BOUSFIELD LOCALIZATION AND ALGEBRAS OVER OPERADS

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Thanks for the invitation and thanks to Peter May for all the helpful conversations over the years. More details can be found on my website: dwhite03.web.wesleyan.edu

1. Abstract

We give conditions on a monoidal model category M and on a set of maps S so that the Bousfield localization of M with respect to S preserves the structure of algebras over various operads. This problem was motivated by an example due to Mike Hill which demonstrates that for the model category of equivariant spectra, even very nice localizations can fail to preserve commutativity. As a special case of our general machinery we characterize which localizations preserve genuine equivariant commutativity. Our results are general enough to hold for non-cofibrant operads as well, and we will demonstrate this via a treatment of when localization preserve strict commutative monoids. En route we will introduce the commutative monoid axiom, which guarantees us that commutative monoids inherit a model structure. If there is time we will say a word about the generalizations of this axiom to other non-cofibrant operads, and about how these generalized axioms interact with Bousfield localization.

2. Pre-Talk: Motivation from Equivariant Homotopy Theory

A big goal of algebraic topology is to understand how homotopy theory interacts with algebraic structure. For today we use model categories as our language for homotopy theory and operads as our language for algebra. We'll work on the subgoal of understanding when Bousfield localization preserves the structure of algebras over an operad.

Many proofs in recent years have demonstrated the value of working on the point-set level rather than in the homotopy category, so that's why we use model categories. Because we need to be doing algebra, we need a monoidal structure. It was already seen in EKMM that there would need to be a compatibility between this monoidal structure and the homotopy theory. This compatibility is encoded in the definition of a monoidal model categories (explored in Hovey's book). We'll be investigating how Bousfield localization (explored by Hirschhorn's book) interacts with algebras over various operads in that context.

This project was motivated by a step in the proof of the Kervaire Invariant One Theorem. The authors needed a 256-periodic $\Omega = D^{-1}MU^{(4)}$ for some D. They were working with MU considered as a commutative equivariant spectrum and needed Ω to be commutative, i.e. have multiplicative norms, i.e. to have $\pi_\star$ forming a Tambara functor. This was needed for reasons related to the spectral sequence computations which occupy the technical details of the proof.

Let G be a finite group and let $\mathscr{S}^G$ be the model category of G-spectra. Recall that for every family of subgroups $\mathscr{F}$ of G there is a model structure on Top^G where weak equivalences and fibrations are maps such that $(-)^H$ is again such a map in Top for all $H \in \mathscr{F}$. For each family there is a universal $\mathscr{F}$-space EF which is a G-CW complex such that $(EF)^H$ is contractible for $H \in \mathscr{F}$ and empty otherwise. The family model structure on Top has generating cofibrations $(G/H \times S^{n-1})_+ \to (G/H \times D^n)_+$ for all n and all $H \in \mathscr{F}$, and the analogous generating trivial cofibrations. These family model structures are also present in $\mathscr{S}^G$, and

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can be defined by similarly changing the generating (trivial) cofibrations. We will make use of these family model structures at the end of the talk.

Sadly, not every localization of an equivariant commutative ring spectrum is commutative.

Example (Hill, Oberwolfach). Let G be a (non-trivial) finite group.

Consider the reduced real regular representation $\overline{\rho} = \mathbb{R}[G]/\mathbb{R}[e]$, i.e. $\overline{\rho}_G = \rho_G - 1$ where 1 means the trivial representation $\mathbb{R}[e]$. Now, ρ_G takes any subgroup H to $[G : H]$ many copies of H so $\rho_G|_H = [G : H]\rho_H$. Putting these facts together, we see that $\overline{\rho}_G|_H = [G : H]\overline{\rho}_H + ([G : H]1 - 1)$.

Now consider the representation sphere $S^{\overline{\rho}}$ and the inclusion $a_{\overline{\rho}} : S^0 \to S^{\overline{\rho}}$. Thinking of S^0 as $\{0, \infty\}$ we see that the only fixed points of this map are 0 and ∞, so it's not equivariantly trivial. Consider the spectrum $E = S^0[a_{\overline{\rho}}^{-1}]$. We will show that this spectrum does not admit maps from the norms of its restrictions, and hence cannot be commutative. The reason is that for all proper $H < G$, $res_H(E)$ is contractible.

The reason $res_H(E)$ is contractible as an H-spectrum is our computation above regarding $\overline{\rho}_G|_H$. Because $[G : H] - 1$ is a number k greater than 0 we have $res_H S^{\overline{\rho}_G} = (S^{\overline{\rho}_H})^{\#[G:H]} \wedge S^k$. This means that as an H-spectrum it is contractible, because there is enough space in the S^k part to deform it to a point.

Now, a key property of commutative equivariant ring spectra is the existence of multiplicative norms. These functors $N_H^G : \mathscr{S}^H \to \mathscr{S}^G$ are left adjoint to the restriction res_H on the category of commutative ring spectra. Thus, if E were commutative we would have a ring homomorphism $* \simeq N_H^G res_H E \to E$. This is not a ring map unless E to be contractible, and we know E is not contractible because $a_{\overline{\rho}}$ fixes 0 and ∞.

