \to T$ be a weak equivalence of commutative monoids. Then $f$ induces a Quillen equivalence $CAlg(R) \cong CAlg(T)$.

**Proof.** Let $f : R \to T$ be a ring homomorphism.

(1) The map $f$ makes $T$ into an $R$-module, and provides the extension of scalars functor from $CAlg(R)$ to $CAlg(T)$, i.e. $N \cong R \otimes_R N \to T \otimes R N$. Because weak equivalences and fibrations are defined in the underlying category, the right adjoint restriction functor preserves (trivial) fibrations. So they form a Quillen pair and the extension functor preserves (trivial) cofibrations.

(2) To check that extension and restriction form a Quillen equivalence in this case, we use Corollary 1.3.16(c) of [21]. First, note that restriction reflects weak equivalences between fibrant objects because the weak equivalences and fibrations in these two categories are the same. Next, suppose $N$ is a cofibrant commutative $R$-algebra. The map $N \cong R \otimes_R N \to T \otimes_R N$ is a weak equivalence because cofibrant objects are flat. Thus, restriction and extension of scalars form a Quillen equivalence.

□

An alternative approach for (2) which avoids the need for cofibrant $R$-modules to be flat is suggested by Theorem 2.4 of [20] in the non-commutative case. Note that we do not require the unit to be cofibrant as Hovey did. This is because we do not obtain our model structure on $CAlg(R)$ from $R$-mod. Rather, we obtain it as the undercategory of $CMon(M)$. Via Remark [3.9] we may view the generating cofibrations of $CAlg(R)$ as $R \otimes \text{Sym}(I)$ where $I$ is the set of generating cofibrations for $M$.

**Theorem 4.2.** Suppose $M$ has a cofibrant unit, satisfies the commutative monoid axiom, and that the domains of the generating cofibrations are cofibrant. Suppose $R$ and $T$ are commutative monoids which are cofibrant in $M$ and suppose $f : R \to T$ is a weak equivalence. Then extension and restriction of scalars is a Quillen equivalence between $CAlg(R)$ and $CAlg(T)$.

**Proof.** We follow the model of Hovey’s proof in [20]. All that must be shown is that for all cofibrant $R$-modules $M$, $M \to M \otimes_R T$ is a weak equivalence. Because $M$ is cofibrant we may write $M = \text{colim} M_\alpha$ where $M_0 = 0$ and $M_\alpha \to M_{\alpha + 1}$ is a pushout of a map in $R \otimes \text{Sym}(I)$. For concreteness we will let $K \to L$ denote the map in $I$ which is used in this pushout.

We show by transfinite induction that $M_\alpha \to M_\alpha \otimes_R T$ is a weak equivalence for all $\alpha$. The base case is trivial because $M_0 = 0$. For the successor case, apply the left adjoint $- \otimes_R T$ to the pushout square defining $M_\alpha \to M_{\alpha + 1}$ and the result will again...
be a pushout square. There is also a map from the former pushout square to the latter, induced by the adjunction. We will apply the Cube Lemma (Lemma 5.2.6 in [21]) to the resulting cube.

\[
\begin{array}{ccc}
R \otimes \text{Sym}(K) & \longrightarrow & R \otimes \text{Sym}(L) \\
\downarrow & & \downarrow \\
M_\alpha & \longrightarrow & M_{\alpha+1}
\end{array}
\quad
\begin{array}{ccc}
T \otimes \text{Sym}(K) & \longrightarrow & T \otimes \text{Sym}(L) \\
\downarrow & & \downarrow \\
M_\alpha \otimes_R T & \longrightarrow & M_{\alpha+1} \otimes_R T
\end{array}
\]

Here we have canceled \( R \otimes_R (-) \) terms in the right-hand square. Because \( \mathcal{M} \) has the commutative monoid axiom and a cofibrant unit, the cofibrancy of \( K \) and \( L \) implies the cofibrancy of \( \text{Sym}(K) \) and \( \text{Sym}(L) \) in \( \mathcal{M} \). Thus, by Lemma 1.1.12 in [21], smashing with these objects preserves weak equivalences between cofibrant objects, so when we apply this to the weak equivalence \( R \to T \) the maps on the upper left and upper right corners in the squares above are weak equivalences. Similarly, the map \( \text{Sym}(K) \to \text{Sym}(L) \) is a cofibration and so because \( R \) and \( T \) are cofibrant the horizontal maps across the top are cofibrations (and hence the bottom horizontals as well, because the are pushouts of cofibrations).

Because all maps \( M_\alpha \to M_{\alpha+1} \) are cofibrations and because \( M_0 \) is cofibrant, all \( M_\alpha \) are cofibrant. Because extension of scalars is left Quillen, the objects in the second square are cofibrant. The inductive hypothesis tells us that the map on the lower left corner is a weak equivalence. The Cube Lemma then guarantees us that the map on the lower right corner is a weak equivalence.

For the limit ordinal case, assume that \( M_\alpha \to M_\alpha \otimes_R T \) is a weak equivalence for all \( \alpha < \lambda \). Then we have a map of sequences

\[
\begin{array}{cccc}
M_0 & \longrightarrow & M_1 & \longrightarrow \cdots \\
\downarrow & & \downarrow & \\
M_0 \otimes_R T & \longrightarrow & M_1 \otimes_R T & \longrightarrow \cdots
\end{array}
\quad
\begin{array}{cccc}
M_\alpha & \longrightarrow & \cdots \\
\downarrow & & \downarrow & \\
M_\alpha \otimes_R T & \longrightarrow & \cdots
\end{array}
\]

where all vertical maps are weak equivalences and all all horizontal maps are cofibrations of cofibrant objects. So Proposition 18.4.1 in [13] proves the colimit map \( M_\lambda \to M_\lambda \otimes_R T \) is a weak equivalence as well.

\[ \square \]

Hovey provides a counterexample which demonstrates that for non-cofibrant \( R \) and \( T \), and without the hypothesis that cofibrant \( R \)-modules are flat, it is not true that \( R \simeq T \) induces a Quillen equivalence of categories of modules. We do not know whether or not Hovey’s example can be generalized to the case of algebras rather than modules. We do know that the spaces considered in Hovey’s example cannot provide such a counterexample for the question of Quillen equivalence between \( \text{CAlg}(R) \) and \( \text{CAlg}(T) \), because commutative monoids in \( \text{Top} \) are generalized Eilenberg-Mac Lane spaces (as discussed in Example 4.4).
The author does not know whether or not it is possible to prove homotopy invariance of $\text{CAlg}(R)$ without the hypothesis that cofibrant objects are flat and without having to assume the objects $R$ and $T$ are cofibrant. Note that Corollary 2.4 of [4] does not apply here because the operads $\text{Com}$, $\text{Com}_R$, and $\text{Com}_T$ are not $\Sigma$-cofibrant.

**Remark 4.3.** The results in this section also hold in the absence of the monoid axiom. By Corollary [3.8] categories of commutative algebras form semi-model categories and the output of the theorem is a Quillen equivalence of semi-model categories. To see this one need only note that the monoid axiom is not used in the proof, and that the semi-model category analog of 1.3.16 in [21] can be found in Section 12.1.8 of [11].

4.2. Rectification. We turn next to the question of rectification. As discussed in [39], categories of algebras over cofibrant operads inherit model structures whenever the monoid axiom is satisfied. Thus, $E_\infty$-algebras in $\mathcal{M}$ will always inherit a model structure in our set-up. There is a weak equivalence $\phi : E_\infty \to \text{Com}$, so it is natural to ask whether or not the pair $(\phi^*, \phi_!)$ forms a Quillen equivalence between $E_\infty$-algebras and $\text{Com}$-algebras. If there is, then rectification is said to occur.

In all model categories of spectra, if commutative ring spectra form a model category then it is Quillen equivalent to $E_\infty$-algebras. We do not have a statement of this kind for general model categories, and in fact the following counterexample demonstrates that this does not come for free:

**Example 4.4.** Let $\mathcal{M}$ be simplicial sets or topological spaces. We will see in the next section that $\mathcal{M}$ satisfies the strong commutative monoid axiom. The monoid axiom and requisite smallness were verified in [36] for simplicial sets, in [20] for compactly generated spaces, and in [43] for $k$-spaces. Thus, commutative monoids inherit a model structure.

For topological spaces the path connected commutative monoids are weakly equivalent to generalized Eilenberg-Mac Lane spaces, i.e. products of Eilenberg-Mac Lane spaces. The fact that there are spaces like $QS = \Omega^\infty \Sigma^\infty S^0$ which is $E_\infty$ but is not a GEM demonstrates that rectification between $E_\infty$ and $\text{Com}$ fails for spaces.

The rectification results of [4] are phrased so as to apply for very general model categories $\mathcal{M}$, including simplicial sets. However, these results do not apply because $\text{Com}$ is not a $\Sigma$-cofibrant operad. If $\mathcal{M}$ satisfies Harper's hypothesis that all symmetric sequences are projectively cofibrant (e.g. if $\mathcal{M} = Ch(k)$ for $k$ a $\mathbb{Q}$-algebra), then $\text{Com}$ is $\Sigma$-cofibrant and so rectification holds.

