LOCALIZATIONS IN TRIANGULATED CATEGORIES AND MODEL CATEGORIES

DAVID WHITE

1. Review of Definitions

Recall that for a triangulated category $\mathcal{T}$, a **Bousfield localization** is an exact functor $L : \mathcal{T} \to \mathcal{T}$ which is coaugmented (there is a natural transformation $Id \to L$; sometimes L is referred to as a pointed endofunctor) and idempotent (there is a natural isomorphism $L\eta = \eta L : L \to LL$). The kernel $\ker(L)$ is the collection of objects X such that $LX = 0$. If $\mathcal{T}$ is closed under coproducts, it's a **localizing subcategory** because L is a left adjoint.

An object t is called **L-local** if $\mathcal{T}(s,t) = 0$ for all $s \in \ker(L)$. If L is inverting $\mathcal{S}$, denote by ${}^{\perp}\mathcal{S}$ the full subcategory of $\mathcal{S}$-local objects. The image of L is contained in the L-local objects. Call a map f an **L-local equivalence** if $\mathcal{T}(f,X)$ is a bijection for all L-local X. These are the maps which are turned into isomorphisms by L. The universal property of killing $\ker(L)$ is equivalent to the universal property of inverting the L-local equivalences. Even in general category theory a morphism $f : X \to Y$ and an object Z in C are called **orthogonal** if the induced map $C(f,Z) : C(Y,Z) \to C(X,Z)$ is a bijection. We introduce this notion because it will be needed when we do localization unstably (and we can't say a map f is an isomorphism iff $cof(f)$ vanishes).

Definition 9.1.1 (Neeman book): Let $\mathcal{S}$ be thick. A **Bousfield localization exists** for $\mathcal{S} \subset \mathcal{T}$ if there is a right adjoint to the natural functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$. Indeed, this occurs iff $\mathcal{S} \hookrightarrow \mathcal{T}$ has a right adjoint (as we'll see below). If Bousfield localization exists then ${}^{\perp}\mathcal{S}$ is equivalent to $\mathcal{T}/\mathcal{S}$ (this is Theorem 9.1.16). To see this, use the functor ${}^{\perp}\mathcal{S} \subset \mathcal{T} \to \mathcal{T}/\mathcal{S}$. It's fully faithful. Morphisms between $\mathcal{S}$-local objects are the same in $\mathcal{T}$ and $\mathcal{T}/\mathcal{S}$ because $\mathcal{T}(x,y) = \mathcal{T}/\mathcal{S}(x,y)$ for $x \in \mathcal{T}, y \in ({}^{\perp}\mathcal{S})$.

Theorem 9.1.18 shows that **Bousfield localization exists iff the inclusion** $I : \mathcal{S} \to \mathcal{T}$ **has a right adjoint**. We have seen the ($\Rightarrow$) direction already, because map $\mathcal{T} \to \mathcal{T}/({}^{\perp}\mathcal{S})$ has a left adjoint which allows us to see the embedding of $\mathcal{T}/({}^{\perp}\mathcal{S})$ as $\mathcal{S} = ({}^{\perp}\mathcal{S})^{\perp}$. So the embedding has a right adjoint $\mathcal{T} \to \mathcal{T}/({}^{\perp}\mathcal{S})$ which we can write explicitly. Conversely, given a right adjoint $J : \mathcal{T} \to \mathcal{S}$ to the embedding $I : \mathcal{S} \to \mathcal{T}$ we can use the unit of the adjunction $IJt \to t \to z \to \Sigma IJt$. Adjointness gives $\mathcal{T}(Ix,t) = \mathcal{S}(x,Jt) = \mathcal{T}(Ix,IJt)$ (using that I is fully faithful). The long exact sequence of this triangle gives that $\mathcal{T}(Ix,z) = 0$ for all $x \in \mathcal{S}$. This means $z \in ({}^{\perp}\mathcal{S})$ so we have a triangle $IJt \to t \to z$ with $IJt \in \mathcal{S}$ and $z \in ({}^{\perp}\mathcal{S})$ both depending on t. So there is a Bousfield localization, because the map $t \to z$ is in $Mor_{\mathcal{S}}$ and hence becomes an isomorphism in the quotient so $\mathcal{T}/\mathcal{S}(x,z) = \mathcal{T}/\mathcal{S}(x,t)$. But $z \in ({}^{\perp}\mathcal{S})$, so $\mathcal{T}/\mathcal{S}(x,t) = \mathcal{T}(x,z)$. Together this gives a natural isomorphism $\mathcal{T}/\mathcal{S}(-,t) = \mathcal{T}(-,Gt)$ for $Gt = z$. So Gt is unique up to canonical isomorphism and G extends to a functor, adjoint to $F : \mathcal{T} \to \mathcal{T}/\mathcal{S}$. Defining G this way is a useful trick. We will see something similar later when we connect localization to Brown representability.