Here is an equivalent approach, which Hill presented at Oberwolfach. Let $\mathscr{F}$ be the family of proper subgroups of G and let $\widetilde{E\mathscr{F}}$ be the cofiber of the natural map from the classifying space $E\mathscr{F}_+$ to S^0. This $\widetilde{E\mathscr{F}}$ is a localization of S^0 obtained by killing all maps from induced cells. If G is finite then it is our E. It's not contractible because $E\mathscr{F}_+$ is not homotopy equivalent to S^0, because $\mathscr{F}$ doesn't contain G. So while any restriction to a proper subgroup views them to be homotopy equivalent, they are not homotopy equivalent in $\mathscr{S}^G$.

In this second approach it becomes clear that this example generalizes to other families of subgroups, proving that in any family model structures (other than $\mathscr{F} = \{e\}$, which recovers naive spectra) one can similarly disprove the preservation of commutativity by localization.

□

The take-away message from this example is that we need a hypothesis on the maps being localized so that equivariant commutativity is preserved. Viewed a certain way, what is failing above is the ability of the localization functor to commute with equivariant suspension with respect to certain representation spheres (namely, those which don't see all the information in G, but only see subgroup information). When localizations kill representation spheres bad things happen. A similar example, due to Carles Casacuberta, proves that not all localizations of spectra preserve ring structure. This is the example of the Postnikov Section:

The n^{th} Postnikov section functor P_n is a homotopical localization for all n but does not commute with suspension. Furthermore, if R is nonconnective, then P_1R does not admit a ring spectrum structure (not even the structure of a ring in the homotopy category). The reason is that if it were a ring then multiplication by the unit S would need to be a homotopy equivalence. But the unit map $\nu : S \to P_1R$ is null since $\pi_0(P_1R) = 0$. The real issue here is that suspension and localization do not commute, and nonconnective ring spectra can feel the difference. We've chopped off the dimension where the unit is supposed to live.

Casacuberta gets around this by placing hypotheses on the localization (he calls the well-behaved localizations "closed") and similarly Hill and Hopkins get around Hill's example by placing hypotheses on the maps:

Theorem 1 (Hill-Hopkins)**.** *If for all acyclics Z for a localization L and for all subgroups H, $N_H^G Z$ is acyclic, then for all commutative G-ring spectra R, $L(R)$ is a commutative G-ring spectrum.*

Here commutativity can mean either strict commutativity (algebras over the operad Com) or E_∞-structure where E_∞ is the linear isometries operad (a model can be taken with $E_\infty[n] = E_G\Sigma_n$), because in $\mathscr{S}^G$ there is rectification between these operads, as recently proven by Blumberg and Hill in the appendix of their 2013 paper. The hypothesis in the theorem is precisely what is needed to make the EKMM proof (via the skeletal filtration) go through.

This example and theorem open a more general question: find conditions on a general model category $\mathcal{M}$ and on a set of maps C so that the Bousfield localization L_C preserves commutativity. We will answer this question, and when we specialize our machinery to $\mathcal{M} = \mathscr{S}^G$ we'll in fact characterize localizations which preserve commutative structure. This will show us yet another reason why Hill's example is failing. In particular, it will fail because for the set of maps being inverted we have that $C \otimes (G/H)_+$ is not contained in the C-local equivalences (because it contains the zero map, because proper H sees C to be trivial even though it is not). This makes it easy to see the correct condition on C so that L_C viewed in the family model structures preserves commutativity. The condition will be that $C \otimes (G/H)_+ \subset C$-local equivalences for all $H \in \mathscr{F}$.

3. Background: monoidal model categories, operads, Bousfield localization

Recall that we care about model categories $\mathcal{M}$ because the passage to Ho($\mathcal{M}$) works (this functor inverts the weak equivalences $\mathscr{W}$) and we have some control over the resulting maps because of cofibrant and fibrant replacement. Let $\mathscr{Q}$ and $\mathscr{F}$ be the cofibrations and fibrations. If we're going to talk about commutative monoids then we need to have a monoidal structure on $\mathcal{M}$. It turns out that we also need a compatibility hypothesis between $\mathcal{M}$ and the monoidal structure, as explained in chapter 4 of Hovey's book. Let $\otimes$ denote the monoidal product.

Given $f : A \to B$ and $g : X \to Y$, define the *pushout product* $f \square g$ to be the corner map in

$$\begin{tikzcd}
A \otimes X \arrow[r] \arrow[d] & A \otimes Y \arrow[d] \arrow[ddr, bend left] & \\
B \otimes X \arrow[r] \arrow[drr, bend right] \arrow[ur, phantom, "\searrow"] & Q_2 \arrow[dr] & \\
& & B \otimes Y
\end{tikzcd}$$

A *monoidal model category* is a model category which is also a monoidal category and satisfies:

Pushout product axiom: if $f, g \in \mathscr{Q}$ then $f \square g \in \mathscr{Q}$. Additionally, if either is in $\mathscr{W}$ then $f \square g \in \mathscr{W}$.