The key property possessed by good monoidal categories of spectra which allows rectification is

\[ (** ) \forall \text{ cofibrant } X, \text{ the map } (E\Sigma_n)_+ \wedge \Sigma_n X^{\wedge n} \to X^{\wedge n}/\Sigma_n \text{ is a weak equivalence.} \]
This property is certainly related to the commutative monoid axiom, but it is not necessary for strictly commutative monoids to inherit a model structure. In particular, it fails for simplicial sets and for topological spaces, so it is impossible to deduce property (**) from the commutative monoid axiom. We now record the correct analogue of this property (**) in general model categories:

**Definition 4.5.** Let $\mathcal{M}$ be a monoidal model category which is a $\mathcal{D}$-model category in the sense of [21]. View the unit $S$ of $\mathcal{D}$ as an object in $\mathcal{D}^{\Sigma_n}$ with the trivial $\Sigma_n$ action. Let $q: Q_{\Sigma_n} S \to S$ be cofibrant replacement in $\mathcal{D}^{\Sigma_n}$. Then $\mathcal{M}$ is said to satisfy the rectification axiom with respect to operads valued in $\mathcal{D}$ if for all cofibrant $X$ in $\mathcal{M}$, the natural map $Q_{\Sigma_n} S \otimes_{\Sigma_n} X^{\otimes n} \to X^{\otimes n}/\Sigma_n$ is a weak equivalence.

A similar property, requiring certain homotopy orbits to be weakly equivalent to orbits, appears in the axiomatization of good model structures of spectra given by [12]. However, in [12], this condition is equivalent to the condition that all simplicial operads are admissible, and as we have seen that will not be true for general model categories. We pause now to record a proposition about the interplay between the rectification axiom and the commutative monoid axiom which we shall use in Section 5.

**Proposition 4.6.** Suppose $\mathcal{M}$ is a monoidal model category satisfying the rectification axiom. Then $\text{Sym}^n(-)$ takes trivial cofibrations between cofibrant objects to weak equivalences.

**Proof.** Let $f: A \to B$ be a trivial cofibration between cofibrant objects. Note that $f^{\otimes n}: (A)^{\otimes n} \to (B)^{\otimes n}$ is a trivial cofibration in $\mathcal{M}$ because it is the composite $A^{\otimes n} \to A^{\otimes n-1} \otimes B \to A^{\otimes n-2} \otimes B^{\otimes 2} \to \cdots \to B^{\otimes n}$. This follows by iteratively applying the fact that $A \otimes -$ and $B \otimes -$ are left Quillen functors.

Furthermore, $Q_{\Sigma_n} S$ is $\Sigma_n$-cofibrant and so when we take the pushout product of $A^{\otimes n} \to B^{\otimes n}$ with $\otimes \to Q_{\Sigma_n} S$ we obtain a $\Sigma_n$-trivial cofibration, e.g. by Lemma 2.5.2 in [3]. When we pass to $\Sigma_n$-coinvariants we obtain a trivial cofibration $A^{\otimes n} \otimes_{\Sigma_n} Q_{\Sigma_n} S \to B^{\otimes n} \otimes_{\Sigma_n} Q_{\Sigma_n} S$ because $(-)/\Sigma_n$ is left Quillen. Consider the following commutative square, where the bottom horizontal map is $\text{Sym}^n(f)$, the top horizontal map is the map we have just described, and the vertical maps are induced by $Q_{\Sigma_n} S \to S$ and by passage to $\Sigma_n$-coinvariants:

$$
\begin{array}{ccc}
Q_{\Sigma_n} \otimes_{\Sigma_n} A^{\otimes n} & \longrightarrow & Q_{\Sigma_n} \otimes_{\Sigma_n} B^{\otimes n} \\
\downarrow & & \downarrow \\
A^{\otimes n}/\Sigma_n & \longrightarrow & B^{\otimes n}/\Sigma_n
\end{array}
$$

We have shown the top vertical map is a weak equivalence. The vertical maps are weak equivalences by the rectification axiom. Thus, the bottom horizontal map is a weak equivalence by the two-out-of-three property. □
In situations arising from topology, where \( \mathcal{M} \) is spectra and \( \mathcal{D} \) is spaces, the map \( Q_S S \to S \) is the cofibrant replacement of the point and so is \( E\Sigma_n \to * \) in the unpointed setting and \( (E\Sigma_n)_+ \to S^0 \) in the pointed setting. This proposition is used in Section 5 to make sure that a particular Bousfield localization respects the commutative monoid axiom.

We have not undertaken a general study of when rectification between \( \text{Com} \) and \( E_\infty \) holds. The interested reader is encouraged to consult [14], [33], and [30] for more information about rectification for general model categories. In particular, the preprint [30] will soon be available on the arxiv and will focus on rectification in categories of spectra formed from general model categories.

4.3. **Relationship to Bousfield Localization.** We now record a few facts regarding the relationship between the model category axioms we have discussed and (left) Bousfield localization. These results are proven in the author’s thesis [44], and will appear soon in the companion paper [42]. Taken together, the following three results give a list of checkable conditions on a model category \( \mathcal{M} \) and a set of maps \( C \) so that the Bousfield localization \( L_C(\mathcal{M}) \) of \( \mathcal{M} \) with respect to \( C \) satisfies the necessary hypotheses of Theorem 3.2, i.e. so that one may obtain a model structure on the category of commutative monoids in \( L_C(\mathcal{M}) \). It is proven in [44] that these properties imply that commutative \( R \)-algebras are preserved by \( L_C \). Throughout we assume that the maps in \( C \) are cofibrations between cofibrant objects. If they are not, then this can be arranged without loss of generality by taking cofibrant replacements of the maps in \( C \) and applying the factorization axiom to obtain cofibrations between cofibrant objects.

**Theorem 4.7.** Let \( \mathcal{M} \) be a tractable, left proper, monoidal model category where cofibrant objects are flat. Let \( C \) be a set of maps such that the Bousfield localization \( L_C(\mathcal{M}) \) exists. Then \( L_C(\mathcal{M}) \) has cofibrant objects flat and satisfies the pushout product axiom if and only if for all domains and codomains \( K \) of the generating cofibrations, maps in \( C \otimes \text{id}_K \) are \( C \)-local equivalences.

Furthermore, without the tractable hypothesis, we have:

\( L_C(\mathcal{M}) \) has cofibrant objects flat and satisfies the pushout product axiom if and only if for all cofibrant \( K \), maps in \( C \otimes \text{id}_K \) are \( C \)-local equivalences.

Note in particular that under these hypotheses \( L_C(\mathcal{M}) \) also satisfies the unit axiom.

In light of this characterization, we refer to Bousfield localizations satisfying the hypotheses of the theorem as **monoidal Bousfield localizations**. We turn next to the strong commutative monoid axiom.

**Theorem 4.8.** Suppose \( \mathcal{M} \) is a simplicial model category satisfying the strong commutative monoid axiom. Suppose that for all \( n \in \mathbb{N} \) and \( f \in C \), \( \text{Sym}^n(f) \) is a \( C \)-local equivalence. Then \( L_C(\mathcal{M}) \) satisfies the strong commutative monoid axiom.

Because the results in [42] are general enough to hold only in the presence of a semi-model structure on commutative monoids, it is enough for localization to
preserve the pushout product axiom and the commutative monoid axiom. However, we also have a result regarding preservation of the monoid axiom which we record here for the reader’s convenience. First we must introduce a new definition, taken from [1]:

**Definition 4.9.** A map \( f : X \to Y \) is called an \( h \)-cofibration if the functor \( f_! : X/M \to Y/M \) given by cobase change along \( f \) preserves weak equivalences.

\( M \) is said to be \( h \)-monoidal if for each (trivial) cofibration \( f \) and each object \( Z \), \( f \otimes Z \) is a (trivial) \( h \)-cofibration.

If \( M \) is left proper, then an equivalent characterization of an \( h \)-cofibration is as a map \( f \) such that every pushout along \( f \) is a homotopy pushout (this the version of the definition above was independently discovered in [44]). In [1], \( h \)-monoidality is verified for the model categories of topological spaces, simplicial sets, chain complexes over a field (with the projective model structure), symmetric spectra (with the stable projective model structure), and several other model categories not considered in this paper. More examples can be found in [42].

With this definition in hand, it is proven in Proposition 2.5 of [1] that if \( M \) is left proper, \( h \)-monoidal, and the weak equivalences in \((M \otimes I)\)-cell are closed under transfinite composition, then \( M \) satisfies the monoid axiom. We strengthen this result by replacing the third condition with the hypothesis that the (co)domains of \( I \) are finite relative to the class of \( h \)-cofibrations (in the sense of Section 7.4 of [21]). Because this is a statement phrased entirely in terms of \( I \), it is preserved by any Bousfield localization \( L_C \). We therefore are able to prove:

**Theorem 4.10.** Suppose \( M \) is a tractable, left proper, \( h \)-monoidal model category such that the (co)domains of \( I \) are finite relative to the \( h \)-cofibrations and cofibrant objects are flat. Then for any monoidal Bousfield localization \( L_C \), the model category \( L_C(M) \) satisfies the monoid axiom.

### 5. Examples

In this section we verify the strong commutative monoid axiom for the model categories of chain complexes over a field of characteristic zero, for simplicial sets, for topological spaces, and for positive flat model structures on various categories of spectra. We also discuss precisely what is true for positive (non-flat) model structures of spectra. Throughout this section we make use the following lemma, which is proven in Appendix [A]

**Lemma 5.1.** Suppose \( M \) is a cofibrantly generated monoidal model category and that for all \( f \in I \) (resp. \( J \)) we know that \( f^{op}/\Sigma_n \) is a (trivial) cofibration. Then the strong commutative monoid axiom holds for \( M \).
5.1. **Commutative Differential Graded Algebras in characteristic zero.** Consider a field $k$ and $\mathcal{M} = \text{Vect}(k)$. Then $\mathcal{M}$ satisfies the strong commutative monoid axiom if and only if $\text{char}(k) = 0$. Because $\mathcal{M}^{\Sigma_n} \cong k[\Sigma_n] \mod$, the projective model structure is nicely behaved (i.e. matches the injective model structure) exactly when $k[\Sigma_n]$ is semisimple, i.e. exactly when $k$ has characteristic zero. Indeed, such $\mathcal{M}$ satisfies the stronger condition required in Theorem 3.10. This example generalizes to pertain to $\text{Ch}(R)$ whenever $R$ is a commutative $\mathbb{Q}$-algebra.