Definition. An object t is called **L-colocal** if $\mathcal{T}(t,s) = 0$ for all $s \in \ker(L)$. Denote by $\mathcal{S}^{\perp}$ the full subcategory of $\mathcal{S}$-colocal objects. If $\mathcal{T}$ is closed under products, it's a **colocalizing subcategory**. A map f is a **L-colocal equivalence** if $\mathcal{T}(X,f)$ is a bijection for all L-colocal X. We say **Bousfield colocalization exists** when there is a left adjoint to the natural functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$. This occurs iff $\mathcal{S} \hookrightarrow \mathcal{T}$ has a left adjoint.

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If the maps to be inverted take the form $W \to *$ for some object W then we give the localization a special name and call it W**-nullification** (it's equivalent to localization at the map $* \to W$). We often denote this localization P_W by analogy to the Postnikov section. For all $t \in \mathcal{T}$, the coaugmentation of a localization $t \to Lt$ can be extended to form a triangle $At \to t \to Lt$. The object Lt is $\mathcal{S}$-local and the object At is $(^{\perp}\mathcal{S})$-colocal (it is the acyclization of t; note that it's colocal but not with respect to the same L). Similarly, the augmentation of a colocalization $Ct \to t$ can be extended to a triangle $Ct \to t \to Pt$ where Pt is local (it is the nullification of t).

Example: Work p-locally. In the stable homotopy category, the category of rational spectra is both localizing and colocalizing. It's localizing because it's a colocalization with respect to the rationalization of the sphere. It's colocalizing because it's the image of a localization, namely the map which kills the mod p Moore space. Here we have $A(X) \to X \to L_0(X)$ and also have $Cell(X) \to X \to P(X)$. The categories $im(L_0)$ and $im(Cell)$ are the same, but for an individual X, $L_0(X)$ is far from equaling $Cell(X)$.

Remark: The most general form of localization, which we will call **categorical localization**, is an idempotent coaugmented functors (not necessarily on a triangulated category). Indeed, the data of a localization functor is equivalent to the data of an idempotent monad (T, η, μ) and both are equivalent to the data of an orthogonal pair. Given an idempotent functor T, one can form a monad $C \xrightarrow{T} \mathcal{D}(T) \xrightarrow{K} C$ where $\mathcal{D}(T)$ is the Eilenberg-Moore category of T and K is defined by realizing that $\mathcal{D}(T)$ is equivalent to the Kleisli category of T. Next, given an idempotent monad T, let $\mathcal{S}$ be the class of morphisms which T sends to isomorphisms. By construction, $(\mathcal{S}(T), \mathcal{D}(T))$ is an orthogonal pair (using that T is idempotent), and $\mathcal{D}(T)$ is the full image of T (i.e. the objects Y such that $Y \cong TY$). Finally, given an orthogonal system, define a localization to kill the $\mathcal{S}$-local objects. So we see that the theory of orthogonal pairs is intimately linked with that of Bousfield localization. This will come back later.