Unit Axiom: If Z is cofibrant then $QS \otimes Z \to S \otimes Z \cong Z$ is a weak equivalence.

These axioms assure you that $Ho(\mathcal{M})$ is a monoidal category. We'll be studying objects in $\mathcal{M}$ which carry the additional algebraic structure encoded by a symmetric operad, e.g. monoids, commutative monoids, A_∞ or E_∞ algebras, Lie algebras, etc. All operads today are symmetric.

Recall that an operad in $\mathcal{M}$ is a symmetric sequence $P = (P(n))_{n\in\mathbb{N}}$ of objects in $\mathcal{M}$ (i.e. each $P(n)$ is in $\mathcal{M}^{\Sigma_n}$ i.e. has an action of the symmetric group Σ_n) satisfying some axioms. The object $P(n)$ can be thought of as parameterizing maps of arity n. There is a notion for cofibrancy of an operad which comes down to requiring the left lifting property of $\emptyset \to P$ with respect to maps which are levelwise trivial fibrations in $\prod_{n\in\mathbb{N}} \mathcal{M}^{\Sigma_n}$ where $\mathcal{M}^{\Sigma_n}$ is the model category of objects in $\mathcal{M}$ with a Σ_n action.

Formally...

A *symmetric operad* is a sequence $(P(n))_{n\in\mathbb{N}}$, with a right action $*$ of the symmetric group Σ_n on $P(n)$, an identity element $1 \in P(1)$ and compositions maps $\circ$ satisfying the above associative and identity axioms, as well as equivariance, i.e. given $s_i \in \Sigma_{k_i}$ and $t \in \Sigma_n$:

$$(\theta * t) \circ (\theta_{t1}, \dots, \theta_{tn}) = (\theta \circ (\theta_1, \dots, \theta_n)) * t;$$
$$\theta \circ (\theta_1 * s_1, \dots, \theta_n * s_n) = (\theta \circ (\theta_1, \dots, \theta_n)) * (s_1, ..., s_n)$$

A morphism of symmetric operads $f : \mathcal{P} \to \mathcal{Q}$ consists of a sequence $(f_n : P(n) \to Q(n))_{n\in\mathbb{N}}$ such that $f(1) = 1$, $f(\theta \circ (\theta_1, \dots, \theta_n)) = f(\theta) \circ (f(\theta_1), \dots, f(\theta_n))$, and $f(x * s) = f(x) * s$.

From now on we use the word operad to refer to symmetric operads. Both types of operad come with maps $\circ_i : P(m) \times P(n) \to P(m-1+n)$ which take (f, g) to a function where g is plugged into the i-th spot of f, i.e. evaluate f on the first $i-1$ variables, $f_i \circ g$ on the next n variables, and f on the rest of the variables.

Examples:

(1) Ass is the operad encoding associativity. $Ass[n] = \Sigma_n$
(2) Com is the operad encoding strict commutativity. $Com[n] = *$
(3) $\mathscr{L}$ is the linear isometries operad. If we fix a universe U then the n^{th} space of $\mathscr{L}$ is $\mathscr{L}(U^n, U)$, the space of linear isometries from U^n to U.
(4) An E_∞ operad has $P(n)$ contractible and Σ_n acts freely. So the linear isometries operad and little cubes operad are both E_∞.

I mostly care about operads for the categories of algebras they encode. An *algebra over an operad* is an object $A \in C$ equipped with coherent maps $P(n) \times A^n \to A$. More compactly, an algebra over an operad is a map of operads from $\mathcal{P} \to End_A = (C(A^n, A))_{n\in\mathbb{N}}$. Even more compactly, these objects come with morphisms $P \circ A \to A$, but I don't want to get into the circle product. These objects form a category, with morphisms P-algebra homomorphisms (maps which respect this structure).

If I want to do homotopy theory with P-algebras then I'll want them to inherit a model structure. Thankfully, algebras over an operad are encoded by a certain monad:

$P : \mathcal{M} \leftrightarrow P - alg(\mathcal{M}) : U$ where $P(X) = \coprod_n (P(n) \otimes X^{\otimes n})$

Here U is the forgetful functor and P is the free algebra functor. If we wish to place a model structure on P-alg we will want it to be compatible with the model structure on $\mathcal{M}$. In particular, we want the forgetful functor to be right Quillen. So we need the model structure on P-alg to have weak equivalences and fibrations maps which are such as maps in $\mathcal{M}$. Cofibrations are therefore determined by the lifting property.

It's not always true that the model structure on $\mathcal{M}$ can be passed across this adjunction. Sometimes it can. At the bare minimum we need $\mathcal{M}$ to be cofibrantly generated, and if the generating maps are I and J then the generators for P-alg are $P(I)$ and $P(J)$. Let's work through an example to see what kind of hypotheses are needed on $\mathcal{M}$ and P for this to work. Consider the following general lemma from Schwede-Shipley:

Lemma 2 (Lemma 2.3)**.** *Suppose $\mathcal{M}$ is cofibrantly generated and T is a monad which commutes with filtered direct limits. If the domains of $T(I)$ and $T(J)$ are small relative to $T(I)$-cell and $T(J)$-cell respectively and EITHER*

(1) $T(J)$−cell$\subset \mathscr{W}$, or

(2) All objects are fibrant and every T-algebra has a path object (factoring $\delta : X \to X \otimes X$ into $\overset{\simeq}{\hookrightarrow}\twoheadrightarrow$)

then T-alg inherits a cofibrantly generated model structure with fibrations and weak equivalences created by the forgetful functor to $\mathcal{M}$.