The commutative monoid axiom fails over $\mathbb{F}_2$ because $\mathbb{F}_2[\Sigma_2]$ is not projective over $\mathbb{F}_2$ (because now Maschke’s Theorem does not hold) and so the cokernel of $f^{\Sigma_n}$ does not have a free $\Sigma_n$ action, and this will be an obstruction to $f^{\Sigma_n}/\Sigma_n$ being a cofibration.

That $\text{CDGA}(k)$ cannot inherit a model structure for $\text{char}(k) = p > 0$ has been known for many years. The fundamental problem is that $\text{Sym}(-)$ does not preserve weak equivalences between cofibrant objects and so cannot be a left Quillen functor. This is because for example $\text{Sym}^p(D(k))$ will not be acyclic even though the disk $D(k)$ is acyclic.

5.2. **Spaces.**

**Theorem 5.2.** The category of simplicial sets satisfies the strong commutative monoid axiom but does not satisfy Lurie’s axiom or the rectification axiom.

**Proof.** To see that the rectification axiom fails, consider $X = \Delta[0]$. Then the rectification axiom is asking $B\Sigma_n$ to be contractible. To see that Lurie’s axiom fails, consider $f^{\Sigma_2}$ where $f : S^0 \to D^1$. This map is not a $\Sigma_2$-cofibration because the action on the cofiber of $f^{\Sigma_2}$ is not free. However, to show that we get a cofibration after passing to $\Sigma_2$ coinvariants is easy, because the map is a monomorphism. Furthermore, this line of reasoning generalizes to show that $f^{\Sigma_n}/\Sigma_n$ is a cofibration whenever $f$ is a generating (trivial) cofibration. To check that it’s also a weak equivalence if $f$ is a generating trivial cofibration, we use the following theorem of Casacuberta [6]:

**Theorem 5.3.** If $f$ is any map in $s\text{Set}$, then $\text{Sym}(-)$ preserves $f$-equivalences.

Obviously, this proves much more than we needed, and in fact we use the proof of this theorem in [42] to see that any monoidal Bousfield localization of $s\text{Set}$ also satisfies the strong commutative monoid axiom. The key point in the proof of this theorem is due to an observation of Farjoun [10] which says that for any $X$, $\text{Sym}^n(X)$ can be written as a homotopy colimit of a free diagram formed by the orbits of $\Sigma_n$ where each quotient $\Sigma_n/H$ is sent to the fixed-point subspace $(X^H)^H$. It is then not too much work to see that $\text{Sym}^n(-)$ preserves weak equivalences (and more generally $f$-equivalences). We refer the reader to [6] for the details.
Observe that the counterexample displaying the failure of Lurie’s axiom and the rectification axiom also applies to $\text{Top}$, $s\text{Set}^G$, and $\text{Top}^G$.

**Theorem 5.4.** The category of compactly generated topological spaces satisfies the strong commutative monoid axiom.

**Proof.** In $\text{Top}$, cofibrations are no longer monomorphisms, but the strong commutative monoid axiom still holds. This may be verified by either checking it directly on the generating maps $S^{n-1} \to D^n$ and $D^n \to D^n \times [0,1]$ (a valuable exercise), or by transporting the strong commutative monoid axiom on $s\text{Set}$ to $\text{Top}$ via the geometric realization functor. From [20] we see that $\text{Top}$ satisfies the necessary smallness hypotheses, so Theorem 3.2 applies. □

In case the reader is interested in checking the commutative monoid axiom on $\text{Top}$ directly, we remark that the interpretation of Farjoun’s work in [6] makes clear that the only property of simplicial sets being used in the argument is that the fixed point subspaces of actions of subgroups of $\Sigma_k$ on $X_k$ are homeomorphic to spaces $X^n$ for some $n \leq k$. So one could apply Farjoun’s work just as well in $\text{Top}$ as in $s\text{Set}$. Indeed, Farjoun’s work provides a way to “free up” any diagram category and view the colimit of a diagram as the homotopy colimit of a different diagram (indexed by the so-called orbit category). In this way good homotopical properties can be achieved in a great deal of generality. The fact that the same argument works in both $\text{Top}$ and $s\text{Set}$ leads us to make the following conjecture.

**Conjecture 5.5.** Suppose that $\mathcal{M}$ is a concretizable Cartesian closed model category in which cofibrations are closed under the operation $(\cdot)^{\Sigma_n}/\Sigma_n$. Then the strong commutative monoid axiom holds in $\mathcal{M}$.

We now turn to equivariant spaces.

**Theorem 5.6.** Let $G$ be a finite group. Then $s\text{Set}^G$ and $\text{Top}^G$ satisfy the strong commutative monoid axiom.

**Proof.** We begin with $s\text{Set}^G$. Note that just as for $s\text{Set}$, cofibrations are monomorphisms. Thus, the same proof as for $s\text{Set}$ applies. In particular, when applying Farjoun’s trick on $(X^n)^H$ where $H < \Sigma_n$, we simply use the fact that the $G$ action and the $\Sigma_n$ action commute.

To handle the situation of $\text{Top}^G$ we may again transfer the strong commutative monoid axiom via geometric realization. Here we really need $G$ to be a finite group. For any simplicial group $G$, a $G$ action on $X \in s\text{Set}$ is taken to an action of $|G|$ on $|X|$ by geometric realization. If $G$ is finite then $G = \text{Sing}[G]$ acts on $\text{Sing}[X]$ and we can prove $s\text{Set}^G$ is Quillen equivalent to $\text{Top}^{|G|}$. However, for non-finite $G$ we do not know whether or not every subgroup $K$ of the topological group $|G|$ is realized as some $|H|$ for $H < G$, so there may be fewer weak equivalences in $\text{Top}^{|G|}$ than in $s\text{Set}^G$. □
5.3. **Symmetric Spectra.** The obstruction noticed by Gaunce Lewis and discussed in [24] guarantees that commutative monoids in the usual model structure on symmetric spectra cannot inherit a model structure, because the unit is cofibrant and because the fibrant replacement functor is symmetric monoidal. This second property cannot be changed, but there are model structures on symmetric spectra in which the unit is not cofibrant. The positive model structure was introduced in [22] and [28] and this model structure breaks the cofibrancy of the sphere by insisting that cofibrations be isomorphisms in level 0 (though in other levels they are the same as the usual cofibrations of symmetric spectra). In [38], Shipley found a more convenient model structure which is now called the positive flat model structure. In this model structure the cofibrations are enlarged to contain the monomorphisms, and then the condition in level 0 is applied. The result is a model structure in which commutative ring spectra inherit a model structure and in which cofibrations of commutative ring spectra forget to cofibrations of spectra.

Note that in [25], Lurie’s axiom is claimed to hold for positive flat symmetric spectra. This is an error, as acknowledged in [26]. Indeed, the example given in Proposition 4.2 of [38] demonstrates this failure conclusively, for both the positive and the positive flat model structures. We will now show that the commutative monoid axiom holds for positive flat (stable) symmetric spectra, and a slight weakening holds for positive (stable) symmetric spectra.

5.3.1. **Positive Flat Stable Model Structure.**

**Theorem 5.7.** The strong commutative monoid axiom holds for the positive flat stable model structure on symmetric spectra.

**Proof.** By Lemma [5.1], it’s sufficient to check the strong commutative monoid axiom on the generating (trivial) cofibrations. We focus first on the the generating cofibrations, which take the form \( SI^{\ell} = S \otimes I^{\ell} = S \otimes \bigcup_{m>0} G_m(I_{\Sigma_m}) \) where \( G_m \) is the left adjoint to \( E_{\Sigma_m} \) and \( I_{\Sigma_m} \) is the set of generating cofibrations for \( sS eK_m \). First observe that \((S \otimes f)^{\Sigma_n}\) is itself an iterated pushout product of \( f \) with \( \emptyset \to S \) and because the pushout product is symmetric we can pull the \( S \)'s to one side. There we can smash them together because \( S \) is the unit. So \((S \otimes f)^{\Sigma_n} \cong S \otimes f^{\Sigma_n} \).

Next, \( f \) is some \( G_m(i) \) where \( i : \partial \Delta \to \Delta \). We will prove that \( f^{\Sigma_n} \) is \( G_{nm}(f^{\Sigma_n}) \). First observe that the domain of \( f^{\Sigma_n} \) is a colimit built out of terms of the form \( G_m(X)^j \land G_m(Y)^k \), and the codomain is \( G_{nm}(Y)^n \). We next use Proposition 2.2.6 in [22] to rewrite \( G_m(X) \land G_m(Y) \) as \( G_{2mn}(X \land Y) \). So the domain can be written as colimit built from terms of the form \( G_{nm}(X)^j \land Y^k \) and the range as \( G_{nm}(Y)^n \).