Localizations are characterized by each of two universal properties:
(i) $\eta_X : X \to LX$ is initial among morphisms from X to an L-local object
(ii) $\eta_X : X \to LX$ is terminal among L-equivalences going out of X.

A map is an L-equivalence if and only if it is orthogonal to all L-local objects, and an object is L-local if and only if it is orthogonal to all L-equivalences. Of course the same holds for $\mathcal{S}$-equivalences and $\mathcal{S}$-local objects (it's just different notation for the same thing).

2. Not all localizing subcategories give a Bousfield localization

Is every localizing subcategory the kernel of a localization? (A similar question, but in the special case where $\mathcal{T}$ is a stable homotopy category, was asked in HPS page 35)

The following demonstrates that the answer is no for a general $\mathcal{T}$. From Casacuberta-Neeman: Brown Representability Does Not Come For Free.

Example: Freyd's book *Abelian Categories*, Exercise A, Chapter 6 constructs the following abelian category. Let I be the class of all small ordinals and let $R = \mathbb{Z}[I]$. It's a large ring. Let $\mathcal{A}$ be the abelian category of all small R-modules (i.e. small abelian groups with compatible endomorphisms indexed by I). For any M, N in $\mathcal{A}$ we have an inclusion $\mathcal{A}(M, N) \subset Hom_{Ab}(M, N)$ so this category has small hom sets. For any ring, R-mod has exact products and coproducts, and filtered colimits are exact. This is a property of Grothendieck abelian categories generally. This category fails to have a small generator, and also cannot have enough injectives/projectives.

Consider $K(\mathcal{A})$ and let $A(\mathcal{A})$ be the full subcategory of acyclic complexes. Both are triangulated categories with small Hom-sets (morphisms are homotopy equivalence classes of chain maps), because $K(\mathcal{A})$ is always triangulated for $\mathcal{A}$ abelian, and complexes of acyclics are always triangulated subcategories (exactness is easy to verify via the 5-lemma and maps to the all zero triangle).

Looking at the proof of Verdier's Theorem, we find a construction of the thick subcategory $\widehat{C}$ generated by $C \subset T$. We see in particular that $\widehat{C}$ is the full subcategory consisting of direct summands of objects of C. I want to show that $A(\mathcal{A})$ is thick. The proof in Neeman is the next paragraph and holds in more generality (i.e. whenever idempotents split), but a simpler proof is to observe that if a direct sum is in $A(\mathcal{A})$ then $H_*(X \oplus Y) = 0$ so $H_*(X) \oplus H_*(Y) = 0$ and this proves X and Y are in $A(\mathcal{A})$.

Lemma 1.6.8 (Neeman): if all idempotents in C split, then $\widehat{C} = C$. Note that if C is closed under coproducts then all idempotents split. This takes Neeman some hard work to prove. I'll sketch the proof. He uses homotopy colimits of the "all e" and "all $(1-e)$" telescopes of X's, takes the direct sum, and uses that $\begin{pmatrix} e & 1-e \\ 1-e & e \end{pmatrix}$ is its own inverse on $X \oplus X$. Then one can decompose the telescope of the $X \oplus X$'s (which converges to $Y \oplus Z$) into the direct sum of "all 1" and "all 0" sequences. Thus, $Y \oplus Z \cong X \oplus 0$ and we get $Y \to X$. Composition along the chain gives $X \to Y$ and one can show $g \circ f = e$ and $f \circ g = 1_Y$. Similarly, get the same for Z and $1-e$.