One half of lifting comes for free, 2 out of 3 and retracts are inherited from $\mathcal{M}$, so only factorization must be proven. If P preserves smallness then the small object argument is used to get the generators above and to get cofibration-trivial fibration factorization. For the other factorization axiom we need to know that when every homomorphism p which is a transfinite composition of pushouts of coproducts of maps of the form $P(f)$ where f is a trivial cofibration in $\mathcal{M}$ has p being a weak equivalence in $\mathcal{M}$ (hence in $P - alg(\mathcal{M})$). Once you have this you get the other half of lifting by the retract argument.

Schwede-Shipley prove that this extra condition can be deduced if every object of $\mathcal{M}$ is fibrant and if every P-algebra has a path object (using the retract argument). They also generalize Crans's result to algebras over a monad. A great deal of this theory has been worked out in the case where all objects are fibrant by Berger and Moerdijk. Of course, this fails in sSet and all the categories of spectra so I'm more interested in the other approach. Let's work out an example:

The simplest P is Ass. In that case the free algebra functor is $T(X) = S \wedge X \wedge X^2 \wedge \dots$. If we have a trivial cofibration $f : K \to L$ then applying this functor gives $T(K) \to T(L)$ and we need to look at pushouts of this map in the category of monoids: $X \leftarrow T(K) \to T(L)$. Call the pushout P

The trick is to factor $X \to P$ as $X = P_0 \to P_1 \to \dots$. Because of the structure of T we can define each map $P_{n-1} \to P_n$ inductively. Let Q_n denote the colimit of the punctured n-dimensional cube with vertices $X \wedge K \wedge X \wedge K \wedge \dots X$ and with varying numbers and placements of L's. Then we have

$$\begin{tikzcd} Q_n \arrow[r] \arrow[d] & (X \wedge L)^n \wedge X \arrow[d] \\ P_{n-1} \arrow[r] & P_n \end{tikzcd}$$

We can then shuffle the X's to the side and we see that exactly the condition needed on $\mathcal{M}$ for this argument to work is the following: $(\mathcal{M} \wedge TrCof) - cell \subset \mathscr{W}$. The elements in this collection of maps are $Z \wedge f$ where Z is an object of $\mathcal{M}$ and f is a trivial cofibration. Applying cell means taking transfinite compositions of pushouts. Indeed, only countable transfinite compositions are necessary.

A recurring theme in this talk will be that there is a cofibrancy price to pay in order to pass this model structure across this adjunction. For example, consider the following theorem of Spitzweck:

Theorem 3. *Suppose P is a Σ-cofibrant operad and $\mathcal{M}$ is a monoidal model category. Then P-alg is a semi-model category which is a model category if P is cofibrant and $\mathcal{M}$ satisfies the monoid axiom.*

In a semi-model category lifting of a trivial cofibration against a fibration only holds if the domain is cofibrant. Everywhere we've applied fibrant replacement it's been to an object which is cofibrant in the underlying category, so that's no problem. The lifting argument is for a map which has cofibrant domain, so that's fine too. Thus, even if the monoid axiom is not preserved we can still say P-alg is a semi-model category. So all objects admit cofibrant replacement, but only cofibrant objects admit fibrant replacement. These semi-model structures arise when you can't check all the hypotheses of a model category because Crans's condition fails.

The model structure on the category of operads is obtained via the transfer principle applied to the adjunction $F : Coll(\mathcal{M}) \leftrightarrow Op(\mathcal{M}) : U$ where $Coll(\mathcal{M}) = \Pi\, \mathcal{M}^{\Sigma_n}$ is the category of collections. This transfer doesn't always work, but even if operads don't form a model category you can still talk about Σ-cofibrant operads as operads which are cofibrant as collections. Even more generally you can talk about operads whose underlying collection is cofibrant. Even more generally there are levelwise cofibrant operads.

Examples:

Ass is Σ-cofibrant, A_∞ is cofibrant

Com is levelwise cofibrant but not Σ-cofibrant. Any E_∞ operad is a Σ-cofibrant replacement. Morally this is good enough to be a "cofibrant replacement" for Com. If you want an honestly cofibrant operad you need to use the Fulton MacPherson operad. The algebras over all E_∞ operads are Quillen equivalent because any two homotopy equivalent Σ-cofibrant operads have Quillen equivalent categories of algebras.

Thanks to Spitzweck's result, we don't need to be overly careful about the difference between cofibrant and Σ-cofibrant in order to conclude preservation of algebra structure by localization, so we'll choose the E_∞ operad with $E_\infty[n] = E\Sigma_n$.