Finally, \( G_{nm} \) commutes with colimits because it is a left adjoint, so the map \( f^{\Sigma_n} \) takes the form \( G_{nm}(f^{\Sigma_n}) : G_{nm}(Q_n) \to G_{nm}(Y^{\Sigma_n}) \). Thus, because spaces and \( \Sigma_n \)-spaces satisfy the strong commutative monoid axiom, \( f^{\Sigma_n}/\Sigma_n \) is a (trivial) cofibration if \( i \) is. Because \( G_{nm} \) is left Quillen and commutes with \((-)/\Sigma_n \) we’re done.

In particular, Lemma [5.1] now implies the class of cofibrations is closed under the operation \((-)^{\Sigma_n}/\Sigma_n \), in either the levelwise or stable model structures. The same
argument works for maps in $SJ_*^+$ (the generating trivial cofibrations for the levelwise model structure), and proves that applying $(-)^{cn}/\Sigma_n$ takes such maps to trivial cofibrations.

To complete the proof we must now prove the commutative monoid axiom is satisfied by the other maps in the set of generating trivial cofibrations for the stable model structure. Recall from Theorem 2.4 in [38] and from Definition 3.4.9 in [22] that the stable model structure is obtained via a Bousfield localization of the levelwise model structure with respect to the set of maps $C = \{s_m : G_{m+1}S^1 \to G_mS^0\}$, where $s_m$ is adjoint to the identity map of $S^1$. One could equivalently invert maps in $C' = \{Qf \mid f \in C\}$, where $Qf$ is chosen to be a cofibrant replacement that replaces maps with cofibrations. The generating trivial cofibrations take the form $S_*^+J = S_*^+ \cup K^+$ where $K^+ = \bigcup_{m>0} K_m$, $K_m = c_m \Box I$, and $c_m$ is a stable cofibrant replacement of $s_m$.

Every $f \in K^+$ is a cofibrations, so $f^{cn}/\Sigma_n$ is still a cofibration. That it is also a stable equivalence follows from Theorem 4.8. To see that $\text{Sym}^n(s_m)$ is a $C$-local equivalence for all $n, m$, apply Proposition 4.6 with the input model category being $L_{C'}(S p_+)$ (where $S p_+$ is the positive flat model structure), with $D = \text{Sym}^n$, and using the observation that in $L_{C'}(S p_+)$ the maps in $C'$ are trivial cofibrations between cofibrant objects. Observe that this proof is also using the fact that $L_{C'}$ respects the monoidal structure on $S p_+$, but this can easily be checked using Theorem 4.7. This completes the proof.

We remark that this theorem together with Lewis’s example demonstrate that the commutative monoid axiom need not be preserved by monoidal Quillen equivalences, since the positive flat stable model structure is monoidally Quillen equivalent to the canonical stable model structure. This can be seen via Proposition 2.8 in [38], together with the fact that stable cofibrations are contained in flat cofibrations (Lemma 2.3 in [38]) and the fact that the two model structures have the same weak equivalences. We do not know of a similar example which would demonstrate that the monoid axiom need not be preserved by monoidal Quillen equivalence.

5.3.2. Positive Stable Model Structure. Shipley proves in [38] that positive symmetric spectra do not satisfy the property that cofibrations of commutative monoids forget to cofibrations of symmetric spectra. Thus, this model structure cannot satisfy the strong commutative monoid axiom. However, Proposition 4.2 in [38] proves that a cofibration of commutative $R$-algebras forgets to a positive $R$-cofibration (and hence to an $R$-cofibration) even though it is not a positive cofibration in the sense of [28]. This suggests the following result:

**Proposition 5.8.** Let $f$ be a (trivial) cofibration in the positive stable model structure. Then $f^{cn}/\Sigma_n$ is a (trivial) cofibration in the positive flat stable model structure. Furthermore, commutative monoids inherit a model structure in the positive stable model structure.
Proof. The proof is identical to the proof that the positive flat stable model structure satisfies the strong commutative monoid axiom. This is because positive cofibrations form a subclass of positive flat cofibrations. For the statement regarding trivial cofibrations, the same logic used above holds, because it is a Bousfield localization with respect to the same class of maps, and the weak equivalences of both the positive stable and positive flat stable model structures are the same. In particular, this observation proves that the positive (stable) model structures satisfy the weak form of the commutative monoid axiom discussed in Remark 3.3, so commutative monoids inherit a model structure.

Shipley provides a counterexample which demonstrates that \( \text{Sym}(F_1S^1) \) is not positively cofibrant (only positively flat cofibrant) because \((F_1S^1)^{(2)}/\Sigma_2 \neq (S^1 \wedge S^1)/\Sigma_2 \) and this is not \( \Sigma_2 \)-free. Thus, Proposition 3.5 cannot hold as stated. However, for the same reasons as in the proof above (namely, the containment of positive cofibrations in positive flat cofibrations) we can obtain the following weakened form of Proposition 3.5.

**Proposition 5.9.** Let \( M \) be the positive stable model structure on symmetric spectra, and let \( \text{CAlg}(R) \) be the model structure passed from \( M \) to the category of commutative \( R \)-algebras (where \( R \) is a commutative monoid in \( M \)). Suppose \( f \) is a cofibration in \( \text{CAlg}(R) \) whose source is cofibrant in \( M \). Then \( f \) forgets to a cofibration in the positive flat stable model structure.

### 5.4. General Diagram Spectra.

In [28], a general theory of diagram spectra is introduced which unifies the theories of \( S \)-modules, symmetric spectra, orthogonal spectra, \( \Gamma \)-spaces, and \( W \)-spaces. For the first, homotopy-coherence is built into the smash product, so commutative monoids immediately inherit a model structure and there is rectification between \( \text{Com-alg} \) and \( \text{E}_\infty \)-algebras. For the next two, positive model structures are introduced which allow strictly commutative monoids to inherit model structures. The rectification axiom is then proved and rectification is deduced as a result.

**Theorem 5.10.** The positive flat stable model structure on (equivariant) orthogonal spectra satisfies the strong commutative monoid axiom and the rectification axiom. The positive stable model structure satisfies the weak commutative monoid axiom, Proposition 5.8, and Proposition 5.9.

Proof. For the positive flat stable model structure on (equivariant) orthogonal spectra, proceed as in the proof of Theorem 5.7, but using (equivariant) topological spaces rather than simplicial sets. The rectification axiom is proven in [28] (and in [51] for the equivariant case). For the positive stable model structure proceed as in Proposition 5.8 and Proposition 5.9.

We turn now to \( W \)-spaces and \( \Gamma \)-spaces. Recall that \( W \) is the category of based spaces homeomorphic to finite CW-complexes, \( \Gamma \) is the category of finite based sets, and \( \mathcal{D} \)-spaces are functors from \( \mathcal{D} \) to \( \text{Top} \) (where \( \mathcal{D} \) is either \( W \) or \( \Gamma \)). The
indexing category for $\Gamma$-spaces is a subset of $W$. First, Lewis’s counterexample \cite{24} still applies to rule non-positive model structures out from consideration. This is discussed in the context of $\Gamma$-spaces in Remark 2.6 of \cite{34}. The author has not been able to find a place where this is written down for $W$-spaces, but it is clear that the same counterexample applies for $W$-spaces. We must work in positive model structures on $W$-spaces and $\Gamma$-spaces. Such positive model structure are introduced in Section 14 of \cite{28}.

While the author cannot find a reference for positive flat model structures (also known as convenient model structures), he believes they can be constructed in the same way as for symmetric spectra. For instance, one can carry out the program of \cite{38} for $\Gamma$-spaces (e.g. following the work in \cite{32} and making use of the relationship between $\Gamma$-spaces and symmetric spectra as explored in \cite{35}) to obtain the necessary mixed model structure on spaces. From there it is purely formal to construct the appropriate levelwise model structure on diagrams, e.g. using Theorem 6.5 in \cite{28}. The generating cofibrations for $W$-spaces take the form $F_W I = \{ F_{d}(i) \mid d \in \text{skel } W, i \in I \}$ where $F_{d}(-)$ is $W(d, -) : W \to \text{Top}$. The indexing category for $\Gamma$-spaces is a subset of $W$, so an analogous construction works for $\Gamma$-spaces.

The monoidal product is computed levelwise. Passage from the levelwise structure to the positive flat model structure is again formal, and is accomplished by truncating the levelwise cofibrations to force levelwise cofibrations to be isomorphisms in degree 0. Finally, passage to the positive flat stable model structure may be accomplished via Bousfield localization, just as in Section 8 of \cite{28}.

**Theorem 5.11.** The positive flat model structures on $W$-spaces and $\Gamma$-spaces satisfy the strong commutative monoid axiom. The positive model structure on $W$-spaces and $\Gamma$-spaces satisfies the weak commutative monoid axiom. So commutative monoids inherit model structures in both settings.

The verification of the strong commutative monoid axiom proceeds precisely as for the positive flat model structure on symmetric spectra. In particular, one can reduce the verification to a verification in spaces. We leave the details to the reader. The difficulty comes in the part of the proof when one attempts to pass the commutative monoid axiom to the stable model structure, and that is why the adjective stable is not in the statement of the theorem. In particular, the difficulty is that the rectification axiom is not known to hold for $D$-spaces (where $D$ is either $W$ or $\Gamma$). Indeed, we can show that the rectification axiom cannot hold.

First, if the rectification axiom held, then the proof that the strong commutative monoid axiom holds for positive flat stable symmetric spectra (i.e. via Theorem 4.8) would prove that $D$-spaces satisfy the commutative monoid axiom. Secondly, because of the rectification axiom the rest of the work in \cite{28} and \cite{38} would prove that commutative $D$-rings were Quillen equivalent to $E_{\infty}$-algebras and this would contradict the main theorem of Tyler Lawson’s paper \cite{23}.
Lawson produces an $E_{\infty}$-algebra in $\Gamma$-spaces which cannot be strictified to a commutative $\Gamma$-ring. Together with the monoidal functor from $\Gamma$-spaces to $W$-spaces (developed in [28]), this same counterexample proves that not all $E_{\infty}$-algebras in $W$-spaces can be strictified to commutative $W$-rings.