Because $A(\mathcal{A})$ is thick (hence localizing and colocalizing, and clearly an ideal) we can form the Verdier quotient $D(\mathcal{A}) = K(\mathcal{A})/A(\mathcal{A})$. We will show it cannot be a Bousfield localization because it does not have small hom-sets. In particular, $D(\mathcal{A})(\mathbb{Z}, \Sigma\mathbb{Z}) \cong \mathrm{Ext}^1_{\mathcal{A}}(\mathbb{Z}, \mathbb{Z})$ is a proper class, because we have a proper class of non-isomorphic M_i which satisfy $0 \to \mathbb{Z} \to M_i \to \mathbb{Z} \to 0$. They are all isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ as abelian groups, but the different endomorphisms ϕ_i are determined by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, all are non-isomorphic as modules because the element for which the endomorphism is non-zero changes as we change i.

Finally, note that $D(\mathcal{A})$ is a Bousfield (co)localization iff $i : A(\mathcal{A}) \to K(\mathcal{A})$ has a right (resp left) adjoint, by Prop 9.1.18. In our case this means that if Bousfield localization existed then $D(\mathcal{A})$ would be equivalent to a full subcategory ${}^{\perp}A(\mathcal{A}) \subset K(\mathcal{A})$, namely the X such that $K(\mathcal{A})(A, X) = 0$ for all $A \in A(\mathcal{A})$. But this cannot be the case because we have just shown $D(\mathcal{A})$ does not have small hom sets. Dually, there cannot be a Bousfield colocalization.

Corollary: Neither $A(\mathcal{A})$ nor its dual satisfy Brown representability. If it did, then $i : A(\mathcal{A}) \to K(\mathcal{A})$ would have a right adjoint G, defined by representability with respect to the functor $T(i(-), t) = A(\mathcal{A})(-, Gt)$. Given a map $t \to t'$, do representability for both and get $S(-, Gt) \to S(-, Gt')$. Apply Yoneda's lemma to get a unique map $Gt \to Gt'$ making G a functor. Similarly, if Brown representability held for the dual then there would be a left adjoint.

Note: if you think this example is cheating by making the difficulty be with the difference between a set and a class, go check out Christiensen-Keller-Neeman to see Brown representability fail because of a spectral sequence Ext computation. Technically, I'd say Adams representability fails, or perhaps Brown representability for homology, just to be precise.

3. Bousfield localization of model categories

Let's focus on the specific case where $\mathcal{T}$ is the stable homotopy category, i.e. the homotopy category of the model category $\mathcal{S}$ of spectra. The motivating example of a Bousfield localization is **homological localization**. Let E be a generalized homology theory and consider the maps f such that $E_*(f)$ is an isomorphism. Call these maps ***E*-equivalences** and say an object Z is ***E*-local** if $f^* : [Y, Z] \to [X, Z]$ is an isomorphism for all E-equivalences $f : X \to Y$. Bousfield proved that there is a model structure $L_E\mathcal{S}$ with weak equivalences the E_*-isomorphisms and cofibrations the same as in $\mathcal{S}$. This works if we replace $\mathcal{S}$ by spaces too (and in fact Bousfield proved this version earlier). To see why we changed from testing orthogonality $\mathcal{M}(Y, Z) \to \mathcal{M}(X, Z)$ note that we're doing homotopy theory now so we need to know about homotopy-orthogonality and not strict orthgonality.

A more subtle question is whether or not cohomological localizations exist. This question is open, though it has been resolved by Casacuberta and Chorny under the assumption of Vopenka's Principle, which is a statement about supercompact cardinals. Vopenka is equivalent to the statement that for $\mathcal{C}$ a locally presentable category, every full subcategory $\mathcal{D}$ which is closed under colimits is a coreflective subcategory. It's also equivalent to the statement that every cofibrantly generated $\mathcal{M}$ is Quillen equivalent to some combinatorial $\mathcal{M}$. Vopenka's principle lets you get from a class worth of maps to a set-worth (and then to just one, via coproduct). The reason to introduce these axioms outside ZFC is to eventually prove that the existence of cohomological localization is outside the power of ZFC to prove (i.e. to show that the question is set-theoretical).