When two operads O and P have the property that their categories of algebras are Quillen equivalent then rectification is said to occur (e.g. P rectifies to O). When we move away from associative operads to commutative this notion becomes important, because homotopy coherent commutativity is encoded by the cofibrant replacement for the Com operad in the model category of operads (also obtained by the transfer principle).

Example: In any modern category of spectra where both Com-algebras and E_∞-algebras form model categories, these two admit rectification. In S-modules this is built into the product. In fact, I don't know how to discuss strictly commutative S-algebras at all. For orthogonal and symmetric spectra, strictly commutative monoids do not form a model category, due to the well-known obstruction of Gaunce Lewis.

Moore proved that: Connected commutative topological monoids are product of Eilenberg-Mac Lane spaces.

Lewis: if commutative monoids formed a model category then taking the zeroth space of the cofibrant replacement of the sphere in that category would give such a space, but this implies there are no homotopy operations present in the stable homotopy groups of spheres, contradiction.

Moving to the positive model structure fixes this (by breaking the cofibrancy of the unit) and rectification occurs because symmetric powers are weakly equivalent to homotopy symmetric powers, i.e. the smash product can't see the difference between the free algebra functors over these two operads. Different choices for E_∞ make no difference to the resulting algebras.

For equivariant spectra we will see that care has to be taken with the notion of E_∞. If one uses the notion above then the action of G is ignored and the resulting operad is not cofibrant. So rectification with naive E_∞ fails but rectification with genuine E_∞ works, as proven by Blumberg-Hill 2013. We'll get there.

Equivariantly, this operad encodes naive E_∞ structure. Genuine E_∞ structure is encoded by any operad P where $P(n)$ is an $E_G\Sigma_n$, i.e. a space with a $G\times\Sigma_n$-action which is characterized up to $G\times\Sigma_n$-weak equivalence by the property that for $H < G\times\Sigma_n$, we have $(E_G\Sigma_n)^H = \varnothing$ if $H\cap\Sigma_n \neq \{e\}$ and $(E_G\Sigma_n)^H \simeq *$ otherwise. This space $E_G\Sigma_n$ can be defined as the total space of the universal G-equivariant principle Σ_n-bundle. The following result is joint with Javier Gutierrez, but may have been known previously:

Theorem 4. *The category of simplicial (resp. topological) G-operads can be given a model structure via transfer from the category of collections on G-spaces. Neither Com nor the naive E_∞ operads are cofibrant. Their Σ-cofibrant replacement E_∞^G can be described by $E_\infty^G[n] = E_G\Sigma_n$.*

In model categories other than spectra rectification does not hold, so it becomes important to consider strictly commutative monoids in their own right. For example, rectification fails for topological spaces or simplicial sets, for the same reason as in Lewis's example. Does rectification hold for Ch(k)?

Let's talk about when commutative monoids inherit a model structure. For monoids this is done by Schwede-Shipley and the hypothesis needed on $\mathcal{M}$ is the *monoid axiom*, which says that for all objects X, $(id_X\otimes(\mathcal{Q}\cap\mathcal{W}))-cell \subset \mathcal{W}$. Here applying cell to a class of maps means taking its closure under transfinite compositions and pushouts. For commutative monoids the correct hypothesis is the *commutative monoid axiom*: If g is a (trivial) cofibration then $g^{\square n}/\Sigma_n$ is a (trivial) cofibration.

Theorem 5. *If a monoidal model category satisfies the monoid axiom and the commutative monoid axiom then commutative monoids form a model category and the forgetful functor is right Quillen.*

Proof. This goes basically the same way as the SS00 result. Now we use the functor $Sym(X) = S \wedge X \wedge X^2/\Sigma_2 \wedge \dots$. Again we take a pushout of $Sym(K) \to Sym(L)$ in the category of commutative monoids and again we factor $X \to P$ into a transfinite composition. Letting $\mathrm{Sym}^n(L;K)$ denote the colimit of the punctured cube defined by n-length products of L and K, we see that the pushout in question is $X = P_0 \to P_1 \to \cdots \to P$ where $P_{n-1} \to P_n$ is defined by

$$\begin{tikzcd}
X \otimes \mathrm{Sym}^n(L;K) \arrow[r] \arrow[d] & X \otimes \mathrm{Sym}^n(L) \arrow[d] \\
P_{n-1} \arrow[r] \arrow[ur, phantom, "\Searrow"] & P_n
\end{tikzcd}$$

The commutative monoid axiom ensures us that the part of this map after the $X \otimes -$ is a trivial cofibration. The monoid axiom ensures us that taking transfinite compositions and pushouts do not ruin this. □

This result generalizes a theorem of Lurie's from DAGIII, i.e. my hypothesis is weaker.

Examples:

(1) Ch(k) where $char(k) = 0$. Lurie had this too. More generally, can get any $\mathbb{Q}$-algebra

(2) sSet - this fails Lurie's hypothesis. My proof uses the fact that cofibrations are monomorphisms to get the bit about cofibrations. For the weak equivalences part we rely on a clever trick of Dror Farjoun.