5.5. Other Examples. We have not included a proof that simplicial presheaves satisfy the strong commutative monoid axiom, but this should follow from general facts about diagram categories. Indeed, we hope that this can generalize further to the so-called excellent model categories introduced in [25].

We have not addressed positive model structures on motivic symmetric spectra. We understand that these examples are central to the work of [30], which will appear soon.

There are several other examples which we have not investigated and which we would be curious to learn more about. We list them here:

- Stable module categories over $Q$-algebras.
- Comodules over a Hopf algebroid
- The model for spectra consisting of simplicial functors, in the style of [27].

APPENDIX A. Sufficiency of Commutative Monoid Axiom on Generators

We prove that if the strong commutative monoid axiom holds for the generating (trivial) cofibrations $I$ and $J$ then it holds for all (trivial) cofibrations.

Lemma A.1. Suppose $\mathcal{M}$ is a cofibrantly generated monoidal model category and that for all $f \in I$ (resp. $J$) we know that $f^{I_n}/\Sigma_n$ is a (trivial) cofibration. Then the strong commutative monoid axiom holds for $\mathcal{M}$.

We will prove that the class of maps satisfying the condition in the strong commutative monoid axiom is closed under retracts, pushouts, and transfinite compositions. The first two are easy, but the third will require an induction. So we must introduce some new notation, following [16]. Let $f : X \to Y$ and consider the $n$-dimensional cube in which each vertex is a word of length $n$ on the letters $X$ and $Y$.

Recall the action of $\Sigma_n$ on the diagram which defines $Q_n$. The vertices of the cube correspond to subsets $D$ of $\{1, 2, \ldots, n\}$ where a vertex $C_1 \otimes \cdots \otimes C_n$ has $C_i = X$ if $i \notin D$ and $C_j = Y$ if $j \in D$. Any $\sigma \in \Sigma_n$ sends the vertex so defined to the vertex corresponding to $\sigma(D) \subseteq \{1, 2, \ldots, n\}$ using the action of $\Sigma_{|D|}$ on the $X$’s and $\Sigma_{n-|D|}$ on the $Y$’s. Clearly, this action descends to an action on the colimit $Q_n$.

For inductive purposes, we will need to consider subdiagrams whose vertices consist of words with $\leq q$ copies of the letter $Y$. This subdiagram consists of all vertices of distance $\leq q$ from the initial vertex $X^{\otimes n}$. We denote the colimit of this subdiagram by $Q_{n}^{q}$. The superscript $n$ refers to the fact that this is a subdiagram of the $n$-dimensional cube, so in particular each vertex is a word on $n$ letters. In
particular, \( Q^n_n = X^\otimes n \) and \( Q^n_n = Y^\otimes n \). Observe that \( Q^n_{n-1} \) is the domain of \( f^\Sigma_n \), which we have formerly denoted by \( Q_n \). For the purposes of this proof we will now write it as \( Q_{n-1}^n(f) \) (or \( Q_{n-1}^n \) if the context is clear).

The induction will make use of the maps of colimits \( Q^n_{q-1} \rightarrow Q^n_q \) which are induced by inclusion of subdiagram. The \( \Sigma_n \)-action on the cube clearly preserves the size of the subset \( D \subset [n] \) and so it restricts to an action of \( \Sigma_n \) on each \( Q^n_q \). Because this action is a restriction of the \( \Sigma_n \)-action on the full cube, the map of colimits \( Q^n_{q-1} \rightarrow Q^n_q \) is automatically \( \Sigma_n \)-equivariant. Indeed, the map of colimits \( Q^n_{q-1} \rightarrow Q^n_q \) can be realized by the following pushout:

\[
\begin{array}{ccc}
\Sigma_n \cdot \Sigma_{n-q} \times \Sigma_q \times \Sigma_{q-n} & \xrightarrow{X^\otimes (n-q) \otimes \Sigma_q} & Q^n_{q-1} \\
\downarrow & & \downarrow \\
\Sigma_n \cdot \Sigma_{n-q} \times \Sigma_q \times \Sigma_{q-n} & \xrightarrow{X^\otimes (n-q) \otimes Y^\otimes q} & Q^n_q
\end{array}
\]

where the left vertical map is induced by \( f^\Sigma_q \) (see Section 7 of [16] and Remark 4.15 of [15] for a toy case). To explain the notation \( \Sigma_n \cdot \Sigma_{n-q} \times \Sigma_q \cdot \Sigma_q \cdot \Sigma_{q-n} \cdot (-) \), first note that for any set \( G \) and any object \( A, G \cdot A = \bigcup_{g \in G} A \). When \( G = \Sigma_n \) this object inherits a \( \Sigma_n \)-action by permuting the \( A^\otimes n \) objects in the coproduct. When we write \( \Sigma_n \cdot \Sigma_k \times \Sigma_q \cdot \Sigma_q \cdot \Sigma_{q-n} \cdot (-) \) we are quotienting out by the \( \Sigma_k \times \Sigma_q \) action on this object in \( \mathcal{M}^{\Sigma_q} \). The result is a coproduct with \( n!/(k!q!) \) terms because the order of the \( k! \) terms to the left of the product (and of the \( q! \) terms to the right) do not matter. In particular, applying \( \Sigma_n \cdot \Sigma_k \times \Sigma_q \cdot \Sigma_q \cdot \Sigma_{q-n} \cdot (-) \) has the effect of equivaraintly building in additional layers of the cube. With this notation in hand we proceed to the proof.

**Proof.** Let \( \mathcal{P} \) denote the class of cofibrations \( f \) for which \( f^\Sigma_n \) is also a cofibration. Let \( \mathcal{P}' \) denote the same for trivial cofibrations. We must prove that if \( I \subset \mathcal{P} \) then all cofibrations are in \( \mathcal{P} \) (and the same for \( I \subset \mathcal{P}' \)). We will do so by proving the classes \( \mathcal{P} \) and \( \mathcal{P}' \) are closed under retracts, pushouts, and transfinite compositions.

The simplest to verify is closure under retracts, which follows from the fact that \((-)^{\Sigma_n} / \Sigma_n \) is a functor on \( \text{Arr}(\mathcal{M}) \) so if \( f \) is a retract of \( g \) (with \( g \in \mathcal{P} \) or \( \mathcal{P}' \)) then \( f^\Sigma_n / \Sigma_n \) is a retract of \( g^\Sigma_n / \Sigma_n \) and hence a (trivial) cofibration.

We next consider closure under pushouts. Suppose \( f : X \rightarrow Y \) is a pushout of \( g : A \rightarrow B \) and \( g \in \mathcal{P} \) or \( \mathcal{P}' \). Then we have a \( \Sigma_n \)-equivariant pushout diagram

\[
\begin{array}{ccc}
Q_n(g) & \xrightarrow{\cdot} & B^n \\
\downarrow & & \downarrow \\
Q_n(f) & \xrightarrow{\cdot} & Y^n
\end{array}
\]

by Proposition 6.13 in [15]. When we pass to \( \Sigma_n \)-coinvariants we see that \( f^\Sigma_n / \Sigma_n \) is a pushout of \( g^\Sigma_n / \Sigma_n \), e.g. by commuting colimits. Indeed, for any \( X \in \mathcal{M}^{\Sigma_n} \),
$X \otimes_{\Sigma_n} f^{\Sigma_n}$ is a pushout of $X \otimes_{\Sigma_n} g^{\Sigma_n}$. So if the latter is assumed to be a (trivial) cofibration because $g \in \mathcal{P}$ or $\mathcal{P}'$ then the former will be as well.

Composition is harder, so we begin with the case of two maps $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{P}$ or $\mathcal{P}'$. We will prove that $Q_{n-1}^n(gf)/\Sigma_n \to Z^{\Sigma_n}/\Sigma_n$ is a (trivial) cofibration. First note that this map factors through $Q_{n-1}^n(g)/\Sigma_n$ and the hypothesis on $g$ guarantees that $Q_{n-1}^n(g)/\Sigma_n \to Z^{\Sigma_n}/\Sigma_n$ is a (trivial) cofibration. So we must only prove that $Q_{n-1}^n(gf)/\Sigma_n \to Q_{n-1}^n(g)/\Sigma_n$ is a (trivial) cofibration.

We proceed by realizing both colimit diagrams as subdiagrams of the same diagram, which is a $n$-dimensional cube featuring $3^n$ vertices which are words of length $n$ in the letters $X, Y, Z$. Formally, this cube is an element of the rectangular diagram category $\text{Fun}((0 \to 1 \to 2)^\times, M)$, and every time we write subdiagram we mean with respect to this cube with $3^n$ vertices. The domain $Q_{n-1}^n(gf)$ of the map we care about is the colimit of the $X - Z$ subdiagram, i.e. the punctured cube formed from vertices which are words in $X$ and $Z$, where all maps are compositions $gf$. The codomain $Q_{n-1}^n(g)$ of the map we care about is the colimit of the $Y - Z$ subdiagram, i.e. the punctured cube formed from vertices which are words in $Y$ and $Z$. So we must again introduce new notation to build this map one step at a time.

