Baez-Dolan Stabilization via (Semi-)Model Categories of Operads

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Abstract: We describe a proof of the Baez-Dolan Stabilization Hypothesis for Rezk's model of weak n-categories. This proof proceeds via abstract homotopy theory, and en route we discuss a version of left Bousfield localization which does not require left properness. We also discuss conditions under which various categories of operads can be made left proper, but these conditions are difficult to satisfy as a counterexample in the context of simplicial sets demonstrates.

1 Introduction

In this talk I described the Baez-Dolan Stabilization Hypothesis, reported on Michael Batanin's work to prove it for Rezk's model of weak n-categories, and described my contributions to this program. The main new result I presented was

Theorem 1 *Rezk's model of weak n-categories satisfies both a weak and a strong form of the Baez-Dolan Stabilization Hypothesis.*

Observe that the methods of ∞-categories are of limited use here, because the statement is about weak n-categories not (∞, n)-categories. Proving this result made use of the following:

Theorem 2 *Suppose $(\mathcal{M}, \otimes, I)$ is a cofibrantly generated monoidal model category with cofibrant unit. Then the categories $SOp(\mathcal{M})$ and $Op_n(\mathcal{M})$ of symmetric operads and n-operads inherit projective cofibrantly generated semi-model structures from $\mathcal{M}$. This means the weak equivalences (resp. fibrations) are maps which are componentwise weak equivalences (resp. fibrations). If in addition $\mathcal{M}$ is a compactly generated, strongly h-monoidal model category satisfying a strengthened version of the commutative monoid axiom then these semi-model structures can be made into left proper model structures.*

Left properness was important because at the start of our collaboration, Batanin had reduced the strong version of the Baez-Dolan Stabilization Hypothesis to the existence of these transferred left proper model structures where $\mathcal{M}$ is Rezk's category $tr_{\leq n}(\Theta_n$-spaces) of weak n-categories. Unfortunately, it is not true that symmetric operads valued in simplicial sets are left proper, as a counterexample presented in [**6**] demonstrates. Nevertheless, we are able to prove the strong version of the Baez-Dolan Stabilization Hypothesis using

Research Perspectives CRM Barcelona, Spring 2015, vol. 5, in Trends in Mathematics
Springer-Birkhäuser, Basel

Theorem 3 *If $\mathcal{M}$ is a locally presentable, cofibrantly generated (semi-)model category in which the domains of the generating cofibrations are cofibrant then for any set of morphisms $\mathcal{C}$ in $\mathcal{M}$, there exists a cofibrantly generated semi-model structure $L_{\mathcal{C}}(\mathcal{M})$ on $\mathcal{M}$ with weak equivalences defined to be the $\mathcal{C}$-local equivalences, and cofibrations the cofibrations of $\mathcal{M}$.*

The hypotheses of these two theorems are satisfied by the Quillen model structure on simplicial sets, by Rezk's Θ_n-spaces, and by truncations of these model structures (proven via the machinery of [**10**]), so left Bousfield localizations of $SOp(\mathcal{M})$ and $Op_n(\mathcal{M})$ exist even without left properness.

2 Baez-Dolan Stabilization Hypothesis

This hypothesis was introduced in 1995 in [**1**], where Baez and Dolan attempted to describe n-dimensional topological quantum field theories as (weak) n-category representations, generalizing the notion of a group representation. In doing so, it became important to understand what happens as n varies. Let $nCat$ denote the category of weak n-categories.

Given $C \in (n+1)Cat$ with one object, we can regard it in $nCat$ via reindexing (the new objects are the old morphisms, the new morphisms are the old 2-morphisms, etc). This new n-category has extra structure, e.g. a composition law for objects. Given an $(n+k)$-category with a unique cell in dimensions $\leq k-1$, we can regard it as a n-category with extra structure, which Baez and Dolan call a **k-tuply monoidal n-category**. Let $nCat_k$ denote the category of k-tuply monoidal n-categories.

For example, the Eckmann-Hilton argument tells us that a 2-category with one object x and one morphism 1_x gives a commutative monoid structure on the object $\hom(1_x, 1_x)$ obtained after twice reindexing. Interestingly, a k-category with one cell in each dimension $\leq k-1$ (for $k > 2$) gives no additional structure on the object which results from reindexing k-times. This and similar observations led to

Stabilization Hypothesis: If $k \geq n+2$ then any k-tuply monoidal weak n-category is a $(k+1)$-tuply monoidal weak n-category.

This is a hypothesis rather than a conjecture, because in 1995 there was not a good definition of a weak n-category. Carlos Simpson proved the hypothesis holds for Tamsamani's notion of a weak n-category [**8**], and we seek to prove it holds for Rezk's notion and to also prove a more general version of stabilization. Batanin has reduced this question to a simpler question in a series of papers over the past several years.