3.1. **A punchline due to Bousfield.** . Let J be a set of primes and let $\mathbb{Z}_{(J)}$ be the subring of $\mathbb{Q}$ where p is invertible iff $p \notin J$. Let $X_{(J)} = X \wedge M(\mathbb{Z}_{(J)})$ for the Moore spectrum. This is arithmetic localization of spectra. Let $X_p^\wedge$ be the p-adic completion of X, i.e. the inverse limit $\lim_n(X \wedge M(\mathbb{Z}/p^n))$. Let $X_J^\wedge = \prod_{p\in J} X_p^\wedge$. This is completion of spectra. We now show that both arithmetic localization and completion are special cases of Bousfield localization and indeed get a number theoretic **characterization for $L_E(X)$ if both X and E are connective**.

Theorem 1. *Let E_* be a connective homology theory and X a connective spectrum. Let J be complementary to the set of primes where E_i is uniquely p-divisible for each i. Then $L_E X = X_J^\wedge$ if each element of E_* has finite order, and $L_E X = X_{(J)}$ otherwise.*

In the same vein, the times p^n map on some X has cofiber $X \wedge M(p^n)$. The homotopical version of completion is the inverse $\text{holim}_n(X \wedge M(p^n))$. This turns out to equal $L_{M(p)}X$. The fact that **completion is a special case of localization** is one of the few places where algebraic topology is simpler than homological algebra.

3.2. **Bousfield Localization more generally.** . As localization is such a handy tool, it's natural to ask whether it can be generalized to model categories other than $sSet$ and Spectra. The answer is yes. A full treatment can be found in Hirschhorn's book. Briefly, the key trick is that it's no longer enough to test isomorphism of homotopy classes of maps $[Y, Z] \to [X, Z]$ because it's not true that f is a weak equivalence iff $[f]$ is an isomorphism in $Ho(\mathcal{M})$. This is only true for maps between cofibrant-fibrant objects. So orthogonality must be defined via **homotopy function complexes**, e.g. **simplicial mapping spaces** (which Hovey proved to exist in every model category $\mathcal{M}$ in the chapter on framings in his book). This is because we need to test homotopy orthogonality on the model category level and we need function complexes that keep track of homotopies, homotopies between homotopies, etc. Orthogonality with respect to homotopy function complexes is a much stronger notion than orthogonality defined in terms of homotopy classes of maps. An equivalent approach is the Hammock localization of Dwyer and Kan.

The idea of framings is to assign to any objects X, Y a fibrant simplicial set $map(X, Y)$ that is homotopy equivalent to $\mathcal{M}(X_*, Y^*)$ in the category of simplicial sets, where $X_* \to X$ is a **cosimplicial resolution** of X and $Y \to Y^*$ is a **simplicial resolution** of Y. The point is that the homotopy type of $map(X, Y)$ does not change if we replace X or Y by weakly equivalent objects, that the assignment of $map(X, Y)$ to X, Y is functorial, and that $\pi_0 map(X, Y) \cong [X, Y]$ as sets. If $\mathcal{M}$ is a simplicial model category, then you can use $Map(QX, FY)$ for $map(X, Y)$, where $Map(,)$ is the simplicial enrichment.

Given a class of maps $\mathcal{S}$ in a model category $\mathcal{M}$, an object Z is **$\mathcal{S}$-local** if $map(s, Z) : map(B, Z) \to map(A, Z)$ is a weak equivalence of simplicial sets for all $s : A \to B$ in $\mathcal{S}$. A map $f : X \to Y$ is an **$\mathcal{S}$-equivalence** if $map(f, Z)$ is a weak equivalence for all $\mathcal{S}$-local Z (this condition implies Bousfield's via taking π_0). An object W is **$\mathcal{S}$-acyclic** if $map(W, Z) \simeq *$ for all $\mathcal{S}$-local Z. A **left Bousfield localization** of a model category $\mathcal{M}$ with respect to $\mathcal{S}$ has weak equivalences being the $\mathcal{S}$-local equivalences and cofibrations the same as in $\mathcal{M}$. Hirschhorn proves that if your model category $\mathcal{M}$ is left proper and either combinatorial or cellular, and if $\mathcal{S}$ is a set, then left Bousfield localization exists. The fibrant objects in this new model category are the L-local objects.