(3) Positive (Flat) model structure on symmetric spectra. Lurie doesn't apply here. He acknowledges his error in DAGIII 4.3.25 in Math Overflow post 146438. My proof needed a technical lemma that it was sufficient to check the commutative monoid axiom on the generators. Luis Pereira proved the same for Lurie's hypothesis

(4) Top - this fails for Lurie. It works for me because the proof of Farjoun generalizes to any Cartesian concrete category, and with a bit more care we don't need cofibrations to be monomorphisms either, because we have our hands on the generators.

(5) Positive orthogonal (equivariant) spectra - using again that it's sufficient to check it on the generators

(6) Positive motivic symmetric spectra - I'm developing this category with Markus Spitzweck.

If we drop the monoid axiom we only get a semi-model structure on *Com*-alg, but that is enough for preservation by localization. I have a theorem about when localization preserves the monoid axiom but it's unnecessary here for this reason. Anyway, in a combinatorial model category this this result adds no hypotheses at all to the maps being inverted. It simply needs that $\mathcal{M}$ is h-monoidal and satisfies a compactness hypothesis on the generating cofibrations I. This hypothesis holds in all the examples.

This commutative monoid axiom generalizes to give a family of axioms. We saw already that if P is cofibrant then basically no hypotheses are needed on $\mathcal{M}$ to get admissibility. Harper has a result that if all symmetric sequences in $\mathcal{M}$ are projectively cofibrant then all operads are admissible. This is a strong hypothesis. I don't know of any examples other than Ch(k) which satisfy it. My result shows that you can pay the cofibrancy price partially on $\mathcal{M}$ and partially on P, e.g. to get levelwise cofibrant P you need for all $X \in \mathcal{M}^{\Sigma_n}$ which are cofibrant in $\mathcal{M}$ one has $X \otimes_{\Sigma_n} f^{\square n}$ is a trivial cofibration. There is also a generalizes version of the regular monoid axiom, which requires that applying cell to a certain class of maps results in a weak equivalence. For details see my research statement.

Theorem 6. *Let $\mathcal{M}$ be a cofibrantly generated monoidal model category. Let f run through the class of (trivial) cofibrations. In each row of the following table, placing the hypotheses in the first column on M gives a good homotopy theory of P-algebras for all P satisfying the hypotheses in the second column.*

Here "good homotopy theory" means at least a relative category, but most often a model category or at least a semi-model category. The differences depend on technicalities in the categorical algebra which are still being worked out. The hypotheses going down the first column are cumulative, e.g. the last row says that if $\mathcal{M}$ is cofibrantly generated, monoidal, satisfies the monoid axiom, and has the property that $\forall X \in \mathcal{M}^{\Sigma_n}$, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration, then all operads are admissible.

Hypothesis on $\mathcal{M}$	Class of operad
$\forall X \in \mathcal{M}^{\Sigma_n}$ projectively cofibrant, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration (this follows from the pushout product axiom)	Cofibrant
Monoid axiom	Σ-cofibrant
$\forall X \in \mathcal{M}^{\Sigma_n}$ cofibrant in $\mathcal{M}$, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration Note: $X = *$ is the commutative monoid axiom	Levelwise cofibrant Special case: $P = \mathrm{Com}$
$\forall X \in \mathcal{M}^{\Sigma_n}$, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration	Arbitrary

The proof works the same as what we've seen, but now we break the pushout down into steps via $O_A[n] \otimes_{\Sigma_n} Q_n \to O_A[n] \otimes_{\Sigma_n} L^n$. These extra axioms ensure that this pushout works. They are satisfied by simplicial sets and Ch(k) at least and likely other places such as the positive flat model structures on spectra.

For the positive flat model structure on symmetric spectra, all operads are admissible (i.e. their algebras form model categories). In joint work with Markus Spitzweck, I study positive model structures in general model categories, and we hope to prove a similar result. We are also interested in rectification in these general positive model structures.

During the talk I'll give conditions so that the pushout product axiom and unit axiom are preserved by Bousfield localization. At the end I'll give a condition so that the commutative monoid axiom is preserved. In the future I hope to study when these intermediate hypotheses are preserved by Bousfield localization. I am also interested in the generalization to colored operads.

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4. The Talk

Recalling our motivation from the beginning of last talk, let's return now to the question of when localization preserves structure over operads. We'll see why we spent so long developing the homotopy theory of algebras over operads. Let's first consider the model category theoretic version of localization, which generalizes the localization in Hill's example and the Hill-Hopkins theorem. This all goes back to work of Bousfield on inverting maps f (of spaces or spectra) seen to be weak equivalences by a homology theory E.