The induction will proceed by moving through the rectangle by adding a single $\Sigma_n$-orbit at a time. So we will need to consider $\Sigma_n$-equivariant subdiagrams of the rectangle which contain the $X - Z$ punctured cube and which contain a new vertex $e$ (and hence its entire $\Sigma_n$-orbit).

In order to build this new vertex into the colimit we will also need to consider the subdiagram of the $X - Y - Z$ box which maps to $e$ (but which does not include $e$ itself). This is collection of vertices sitting under $e$ (i.e. of distance strictly less than $e$ from the initial vertex). As with $e$, we wish to consider the $\Sigma_n$-orbit of this subdiagram, which is equivalently described as all vertices sitting under any vertex in the orbit of $e$. Now that we have a picture of the subdiagram in mind, we denote the colimit of this subdiagram by $Q_e$. By construction there is an induced $\Sigma_n$-equivariant map $Q_e \to e$.

We are now ready to consider the diagrams formed when we adjoin the $Q_e$-diagram with the $X - Z$ punctured cube. Let $Q[0]_{n-1}^n = Q_{n-1}^n(gf)$ be the $X - Z$ punctured cube. Let $Q[1]_{n-1}^n$ denote the colimit of the subdiagram containing the $X - Z$ punctured cube, the orbit of the vertex $e = Y \otimes Z^{\Sigma_n}$, and the vertices in the $Q_e$ subdiagram. Continue inductively, by adding $e = Y \otimes q \otimes Z^{\Sigma_n}$ and vertices below it to the $Q[1]_{n-1}^n$-diagram to get the $Q[q]_{n-1}^n$-diagram. This process terminates with the whole $X - Y - Z$ punctured cube whose $3^n - 1$ vertices contain all words in $X, Y, Z$ except the word $Z^{\Sigma_n}$. The colimit of this diagram is denoted $Q[n]_{n-1}^n$. A cofinality argument shows that this colimit is equal to $Q_{n-1}^n(g)$, because all factors of $X$ which appear are mapped to a factor of $Y$ in the subdiagram and so do not affect the colimit.
The induction will proceed along the maps $Q[q - 1]_{n-1}^n \to Q[q]_{n-1}^n$ induced by containments of subdiagrams. This induction can be thought of as stepping through shells in the cube of increasing distance from the initial vertex $X^{\otimes q}$ until the information from the entire diagram has been built into the colimit.

Because each step $Q[q - 1]_{n-1}^n \to Q[q]_{n-1}^n$ builds in the information of one new vertex (and its orbit under the $\Sigma_n$ action on the cube), we may apply Proposition A.4 from [31] with $e = Y^q \otimes Z^{n-q}$ to write the following pushout diagram:

(2) $\Sigma_n \cdot \Sigma_n \otimes \Sigma_n \cdot Q_e \xrightarrow{\Sigma_n \cdot \Sigma_n \otimes \Sigma_n \cdot Q_e} Q[q]_{n-1}^n$

The left vertical map is induced by $Q_e \to Y^{\otimes q} \otimes Z^{n-q}$ and this is in turn induced by $f^{\text{cl}} \Box g^{\text{cl}(n-q)}$ because

(3) $Q_e \cong Y^{q} \otimes Q_{n-q-1}^{n-q}(g) \coprod_{Q_{n-q-1}^{n-q}(f) \otimes Z^{n-q}} Q_{n-q}^{n-1}(f) \otimes Z^{n-q}$

To see that the diagram defining $Q_e$ decomposes into a gluing of the diagrams defining $Q_{n-q}^{n-1}(f) \otimes Z^{n-q}$ and $Y^{q} \otimes Q_{n-q-1}^{n-q}(g)$ along the diagram defining $Q_{n-q-1}^{n-q}(f) \otimes Z^{n-q}$, note that every $X$ in the $Q_e$ diagram gets mapped to a $Y$ in the $Q_e$ diagram and so does not affect the colimit. This is the reason why we insisted upon including the vertices under $e$ in our construction of the diagram defining $Q_e$. Furthermore, every $Z$ in the $Q_e$ diagram is the image of some $Y$ and so we may apply a cofinality argument to realize that any map out of the diagram for the left-hand side of (3) must factor through the right-hand side, which completes the proof of (3).

Now pass to $\Sigma_n$-coinvariants in Equation (2). Verifying that the left vertical map is a cofibration reduces to verifying that $f^{\text{cl}}/\Sigma_q \Box g^{\text{cl}(n-q)}/\Sigma_{n-q}$ is a cofibration. This in turn follows from the inductive hypothesis on $f$ and $g$. Thus all the maps $Q[q]_{n-1}^n/\Sigma_n \to Q[q + 1]_{n-1}^n/\Sigma_n$ are pushouts of cofibrations and hence are cofibrations themselves. Hence, their composite $Q_{n-1}^{n-1}(g f)/\Sigma_n \to Q_{n-1}^{n-1}(g)/\Sigma_n$ is a cofibration. This completes the proof that the classes $\mathcal{P}$ and $\mathcal{P}^r$ are closed under composition.

Finally, we cover the case of transfinite composition. First note that the proof for the composition of two maps proves that the vertical maps and the induced pushout corner map in the following square become cofibrations after passing to $\Sigma_n$-coinvariants, by the general machinery of adding a new vertex $e$ containing only
Ys and Zs:

\[ Q^n_{n-1}(f) \rightarrow Q^n_{n-1}(gf) \]

\[ Q^n_i(f) \rightarrow Q^n_i(gf) \]

Indeed, the same is true of the diagram

\[ Q^n_{n-1}(f) \rightarrow Q^n_{n-1}(gf) \]

\[ \gamma^n \rightarrow Z^n \]

This is the analogous result to Corollary A.7 in [31], which begins with power cofibrations and concludes that the diagram represents a projective cofibration in Arr(M^{(n)}). Recall, e.g., from Definition 2.1 in [7] that a square is a projective cofibration if and only if the vertical maps and the pushout corner map are cofibrations. In our situation we pass to \(\Sigma_n\)-coinvariants on the diagram level and in that way achieve a projective cofibration in Arr(M).

Now let \(X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots\) be a \(\lambda\)-sequence in which each \(f_\alpha \in \mathcal{P}\). Let \(f_\infty : X_0 \rightarrow X_1\) be the composite. To prove that \(f_\infty^{(n)}/\Sigma_n\) is a cofibration, we realize this map as the colimit of a particular diagram. Because colimits commute we can pass to \(\Sigma_n\)-coinvariants in the diagram and we will see that the colimit of the resulting diagram (which will be \(f_\infty^{(n)}/\Sigma_n\)) will be a cofibration. First we realize the domain of \(f_\infty^{(n)}/\Sigma_n\) as a colimit along the sequence \(Q^n_{n-1}(f_0) \rightarrow Q^n_{n-1}(f_1,f_0) \rightarrow Q^n_{n-1}(f_2,f_1,f_0) \rightarrow \ldots Q^n_{n-1}(f_\infty)\). Next, we realize \(f_\infty^{(n)}\) as the far right-hand map in

\[ Q^n_{n-1}(f_0) \rightarrow Q^n_{n-1}(f_1,f_0) \rightarrow Q^n_{n-1}(f_2,f_1,f_0) \rightarrow \ldots Q^n_{n-1}(f_\infty) \]

\[ X_0^{(n)} \rightarrow X_1^{(n)} \rightarrow X_2^{(n)} \rightarrow \ldots \rightarrow X_\lambda^{(n)} \]

As in the case for two-fold composition, we pass to \(\Sigma_n\)-coinvariants in this diagram and realize that the resulting diagram is a projective cofibration in the category of \(\lambda\)-sequences \(\mathcal{M}\) because all vertical maps and all pushout corner maps are cofibrations. The colimit of such a diagram must be a cofibration, because colimit is a left Quillen functor from \(\mathcal{M} \rightarrow \mathcal{M}\). This proves that \(f_\infty^{(n)}/\Sigma_n\) is a (trivial) cofibration as desired.

Remark A.2. The author is indebted to Luis Pereira for many helpful conversations as this proof was worked out. The author’s original proof proceeded by constructing a lift to prove that \(Q^n_{n-1}(gf)/\Sigma_n \rightarrow Q^n_{n-1}(g)/\Sigma_n\) has the left lifting property with
respect to all (trivial) fibrations. This proof comes down to constructing an equivariant lift at the level of the cube diagrams, and it appears something similar has been done by [13], though we find the proof presented here conceptually simpler. In [31], Pereira uses a similar proof to prove that it is sufficient to check Jacob Lurie’s axiom on the generating (trivial) cofibrations, at least in the case when the domain $X$ of $f$ is cofibrant. Pereira in fact proves something more general about the intermediate maps $Q[q - 1] \to Q[q]$. The proof presented here avoids the need for $X$ to be cofibrant, even in Pereira’s situation of working with Lurie’s axiom rather than the strong commutative monoid axiom.

Appendix B. Proof of Main Theorem

As described in Section 3, it is sufficient to prove the statements of Theorem 3.2 and Proposition 3.5 for the case $R = S$ of commutative monoids in $M$. Before proceeding to the proof, we fix some notation. Given a map $g : K \to L$ one can form $g^{\Sigma n} : Q_n \to L^{\otimes n}$. This map is a (trivial) cofibration if $g$ is such, by the pushout product axiom. The domain and codomain both have an action of $\Sigma_n$. Modding out by this action gives a map which is denoted by $f^{\Sigma n}/\Sigma_n : Q_n/\Sigma_n \to \text{Sym}^n(L) = L^{\otimes n}/\Sigma_n$.