Definition. If $\mathcal{K}$ is a class of objects, say f is a **$\mathcal{K}$-colocal equivalence** if $map(X, f)$ is a weak equivalence for all $X \in \mathcal{K}$. Say Z is **$\mathcal{K}$-colocal** if $map(Z, F)$ is a weak equivalence for all f as above. Say A is $\mathcal{K}$-coacyclic if $map(W, A) \simeq *$ for all $\mathcal{K}$-colocal W. A **right Bousfield localization** has weak equivalences being $\mathcal{K}$-colocal equivalences and fibrations the maps which are fibrations in $\mathcal{M}$. The cofibrant objects are the $\mathcal{K}$-colocal objects.

Analogously to Bousfield localization and colocalization of triangulated categories we have left Bousfield localization and right Bousfield localization of model categories. The right Bousfield localization is sometimes called **cellularization** because it gives rise to a triangle $CW_T(X) \to X \to P_T(X)$ where $CW_T(X)$ is an approximation to X built from T via homotopy colimits, while P_T is nullification with respect to T.

There is a type of localization which is weaker than Bousfield localization but which has all the categorical properties of a Bousfield localization (the difference being that the definition makes no mention of a set of maps to be inverted).

Definition. A **homotopical localization** on a model category $\mathcal{M}$ with homotopy function complexes map$(-,-)$ is a functor $L : \mathcal{M} \to \mathcal{M}$ that preserves weak equivalences and takes fibrant values, together with a natural transformation $\eta : Id_{\mathcal{M}} \to L$ s.t. for all X:

(1) $L\eta_X : LX \to LLX$ is a weak equivalence
(2) η_{LX} and $L\eta_X$ are equal in Ho($\mathcal{M}$)
(3) $\eta_X : X \to LX$ is a cofibration s.t. map(η_X, LY) :map$(LX, LY) \to$ map(X, LY) is a weak equivalence of simplicial sets for all Y

When passing to $Ho(\mathcal{M})$, homotopical localizations are sent to idempotent functors, since $\pi_0 map(X, Y) \cong [X, Y]$. In addition to orthogonality in $Ho(\mathcal{M})$, L-local objects and L-equivalences are orthogonal with respect to simplicial mapping spaces. The correct notion of orthogonality to use in order to characterize local objects in terms of weak equivalences and vice versa is the latter. A fibrant object is L-local if and only if it is simplicially orthogonal to all L-equivalences; a map is an L-equivalence if and only if it is simplicially orthogonal to all L-local objects.

4. Lifting localizations to the model category level

Farjoun's Conjecture: is every homotopy idempotent functor on $sSet$ equivalent to some homotopical localization on the model category level?

Casacuberta-Scevenels-Smith: Yes, under Vopenka's principle. Reason: every idempotent functor gives rise to an orthogonal pair. Proving it's a localization comes down to finding an adjoint. This is called the Orthogonal Subcategory Problem, and it's exactly why Vopenka's principle gets involved.

Casacuberta and Chorny generalized this work. They show that if we are given a homotopy idempotent functor L in a simplicial model category M and L is **continuous**, then L is equivalent to Bousfield localization with respect to some class of maps (which can be replaced by a set of maps under Vopenka).

A functor F in a simplicial model category is called **continuous** if it is equipped with natural continuous maps of simplicial sets $map(X, Y) \to map(FX, FY)$ preserving composition and identity. Without this assumption, homotopy idempotent functors cannot be lifted, as the next example shows.