What if I want to invert some maps $C \not\subset \mathscr{W}$? Because the homotopy category is nice (admits a calculus of fractions), we can do:

$$\begin{tikzcd} \mathcal{M} \arrow[r, dotted] \arrow[d] & ????? \arrow[d, dotted] \\ \mathrm{Ho}(\mathcal{M}) \arrow[r] & \mathrm{Ho}(\mathcal{M})[C^{-1}] \end{tikzcd}$$

We'd like a model category $L_C\mathcal{M}$ which actually sits above $\mathrm{Ho}(\mathcal{M})[C^{-1}]$. Because all three categories above have the same objects, its objects are determined. It's morphisms will be the same as those in $\mathcal{M}$, but we want maps in C to become isomorphisms in $\mathrm{Ho}(\mathcal{M})[C^{-1}]$ so we need them to be weak equivalences in $L_C\mathcal{M}$. So this category must have a different model structure, where $\mathcal{W}' = \langle C \cup \mathcal{W} \rangle$ and clearly $\mathcal{W} \subset \mathcal{W}'$. You can't change only $\mathcal{W}$ because it'll screw up the axioms. We want to keep the cofibrations fixed so we can build things out of them and have the two model structures related, so we have to shrink the fibrations: $\mathcal{F} \supset \mathcal{F}'$. **Bousfield's Theorem** (1978) says you can do this and you still get a model structure, but you have to be careful with how you generate $\mathcal{W}'$ from C. Details are in Hirschhorn's book.

Formally, define $X \in \mathcal{M}$ to be C-local if X is fibrant and $f^* : Map(B, X) \to Map(A, X)$ is a weak equivalence, for all $f : A \to B$ in C. These objects X look trivial to the eyes of C. Define $g : D \to E$ to be a C-local equivalence if for all C-local X, $Map(E, X) \to Map(D, X)$ is a weak equivalence. This follows the idea in algebra, where a module M is S-local if μ_s is an isomorphism for all $s \in S$. A map is an S-equivalence if applying $Hom(-, M)$ gives an isomorphism for all S-local M. It turns out $R \to R[S^{-1}]$ is an S-equivalence to an S-local object. We'd call that fibrant replacement in $L_C(\mathcal{M})$. Proving this object exists is the major technical difficulty faced by Bousfield, and is the reason hypotheses on $\mathcal{M}$ are needed.

This story works when $\mathcal{M}$ is left proper and either cellular or combinatorial. Left proper means the pushout of a weak equivalence by a cofibration is a weak equivalence. It makes the model category act more like Top. Combinatorial means all objects are small. Cellular means it's cofibrantly generated, the (co)domains of I are compact, the domains of J are small relative to I, and the cofibrations are contained in the effective monomorphisms (i.e. maps $f : X \to Y$ such that $X \to Y \rightrightarrows Y \coprod_X Y$ is an equalizer). We will assume $\mathcal{M}$ is left proper, but we need not assume cellular or combinatorial; only that the Bousfield localization in question exists.

The identity maps $\mathcal{M} \rightleftarrows L_C\mathcal{M}$ are a Quillen adjoint pair and prove that $L_C\mathcal{M}$ satisfies a universal property as the "closest" model category to $\mathcal{M}$ in which C is contained in the weak equivalences. The fibrant objects in $L_C\mathcal{M}$ are the C-local objects, and local equivalences between local objects are weak equivalences in the original model category. Bousfield localization gives a Quillen pair (L_C, U_C), which are both the identity functors on objects and morphisms, and these induce (L_C^H, U_C^H) on the homotopy level.

Our goal is to find conditions on $\mathcal{M}$ and C under which Bousfield localization *preserves* $\mathbb{P}$*-algebra structure*, i.e. if $[E] \in \mathrm{Ho}(\mathcal{M})$ has a representative $E \in \mathbb{P}$-alg then we're asking for $(U_C^H \circ L_C^H)([E])$ to have a representative in $\mathbb{P}$-alg. This means that for all cofibrant $E \in P$−alg, $L_C(E) \in P$−alg and $E \to L_C(E)$ is a P−alg homomorphism. More generally: given $E \in P$−alg, we need $\widetilde{E} \in P$−alg with $L_C(E) \simeq \widetilde{E}$. We will use the fact that Bousfield localization works via the derived functors of the identity, so $L_C(E)$ is $R_C QE$ where R_C be fibrant replacement in $L_C(M)$.

Theorem 7. *Let $\mathcal{M}$ be a monoidal model category and let P be an operad valued in $\mathcal{M}$. If P-algebras in $\mathcal{M}$ and in $L_C(\mathcal{M})$ inherit model structures such that the forgetful functors back to $\mathcal{M}$ and $L_C(\mathcal{M})$ are right Quillen functors, then L_C preserves P-algebras up to weak equivalence. For well-behaved P there is a list of easy to check conditions on $\mathcal{M}$ and C guaranteeing these hypotheses hold.*

Proof. Here "inherit" means that a map of P-algebras f is a weak equivalence (resp fibration) iff f is a weak equivalence (resp fibration) in $\mathcal{M}$.

Let $R_{C,m}$ be fibrant replacement in $P - alg(L_C(M))$, and Q_m be cofibrant replacement in $P - alg(M)$. In our proof, $\widetilde{E}$ will be $R_{C,m}Q_m(E)$. Because Q is the left derived functor of the identity adjunction between M and $L_C(M)$, and R_C is the right derived functor of the identity, we know that $L_C(E) \simeq R_C Q(E)$. We must therefore show $R_C Q(E) \simeq R_{C,m}Q_m(E)$.