The proofs of Theorem 3.2 and Proposition 3.5 follow the proof in [36] that Mon$(M)$ has a model structure inherited from $M$. Because that proof is based on the general theory of monads (c.f. Lemma 2.3) it will go through verbatim if Lemma 6.2 in [36] can be generalized to describe pushouts in CMon$(M)$ rather than in Mon$(M)$. We state the analogue to Lemma 6.2:

Lemma B.1.  
(1) If $M$ satisfies the commutative monoid axiom then in the category CMon$(M)$, Sym$(J)$-cell is contained in the collection of maps of the form $(id_Z \otimes J)$-cell in $M$. If in addition $M$ satisfies the monoid axiom then these maps are weak equivalences in $M$ and hence in CMon$(M)$.

(2) If $M$ satisfies the strong commutative monoid axiom then maps in Sym$(I)$–cell with cofibrant domain (in $M$) are cofibrations in $M$.

As in [36], the proof of this proposition requires a careful analysis of the filtration on pushouts in the category of commutative monoids. In particular, we must prove the following.

Proposition B.2. Given any map $h : K \to L$ in $M$, the commutative monoid homomorphism $X \to P$ formed by the following pushout in CMon$(M)$

$$
\begin{array}{ccc}
\text{Sym}(K) & \longrightarrow & \text{Sym}(L) \\
\downarrow & & \downarrow \\
X & \longrightarrow & P
\end{array}
$$
factors as $X = P_0 \to P_1 \to \cdots \to P$ where $P_{n-1} \to P_n$ is defined by the following pushout in $\mathcal{M}$

$$
\begin{array}{ccc}
X \otimes Q_n/\Sigma_n & \longrightarrow & X \otimes \text{Sym}^n(L) \\
\downarrow & & \downarrow \\
\downarrow & & \\
P_{n-1} & \longrightarrow & P_n
\end{array}
$$

where $Q_n$ denotes the colimit of the $n$-dimensional punctured cube discussed in Appendix A which has one vertex for each $n$-letter word formed from letters $K$ and $L$, except with the $L^n$ word removed.

This filtration is analogous to the one given in [36], and makes use of the decomposition $\text{Sym}(\cdot) = \oplus \text{Sym}^i(\cdot)$. The map $g : K \to X$ needed for the construction of $P_{n-1} \to P_n$ is adjoint to the map $\text{Sym}(K) \to X$. Note that this description of $P_n$ is significantly simpler than the one found in [36] because commutativity means one need not consider words with $X$’s, $K$’s, and $L$’s interspersed. Rather, all the $X$’s can be shuffled to the left and multiplied at the beginning of the process, rather than at the end as is done in [36]. If we were to keep our notation in line with the notation in [36] then what we call $Q_n$ would be denoted $\overline{Q}_n$, but we will avoid this unnecessary shift in notation, because we will have no need for colimits of cubes formed from words in the letters $X, K, L$.

Once we prove this proposition, we will restrict attention to the case when $h = j$ is a trivial cofibration to prove the first statement in Lemma B.1 and we will restrict to when $h = i$ is a cofibration and $X$ is cofibrant for the second statement. This is done at the end of the section.

**Proof of Proposition B.2.** We begin by describing the left vertical map in the diagram which defines the maps $P_{n-1} \to P_n$. This will be done inductively. Because $X \otimes -$ commutes with colimits (since it’s a left-adjoint), the map $X \otimes Q_n/\Sigma_n \to P_{n-1}$ may be defined componentwise on the vertices of the cube defining $X \otimes Q_n$.

For the $n = 1$ case the map $X \otimes K \to X \otimes X \to X = P_0$ is $g$ followed by $\mu_X : X \otimes X \to X$. Let $D$ be a proper subset of $[n] = \{1, \ldots, n\}$ and define $W(D) = C_1 \otimes \cdots \otimes C_n$ where $C_i = K$ if $i \notin D$ and $C_i = L$ if $i \in D$. These are the vertices of the cube defining $Q_n$. Given a vertex $X \otimes W(D)$ define a map by first applying $g$ to all factors of $K$ (call this map $g^*$), then shuffling all the factors of $X$ so obtained to the left by a permutation $\sigma_D$, then multiplying these factors together. This map takes $X \otimes W(D)$ to $X \otimes L^{|D|}$ and hence to $X \otimes \text{Sym}^{|D|}(L)$ by passing to $\Sigma_n$-coinvariants. Induction then gives a map to $P_{|D|}$ and hence to $P_{n-1}$ because $D$ was a proper subset of $[n]$.

The map above is well-defined (i.e. respects the $\Sigma_n$ action on the cube defining $X \otimes Q_n$) because a permutation $\sigma$ which takes $W(D)$ to a different vertex $W(T)$ for
some $T$ of the same size as $D$ yields the following commutative diagram:

$$
\begin{array}{ccc}
X \otimes W(D) & \longrightarrow & X^{\otimes (n-|D|)} \otimes L^{\otimes |D|} \longrightarrow X \otimes L^{\otimes |D|} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X \otimes W(T) & \longrightarrow & X^{\otimes (n-|D|)} \otimes L^{\otimes |D|} \longrightarrow X \otimes L^{\otimes |D|} \\
\end{array}
$$

The left square commutes because the top left horizontal map is $\sigma_D \circ g^*$ and the bottom left horizontal map is $\sigma_T \circ g^*$, so the dotted arrow can be defined as $\sigma|_D$ on the $|D|$ factors of $L$ and as $\sigma|_{n-|D|}$ on the $n-|D|$ factors of $X$ (using the fact that $X$ is commutative). Thus, both ways of going around are simply doing $g^*, \sigma$, and the shuffling of $X$’s to the left. The right pentagon commutes $X$ is commutative (so the order of factors doesn’t matter) and because passage to $\Sigma$-coinvariants means the order of factors of $L$ does not matter either.

These maps from vertices assemble to a map from $X \otimes Q_n \rightarrow P_{n-1}$ because taking $i \notin D$ and defining the map from $X \otimes W(D \cup \{i\}) \rightarrow P_{n-1}$ as above gives a diagram, which we will show commutes:

$$
\begin{array}{ccc}
X \otimes W(D) & \longrightarrow & X \otimes L^{\otimes |D|} \longrightarrow P_{|D|} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X \otimes W(D \cup \{i\}) & \longrightarrow & X \otimes L^{\otimes |D|+1} \longrightarrow P_{|D|+1} \\
\end{array}
$$

The upper left horizontal map is $\mu_X \circ \sigma_D \circ g^*$ so we may factor it as $X \otimes W(D) \rightarrow X^{\otimes (n-|D|-1)} \otimes K \otimes L^{\otimes |D|} \rightarrow X \otimes L^{\otimes |D|}$ where $K$ is the $i^{th}$ factor of the original $W(D)$. Since this factor becomes an $L$ in the bottom row we have the following diagram:

$$
\begin{array}{ccc}
X \otimes W(D) & \longrightarrow & X \otimes K \otimes L^{\otimes |D|} \longrightarrow P_{|D|} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X \otimes W(D \cup \{i\}) & \longrightarrow & X \otimes L^{\otimes |D|+1} \longrightarrow P_{|D|+1} \\
\end{array}
$$

The difference between the two ways of going around the left-hand square is the order of factors in the $L$ component (the order in the $X$ component doesn’t matter). Thus, this square will commute upon passage to $P_{|D|+1}$ because of passage to $\Sigma_{|D|+1}$-coinvariants. Recall that $P_{|D|+1}$ is a pushout of $X \otimes Q_{|D|+1}$, which is itself a pushout of vertices $X \otimes W(R)$. Because a pushout of a pushout is again a pushout, the right-hand square commutes by the basic property of pushouts.

This completes the inductive definition of $P_n$. Setting $P$ to be the colimit of the $P_n$ (taken in $M$) completes the analysis. Note that in [36] the pushout is the free product of $T(L)$ and $X$ over $T(K)$ for the free monoid functor $T$, whereas in the
commutative setting $P$ is the (conceptually simpler) tensor product $\text{Sym}(L) \otimes_{\text{Sym}(K)} X$. In an analogous way to the versions of these statements in [36], we now prove

1. $P$ is naturally a commutative monoid.
2. $X \to P$ is a map of commutative monoids.
3. $P$ has the universal property of the pushout in the category of commutative monoids.

As in [36], the unit for $P$ is the map $S \to X \to P$ and the multiplication on $P$ is defined from compatible maps $P_n \otimes P_m \to P_{n+m}$ by passage to the colimit. These maps are defined inductively using the following pushout diagram (which is simply the product of the two pushout diagrams defining $P_n$ and $P_m$), where for spacing reasons we let $\widetilde{Q}_n$ denote $Q_n/\Sigma_n$ and let $\widetilde{L}^m$ denote $L^m/\Sigma_m$:

$$
\begin{array}{c}
(X \otimes \widetilde{Q}_n) \otimes (X \otimes \widetilde{L}^m) \coprod (X \otimes Q_n \otimes X \otimes \widetilde{Q}_m) (X \otimes \widetilde{Q}_m) \otimes (X \otimes \widetilde{Q}_m) \\
(P_{n-1} \otimes P_m) \coprod (P_{n-1} \otimes P_{m-1}) (P_n \otimes P_{m-1}) \\
P_n \otimes P_m
\end{array}
$$

This is a pushout square by Lemma 4.1 in [29].