Example (due to Casacuberta, private communication). Consider $\mathscr{S}$, the model category of spectra. It is well-known that rational spectra sit inside the subcategory of GEMs (wedges of Eilenberg-Mac Lane spectra). Projecting onto a sub-wedge in a certain range is a localization on $Ho(\mathscr{S})$ (e.g. it lands in truncated rational spectra). Certainly the functor is idempotent and coaugmented (because you can take rationalization and truncate any spectrum). It cannot be a Bousfield localization L_f because if L comes from an L_f then

$im(L)$ is closed under homotopy colimits (since $im(L_f)$ sits above it). But this example is not closed under homotopy colimits, e.g. a telescope can take you out of the range you're supposed to be truncated in.

5. Localization in a monoidal setting

So far we've tested orthogonality with respect to $C(-,-)$, $[-,-]$, and $map(-,-)$. In a closed monoidal model category we also have an internal hom object. Can we test orthogonality with respect to this? What do we get?

Definition. We say that a localization (L,η) on a closed symmetric monoidal category E is a **closed localization** if, for every L-equivalence $f : X \to Y$ and every L-local object Z, the map $Hom_E(f,Z) : Hom_E(Y,Z) \to Hom_E(X,Z)$ is an isomorphism in E.

For the category of spectra, a localization is closed iff $L\Sigma \simeq \Sigma L$. This follows if the class of L-local equivalences is closed under the monoidal product. A slightly more general definition can be made in the setting of enriched model categories, i.e. when $\mathcal{D}$ is a model category and $\mathcal{M}$ is a $\mathcal{D}$-model category (see chapter 4 of Hovey's book)

Definition. An **enriched homotopical localization** is a weak-equivalence preserving functor $L : \mathcal{M} \to \mathcal{M}$ that lands in the fibrant objects, with a natural transformation $\eta : Id_{\mathcal{M}} \to L$ such that, for all $X \in \mathcal{M}$:
(i) $L\eta_X : LX \to LLX$ is a weak equivalence in $\mathcal{M}$.
(ii) η_{LX} and $L\eta_X$ are equal in the homotopy category $Ho(\mathcal{M})$.
(iii) $\eta_X : X \to LX$ is a cofibration and the induced map $hom(LX, LY) \to hom(X, LY)$ is a weak equivalence in $\mathcal{D}$ for all $Y \in \mathcal{M}$.

Clark Barwick has proven that if $\mathcal{M}$ is cotensored over $\mathcal{D}$, left proper, and both $\mathcal{M}$ and $\mathcal{D}$ are combinatorial then the enriched left Bousfield localization with respect to a set of morphisms exists.

Example (Non-closed localizations can behave poorly): The n^th Postnikov section functor P_n is a homotopical localization for all n but is not a closed localization. Furthermore, if R is nonconnective, then P_1R does not admit a ring spectrum structure (not even the structure of a ring in the homotopy category). The reason is that if it were a ring then multiplication by the unit S would need to be a homotopy equivalence. But the unit map $\nu : S \to P_1R$ is null since $\pi_0(P_1R) = 0$. The real issue here is that suspension and localization do not commute, and nonconnective ring spectra can feel the difference.

Mike Hill's example. It's a localization in the equivariant world which is not stable with respect to certain representation spheres. It fails to preserve E_∞ structure. For equivariant spectra this tells us that closed is not the same as $\Sigma L \simeq L\Sigma$, but rather has to do with Σ^V for all the indexing spaces V. I suspect it's a difference between the monogenic and non-monogenic stories, similar to what we observe in the next example.

Example (joint with Carles Casacuberta): Sometimes the difference between closed and non-closed localizations can also be exploited. If M is the model category of motivic symmetric spectra and T is the projective line $S^1_s \wedge S^1_t$ then we've proved that the localization $L_{S_{**}}$ exists. So for a given X we can form a triangle $A_{S_{**}}(X) \to X \to L_{S_{**}}(X)$ and we've proven that $P_T(X) \cong A_{S_{**}}(X)$. Current work in progress attempts to understand whether or not the difference between these two types of localization is really the difference between enriched localization and Hirschhorn-style localization with respect to simplicial mapping objects.

