This document summarizes my work so far. Section 1 contains my thesis work. Sections 2 and 3 are joint work with European coauthors begun while I was a research visitor at the University of Barcelona. More details can be found in the longer version of my research statement hosted on my website.

1. Bousfield localization and algebras over operads

Let $\mathcal{M}$ be a model category with a closed symmetric monoidal product $\otimes$ and unit S. The pushout product of two maps $f : A \to B$ and $g : X \to Y$ is the map $f \square g : A \otimes Y \coprod_{A \otimes X} B \otimes X \to B \otimes Y$. Let $\mathscr{W}$, $\mathscr{Q}$, and $\mathscr{F}$ denote the weak equivalences, cofibrations, and fibrations. Let $Q(-)$ and $R(-)$ denote cofibrant and fibrant replacement functors. We say $\mathcal{M}$ is a *monoidal model category* if it satisfies the pushout product axiom ($\mathscr{Q} \square \mathscr{Q} \subset \mathscr{Q}$ and $\mathscr{Q} \square (\mathscr{Q} \cap \mathscr{W}) \subset \mathscr{Q} \cap \mathscr{W}$) and the unit axiom (whenever X is cofibrant, the map $QS \otimes X \to S \otimes X \cong X$ is a weak equivalence). Let C be a set of maps such that the Bousfield localization $L_C(\mathcal{M})$ exists. My thesis proves:

Theorem 1. *Let $\mathcal{M}$ be a monoidal model category and let P be an operad valued in $\mathcal{M}$. If P-algebras in $\mathcal{M}$ and in $L_C(\mathcal{M})$ inherit model structures such that the forgetful functors back to $\mathcal{M}$ and $L_C(\mathcal{M})$ are right Quillen functors, then L_C preserves P-algebras up to weak equivalence. For well-behaved P there is a list of easy to check conditions on $\mathcal{M}$ and C guaranteeing these hypotheses hold.*

The motivation for this theorem comes from the recent proof of Hill-Hopkins-Ravenel of the Kervaire Invariant One Theorem. This proof requires a particular Bousfield localization of equivariant spectra to preserve commutative structure. Checking the hypotheses of Theorem 1 for the operad $Com = (*)_{n \in \mathbb{N}}$ requires a general theorem for when commutative monoids inherit a model structure. For monoids this is done in a paper of Schwede and Shipley, and the hypothesis needed on $\mathcal{M}$ is the *monoid axiom*, which says that for all objects X, $(id_X \otimes (\mathscr{Q} \cap \mathscr{W})) - cell \subset \mathscr{W}$. Here applying cell means taking closure under transfinite compositions and pushouts. For commutative monoids the correct hypothesis is the *commutative monoid axiom*: If g is a (trivial) cofibration then $g^{\square n}/\Sigma_n$ is a (trivial) cofibration.

Theorem 2. *If a monoidal model category satisfies the monoid axiom and the commutative monoid axiom then commutative monoids form a model category and the forgetful functor is right Quillen.*

Examples: simplicial sets, $Ch(k)$ for $char(k) = 0$, S-modules, and positive model structures on symmetric spectra, orthogonal (equivariant) spectra, and motivic symmetric spectra (see Section 3).

To check the hypotheses of Theorem 1 on $\mathcal{M}$ we must assume $\mathcal{M}$ satisfies the pushout product axiom, the monoid axiom, and the commutative monoid axiom. We also assume $\mathcal{M}$ is cofibrantly generated, left proper, tractable, and the resolution axiom (for cofibrant X, $- \otimes X$ preserves weak equivalences). These hold in all the examples above. We then provide hypotheses on C so that L_C preserves these axioms and counterexamples to show these hypotheses are necessary. For instance,

Theorem 3. *Under the standing hypotheses above:*

(1) $L_C(\mathcal{M})$ satisfies the resolution axiom and pushout product axiom if and only if for all domains and codomains K of the generating cofibrations, maps $C \otimes id_K$ are weak equivalences in $L_C(\mathcal{M})$

(2) If $Sym(-)$ preserves weak equivalences in $L_C(\mathcal{M})$ then $L_C(\mathcal{M})$ satisfies the commutative monoid axiom. Here $Sym(X) = S \coprod X \coprod X^{\otimes 2}/\Sigma_2 \coprod X^{\otimes 3}/\Sigma_3 \coprod \cdots$

(3) If $\mathcal{M}$ is h-monoidal in the sense of Berger-Batanin and if the generating cofibrations have finite domains and codomains then for any C, then $L_C(\mathcal{M})$ satisfies the monoid axiom.