The lower left corner has a map to $P_{n+m}$ by induction. The upper right corner is mapped there by shuffling the middle $X$ to the left-hand side, multiplying the two factors of $X$, passing to $\Sigma_{n+m}$-coinvariants, and using the definition of $P_{n+m}$. To show $P$ is a commutative monoid one must verify the following diagrams:

$$
\begin{array}{ccc}
S \otimes P & \rightarrow & P \otimes P \\
\downarrow & & \downarrow \mu \\
P & \rightarrow & P \otimes P
\end{array}
\quad
\begin{array}{ccc}
P \otimes P \otimes P & \rightarrow & P \otimes P \\
\downarrow \mu \otimes \mu & & \downarrow \tau \\
P \otimes P & \rightarrow & P \otimes P
\end{array}
$$

The leftmost diagram commutes because the left-hand factor of $P$ is $P_0$, coming from a map $S \to X$, and so if we replace the other factors of $P$ by $P_m$ we see that this diagram commutes before passage to colimits. In particular, the diagram defining the map $P_0 \otimes P_m \to P_m$ collapses in the following way. The upper left corner is $X \otimes X \otimes L^m/\Sigma_m \coprod X \otimes X \otimes Q_m/\Sigma_m = X \otimes X \otimes L^m/\Sigma_m$ because $X \otimes Q_0 = X$. The upper right corner is also $X \otimes X \otimes L^m/\Sigma_m$ because $X \otimes L^0 = X$. Thus, the upper horizontal map is the identity. Similarly the bottom horizontal map is the identity on $P_0 \otimes P_m$. Recalling that the $P_0$ comes from a map $S \to X$ where $S$ is the monoidal unit we may write

$$
P_0 \otimes P_m = (P_0 \otimes P_{m-1}) \coprod (S \otimes X \otimes L^m/\Sigma_m) = P_{m-1} \coprod (X \otimes L^m/\Sigma_m) = P_m
$$

Where $P_0 \otimes P_{m-1} = P_{m-1}$ by induction, and the other factors of $S$ disappear because $S$ is the unit for $X$. This proves the commutativity of the leftmost diagram.
The middle diagram also commutes on the level of individual $P_i$. In particular, the two ways of getting from $P_n \otimes P_m \otimes P_k$ to $P_{n+m+k}$ (i.e. via $P_{n+m} \otimes P_k$ and via $P_n \otimes P_{m+k}$) are the same. The key observation to show this is that all maps in the diagram are of the form (Pushout $\otimes$ Identity), and the pushout of a pushout is a pushout. Thus, both ways of going around are pushouts, and the universal property of pushouts shows that they must be isomorphic.

The rightmost diagram also commutes on the level of individual $P_i$, i.e. $P_n \otimes P_m \to P_{n+m}$ is the same as $P_n \otimes P_m \to P_m \otimes P_n \to P_{m+n}$. To see this, look at the diagram defining $\mu_P$ and consider what happens if the $n$ factors and $m$ factors are swapped. This causes no harm to the upper right corner because the map from $(X \otimes L \otimes n/\Sigma n) \otimes (X \otimes L \otimes m/\Sigma m)$ requires passage to $\Sigma_{m+n}$-coinvariants, so changing the order of the $L$ factors has no effect on $\mu_P$. Similarly there is no harm to the lower left corner because of induction. The upper left corner is hardest, but either way of going around to $P_{n+m}$ will render the swapping of factors meaningless. One way around requires passage to $\Sigma_{m+n}$-coinvariants and the other way goes to $P_i \otimes P_j$ factors for $i, j < n, m$ and so will hold by induction. This completes the proof of statement (1).

To verify that the map $X \to P$ is a map of commutative monoids one must only verify that it’s a map of monoids and that the two monoids in question are commutative. This means verifying the commutativity of the following diagrams:

The map $P \otimes P \to P$ is induced by passage to colimits of the multiplication $P_n \otimes P_m \to P_{n+m}$ and so by definition the obvious diagram with $P_n \otimes P_m$, $P_{n+m}$, $P \otimes P$, and $P$ commutes for all $n, m$. The point is that defining $P \otimes P \to P$ requires one to go to $P_n \otimes P_m$, so the commutativity is tautological. In particular, it commutes for $n = m = 0$ and this proves the left-hand diagram above commutes, since $X = P_0$. The right-hand diagram commutes by definition of the map $S \to P$ as coming from $X$. This completes the proof of statement (2).

To prove that $P$ satisfies the universal property of pushouts in the category of commutative monoids requires one to define a map $P \to M$ which completes the following diagram, where $M$ is a commutative monoid, $X \to M$ is monoidal, and $L \to M$ is a map in $M$. The reason one works with $K$ and $L$ rather than $\text{Sym}(K)$ and $\text{Sym}(L)$ is that the data of a map of commutative monoids $\text{Sym}(K) \to M$ is the
same as that of a map from $K$ to $M$, by the free-forgetful adjunction.

![Diagram](image)

The existence of maps $K \to X \to M$ and $L \to M$ defines maps from $X \otimes W(D) \to M$ for all $D$ and all $n$. Commutativity of the outer diagram forces the maps $X \otimes W(D) \to M$ to be compatible, i.e. commutativity of the square diagram featuring $X \otimes W(D), X \otimes W(D \cup \{i\}), M,$ and $M$. This is because the left-vertical map in that diagram is $K \to L$ and the right vertical map is $K \to X \to M$ (which is easy to see when thinking of commutativity of the outer diagram above as defining a word in $M$). Furthermore, these maps respect the $\Sigma_n$ action on the cube defining $Q_n$ because $M$ is commutative. Thus, by induction on $n$ we may define a map $P_n \to M$ because the diagram featuring $X \otimes Q_n/\Sigma_n, X \otimes L^{\otimes n}/\Sigma_n, P_{n-1},$ and $M$ commutes. In this diagram we use induction to define the map $P_{n-1} \to M$ and we using the fact that $M$ is commutative to define the map $X \otimes L^{\otimes n}/\Sigma_n \to M$.

Commutativity of this diagram is due to the fact that $X \otimes W(D) \to M$ factors through $X \otimes L^{\otimes D_l}/\Sigma_{|D|}$ and hence through $P_{n-1}$ via $P_{|D|}$. The unique maps $P_n \to M$ assemble to a unique map $P \to M$.

Commutativity of the triangle featuring $X, P,$ and $M$ follows by definition of $P$ as a colimit and of $X$ as $P_0$. Commutativity of the other triangle follows because it holds with $P_n$ substituted for $P$, for all $n$. This is because commutativity holds in the triangle which defines the map $P_n \to M$ for all $n$, so it holds in the (first) $L$ factor of $X \otimes L^{\otimes n}/\Sigma_n$, i.e. $L \to M$ is the same as $L \to P_n \to M$ for all $n$. This completes the proof of statement (3) and hence of the proposition. 

We move now to homotopy theoretic considerations, and use the proposition to prove Lemma B.1.

**Proof.** To prove statement (1), recall that the commutative monoid axiom tells us that if $h$ is a trivial cofibration then $h^{\otimes n}/\Sigma_n$ is a trivial cofibration for all $n > 0$.

So suppose $h = j : K \xrightarrow{\sim} L$. Because $j$ is a trivial cofibration, the map $j^{\otimes n}/\Sigma_n : Q_n/\Sigma_n \to \text{Sym}^n(L)$ is a trivial cofibration. Thus, the map $X \otimes j^{\otimes n}/\Sigma_n$ is of the form required by the monoid axiom. This means transfinite compositions of pushouts of such maps are weak equivalences, so in particular $X \to P$ is a weak equivalence in $\mathcal{M}$ and hence in $\text{CMon}(\mathcal{M})$. Any map in $\text{Sym}(J)$-cell is a transfinite composite of pushouts of maps in $\text{Sym}(J)$. We have seen that all such pushouts are of the form required by the monoid axiom, and a transfinite composite of a transfinite composite is still a transfinite composite, so the monoid axiom applied again proves
that \( \text{Sym}(J) \)-cell is contained in the weak equivalences. This completes the proof of (1).

For (2), suppose \( h = i : K \hookrightarrow L \) and suppose \( X \) is cofibrant in \( \mathcal{M} \). By the strong commutative monoid axiom, the maps \( i^{\text{cfin}}_{/\Sigma_n} \) are cofibrations for all \( n \), so \( X \otimes i^{\text{cfin}}_{/\Sigma_n} \) are cofibrations for all \( n \). Since pushouts of cofibrations are again cofibrations, the maps \( P_{n-1} \to P_n \) are cofibrations for all \( i \). Because \( P_0 = X \) is cofibrant, this means all the \( P_k \) are cofibrant and also \( X \to P \) is a cofibration (so \( P \) is cofibrant) because transfinite compositions of cofibrations are again cofibrations (see Proposition 10.3.4 in [18]). Every map in \( \text{Sym}(I) \)-cell which has cofibrant domain is a transfinite composite of pushouts of maps of the form above, and so is in particular again a cofibration in \( \mathcal{M} \).

\[\Box\]

In the proof above, we make use of a particular filtration on the map \( X \to P \). We could also have followed [25] and filtered the map \( \text{Sym}(f) \) as

\[\text{Sym}(K) = B_0 \to B_1 \to \cdots \to \text{Sym}(L)\]

where each \( B_n \) is a \( \text{Sym}(K) \)-module. This makes it clear that the map \( X \to P \) is a map of \( X \)-modules, and thus makes it easier to check that \( P \) is in fact a monoid. However, this filtration requires special knowledge of \( \text{Com} \), namely that it is generated by \( \text{Com}(2) \)-swaps (i.e. functions of arity two) so that \( \text{Com} \)-algebras can be multiplied with themselves. The author chose the approach presented here because he is currently working to generalize the proof to hold for other operads.

\section*{References}


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