First, the structure of a k-tuply monoidal n-category can be encoded by something called a k-operad in $nCat$ (a weak version of this theory would ask it to be encoded by specifically an E_k-operad). Indeed, k-operads were primarily introduced to study this phenomenon, as [**3**] makes clear. Let $\operatorname{tr}_n \mathcal{M}$ denote the n-truncation of $\mathcal{M}$ (i.e. for all X,Y the space $map(X,Y)$ is n-truncated in sSet), and let $Op_k(\operatorname{tr}_n \mathcal{M})$ denote the model category of k-operads in $\operatorname{tr}_n \mathcal{M}$ (see [**5**]). Batanin defined a suspension functor $S^* : Op^{loc}_{k+1}(\operatorname{tr}_n \mathcal{M}) \to Op^{loc}_k(\operatorname{tr}_n \mathcal{M})$ on so-called n-locally constant k-operads which captures the suspension studied by Baez and Dolan. This allows for a proof of the Baez-Dolan Stabilization Hypothesis via the following:

Theorem 4 *If $k \geq n+2$ then S^* is a right Quillen equivalent.*

In the talk I focused on the following easier to state result, which is implied by Theorem 4 and implies the classical Baez-Dolan Stabilization Hypothesis:

Theorem 5 ([**4**]) *Suppose the unit of $(\mathcal{M}, \otimes, I)$ is cofibrant. Let 1_k denote the canonical k-operad $1_k(T) = I, T \in Ord(k)$ and let $G_{n,k}$ denote the n-truncation of its cofibrant replacement. If $k \geq n+2$ then $Alg_{G_{n,k}}(\mathrm{tr}_n \mathcal{M})$ is Quillen equivalent to $Alg_{G_{n+1,k}}(\mathrm{tr}_n \mathcal{M})$.*

This statement can be proven by mapping each of $Alg_{G_{n,k}}(\mathrm{tr}_n \mathcal{M})$ and $Alg_{G_{n,k+1}}(\mathrm{tr}_n \mathcal{M})$ to $Alg_{E_\infty}(\mathrm{tr}_n \mathcal{M})$ where E_∞ is a cofibrant replacement of Com taken in $SOp(\mathcal{M})$. As a consequence of our work, both of these statements will be proven when $\mathrm{tr}_n \mathcal{M}$ is Rezk's model of weak n-categories, i.e. the truncation $tr_{\leq n}(\Theta_n\text{-spaces})$.

3 Tools from Abstract Homotopy Theory

The proof of Theorem 2 involves a careful analysis of the free-algebra extensions coming from the free-forgetful adjunctions $S : \mathcal{M}^{\mathbb{N}} \leftrightarrows SOp(\mathcal{M}) : U$ and $F : Coll_n(\mathcal{M}) \leftrightarrows Op_n(\mathcal{M})$. For simplicity we focus our exposition on the former. Here $\mathcal{M}^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \mathcal{M}$ has the product model structure and this adjunction passes through $\prod_{n \in \mathbb{N}} \mathcal{M}^{\Sigma_n}$ with the projective model structure. As always when attempting to transfer a model structure across an adjunction, what must be checked is that for any trivial cofibration $f : K \to L$ in $\mathcal{M}^{\mathbb{N}}$ and for any map $S(K) \to O$ in $SOp(\mathcal{M})$ then the pushout map $O \to P := O \coprod_{S(K)} S(L)$ is a weak equivalence. If this is only true for cofibrant O then the resulting structure is that of a **semi-model category**, which means the lifting of a trivial cofibration f against a fibration and the factorization of a map g into a trivial cofibration followed by a fibration only hold for maps f, g with cofibrant domain. The analysis in Section 9.4 of [**5**] demonstrates that the monad S can be represented by a Σ-cofibrant colored operad. Theorem 6.3.1 in [**12**] then provides the semi-model structure, which Batanin has used to prove Theorem 5 in [**4**].

In order to promote this to a left proper model structure we view $K = (K_n)_{n \in \mathbb{N}}$ as $(K_0) \coprod K_{red}$ where $K_{red} := (K_n)_{n \geq 1}$ and we factor the pushout as two pushouts

$$\begin{tikzcd}
S(K) \arrow[r] \arrow[d] & S(L) \arrow[d] \\
O \coprod O \arrow[r] \arrow[d] & P_z \coprod P_{red} \arrow[d] \\
O \arrow[r] & P
\end{tikzcd}$$

where $P_z = S(L_0) \coprod_{S(K_0)} O$ and $P_{red} = S(L_{red}) \coprod_{S(K_{red})} O$. This reduces us to studying the maps $O \to P_z$ and $O \to P_{red}$ separately. The commutative monoid axiom of [**9**], there used to prove that commutative monoids in $\mathcal{M}$ inherit a model structure, can be generalized to this setting to provide control over the former map. The hypotheses that $\mathcal{M}$ be strongly h-monoidal and compactly generated allow for control over the latter map, as described in [**5**]. However, since a semi-model structure suffices we prefer here to say a word about Theorem 3, which will appear in [**11**].

The main tool to prove Theorem 3 is a semi-model category version of Jeff Smith's theorem for creating combinatorial model structures (presented in [**2**] where Theorem 3 is conjectured). By carefully proving semi-model categorical versions of several results in [**7**], I am able to prove that the set of generating trivial cofibrations $J_{\mathcal{C}}$ produced by Smith's theorem does in fact provide a Bousfield localization semi-model structure which satisfies the universal property of localization and which is Quillen equivalent (as a semi-model category) to the model category localization $L_{\mathcal{C}}(\mathcal{M})$ should it exist. This theorem is applied to $Op_k(\mathrm{tr}_n(\mathcal{M}))$ and completes the proof of Theorem 4.

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