6. Chorny's Example

Not every localization L_T with respect to a set of maps T can be replaced by one with respect to a single map f. Quoting directly:

"Consider the model category which is a product of two copies of the category of simplicial sets, i.e., the category of diagrams of simplicial sets over the discrete category with two objects, equipped with the projective model structure (where fibrations and weak equivalences are objectwise). Take $S = \{f, g\}$ for $f : (\emptyset, \emptyset) \to (*, \emptyset)$ and $g : (\emptyset, *) \to (\emptyset, * \coprod *)$

An object (X, Y) is S-local if and only if X and Y are fibrant, X is contractible and Y is either contractible or empty.

Suppose that there exists a map $h : (A, B) \to (C, D)$ such that any S-local object is also h-local, and vice versa. The object $(X, \emptyset)$ is h-local if and only if X is contractible. This condition implies that both B and D are empty; otherwise, for any simplicial set Z, either contractible or not, the object $(Z, \emptyset)$ would be h-local. But in this case any object (X, Y) with contractible X becomes h-local, hence the contradiction. Note however that, in order to ensure that every set of maps yields the same localization as their coproduct, it is enough to assume that the set of maps $X \to Y$ is nonempty for all X and Y in the model category under consideration."

7. Why does a Bousfield localization of model categories give a triangulated Bousfield localization?

Answer taken directly from Fernando Muro on MathOverflow:

"Let $\widetilde{C}$ be the left Bousfield localization of C. As categories $C = \widetilde{C}$, but $\widetilde{C}$ has more weak equivalences. In particular, the identity functor induces a functor $\varphi\colon \operatorname{Ho}(C) \to \operatorname{Ho}(\widetilde{C})$. Let $\mathcal{L} = \ker \varphi$, i.e. $\mathcal{L} \subset \operatorname{Ho}(C)$ is the full subcategory spanned by the objects which become trivial in $\operatorname{Ho}(\widetilde{C})$. If $\widetilde{C}$ is stable and φ preserves homotopy colimits then φ is an exact functor between triangulated categories, so $\mathcal{L}$ is a thick subcategory of $\operatorname{Ho}(C)$ and φ induces a functor from the Verdier quotient $\overline{\varphi}\colon \operatorname{Ho}(C)/\mathcal{L} \to \operatorname{Ho}(\widetilde{C})$. Let us show that $\overline{\varphi}$ is an equivalence of categories. It is enough to prove that the canonical composition of 'projection' functors $\widetilde{C} = C \to \operatorname{Ho}(C) \to \operatorname{Ho}(C)/\mathcal{L}$ satisfies the universal property of $\widetilde{C} \to \operatorname{Ho}(\widetilde{C})$. Suppose $\psi\colon C \to D$ is a functor which sends all weak equivalences in $\widetilde{C}$ to isomorphisms. Since weak equivalences in C are weak equivalences in $\widetilde{C}$, ψ factors through $C \to \operatorname{Ho}(C)$ in an essentially unique way. Let $\overline{\psi}\colon \operatorname{Ho}(C) \to D$ be the factorizaton. Recall that $\operatorname{Ho}(C)/\mathcal{L}$ is the localization of $\operatorname{Ho}(C)$ inverting those maps whose mapping cone is in $\mathcal{L}$. Any such map is represented by a zig-zag of weak equivalences in $\widetilde{C}$. Since ψ sends weak equivalences in $\widetilde{C}$ to isomorphisms then $\overline{\psi}$ factors through $\operatorname{Ho}(C) \to \operatorname{Ho}(C)/\mathcal{L}$ in an essentially unique way. In particular ψ factors through the composite $\widetilde{C} \to \operatorname{Ho}(C)/\mathcal{L}$. The essential uniqueness of this factorization follows from the aforementioned essential uniqueness of the two intermediate steps."