As a special case, we recover the theorem of Hill and Hopkins used in Hill-Hopkins-Ravenel:

Corollary 4. *Let R be a monoid in the genuine model structure for G-spectra. If $Sym(-)$ preserves R-acyclicity then the Bousfield localization with respect to R-equivalences preserves commutative monoids.*

Whenever $\mathcal{M}$ satisfies the pushout product and monoid axioms, the subcategory of algebras over any cofibrant operads inherits a model structure as in Theorem 1. Thus, Theorems 1 and 3 provide easy to check conditions on $\mathcal{M}$ and C so that L_C preserves P-algebra structure for cofibrant P. Theorem 2 has a generalization which provides for any operad P, a precise hypothesis on $\mathcal{M}$ such that P-algebras inherit a semi model structure. There are additional axioms so that this semi model structure lifts to a model structure. Increasing the strength of the cofibrancy hypothesis on P (from no hypothesis to levelwise cofibrant to Σ-cofibrant to cofibrant) corresponds to decreasing the strength of the hypothesis on $\mathcal{M}$. So while the cofibrancy price must still be paid, it can be paid partially by P and partially by $\mathcal{M}$.

2. Equivariant Homotopy Theory

My thesis began with an example due to Mike Hill demonstrating that Bousfield localization can fail to preserve commutative monoids (equivalently, genuine E_∞-algebras). In order to apply Theorem 1, Javier Gutiérrez and I seek to understand cofibrancy for operads with a G action. We prove:

Theorem 5. *For every family of subgroups F, the category of G-operads admits a semi-model structure with weak equivalences and fibrations maps f such that f^H is such for all $H \in F$. Taking cofibrant replacements for Com yields a tower of operads E_∞^F interpolating between naive and genuine E_∞ structure. When $F = All$, the n-th space is $E_G\Sigma_n$, a contractible free $G \times \Sigma_n$ space defined as the total space of the universal G-equivariant principle Σ_n-bundle. General E_∞^F admit similar descriptions.*

We are now attempting to classify Bousfield localizations which satisfy Theorem 3 (1).

Additionally, I have a project with Mark Hovey which proposes an alternative approach to equivariant homotopy theory avoiding the use of universes. We start with the category GSp of orthogonal spectra X with a G-action, then use the method in Stefan Schwede's lecture notes to recover $X(V)$ for any V in a universe. The idea of our project is to create a family of model structures on GSp such that each is Quillen equivalent to a model structure defined from a universe and such that Bousfield localization functors between them corresponding to change-of-universe functors.

3. Motivic Homotopy Theory

In order to apply the results of Section 1 in motivic symmetric spectra, Marcus Spitzweck and I proved:

Theorem 6. *Hovey's machine for producing symmetric spectra in general model categories can be tweaked to produce a positive model structure in the sense of Shipley, i.e. wherein the category of commutative monoids inherits a model structure.*

In order to understand motivic Bousfield localizations, Carles Casacuberta and I observed that if E is a motivic homology theory, localization at maps f such that $E_{*,*}(f)$ is an isomorphism is different from localization at f such that $E \wedge f$ is a weak equivalence. The latter was known to exist. We proved:

Theorem 7. *$E_{*,*}$-localizations exist, and when E is the sphere spectrum the image of $S_{*,*}$-localization equals cellular motivic spectra. Thus, this category is both localizing and colocalizing.*

This also gives a new way to test for cellularity, namely by testing whether or not the spectrum is $S_{*,*}$-local. Cellular spectra are singly generated by the sphere spectrum as a localizing subcategory.

Theorem 8. *For all motivic homology theories E, the Brown-Comenetz dual IE exists. Furthermore, IS is cellular and cogenerates cellular motivic spectra as a colocalizing subcategory.*

In the future we would like to prove $IS \wedge IS = 0$, to understand which $L_{E_{**}}$ are smashing, to explore the difference between algebraic and topological cellularity, and to explore $L_{E_{**}}$ unstably.