The map $Q_m E \to E$ is a weak equivalence in $P - alg(\mathcal{M})$, hence in $\mathcal{M}$. The map $QE \to E$ is also a weak equivalence in $\mathcal{M}$ and lifting gives a map from $QE \to Q_m E$ (necessarily a weak equivalence in $\mathcal{M}$ by the 2 out of 3 property).

Since $Q_m E$ is a P-algebra in $\mathcal{M}$ it must also be a P-algebra in $L_C\mathcal{M}$, since the monoidal structure of the two categories is the same. We may therefore construct a lift:

$$\begin{tikzcd}
Q_m E \arrow[r, "\simeq_C"] \arrow[d, hook, "\simeq_C"'] & R_{C,m}Q_m E \arrow[d, two heads] \\
R_C Q_m E \arrow[r] \arrow[ur, dotted, "\therefore\simeq_C"] & *
\end{tikzcd}$$

In this diagram the left vertical map is a weak equivalence in $L_C\mathcal{M}$ and the top map is a weak equivalence in $P - alg(L_C\mathcal{M})$. Because this model category $P - alg(L_C\mathcal{M})$ inherits weak equivalences from $L_C\mathcal{M}$ this map is a weak equivalence in $L_C\mathcal{M}$. Therefore, by the 2 out of 3 property, the lift is a weak equivalence in $L_C\mathcal{M}$. Using this lift we can draw the following diagram:

$$\begin{tikzcd}
QE \arrow[rr, "\simeq"] \arrow[dd] & & Q_m E \arrow[dl] \arrow[dd] \\
& R_C Q_m E \arrow[dr, "\simeq_C"] & \\
R_C QE \arrow[ur, "\therefore\simeq_C"] \arrow[rr, "\therefore\simeq_C"] & & R_{C,m}Q_m E
\end{tikzcd}$$

We showed above that $QE \to Q_m E$ is a weak equivalence in $\mathcal{M}$. Thus, $R_C QE \to R_C Q_m E$ is a weak equivalence in $L_C\mathcal{M}$. We then proved $R_C Q_m E \to R_{C,m}Q_m E$ is a weak equivalence in $L_C\mathcal{M}$. Thus, by the 2 out of 3 property, $R_C QE \to R_{C,m}Q_m E$ is a weak equivalence in $L_C\mathcal{M}$. All the objects in the triangle are fibrant in $L_C\mathcal{M}$ so these C-local equivalences are actually weak equivalences in $\mathcal{M}$.

The triangle commutes because the bottom map is defined as the composite. The square commutes in Ho $\mathcal{M}$ and demonstrates that $R_C QE$ is isomorphic in Ho $\mathcal{M}$ to the P-algebra $R_{C,m}Q_m E$. $\square$

This proof also holds if P-algebras only form a semi-model category. In a semi-model category all objects admit cofibrant replacement, but only cofibrant objects admit fibrant replacement. Lifting of a trivial cofibration against a fibration only holds if the domain is cofibrant. Everywhere we've applied fibrant replacement it's been to an object which is cofibrant in the underlying category, so that's no problem. The lifting argument is for a map which has cofibrant domain, so that's fine too. Thus, even if the monoid axiom is not preserved we can still say P-alg is a semi-model category.

It's a bit unfair to just assume P-algebras form a model category. After all, it can be very difficult to get your hands on $L_C\mathcal{M}$. We'd rather have hypotheses on $\mathcal{M}$ and C to make sure this situation happens. For cofibrant operads P we can use the theorem due to Spitzweck. We see then that if we only care about preserving structure over a cofibrant operad P then we only need to know when $L_C(\mathcal{M})$ is a monoidal model category. We can characterize when this occurs. First, we need a new axiom on the model category:

A common strengthening of the unit axiom is the Resolution Axiom, which states that cofibrant objects are flat, i.e. whenever $f \in \mathscr{W}$ and X is cofibrant, then $X \otimes f \in \mathscr{W}$.

Theorem 8. *Assume $\mathcal{M}$ is a left proper, monoidal model category satisfying the resolution axiom. Then*

$L_C(\mathcal{M})$ satisfies the resolution axiom and pushout product axiom if and only if for all cofibrant K, the maps $C \otimes id_K$ are weak equivalences in $L_C(\mathcal{M})$

If $\mathcal{M}$ is tractable then it suffices to check this on K running through the domains and codomains of the generating (trivial) cofibrations

We call such localizations *monoidal localizations*. As a corollary, we discover how to form the *smallest monoidal Bousfield localization* which inverts a given class C. Simply replace C by the class $C \otimes id_K$. This notion is used in my recent preprint with Hovey (An Alternative Approach to Equivariant Stable Homotopy Theory), and also in a forthcoming preprint with Casacuberta (Localization and Cellularization in the Motivic Stable Homotopy Category).

To prove this we first take care of the Resolution Axiom, then deduce the Pushout Product Axiom using tractability. The non-tractable case reduces to this by a standard transfinite induction.

Proposition 9. *Under the standing hypotheses on $\mathcal{M}$ and under the hypothesis that $f \otimes K$ is an f-local equivalence for all domains and codomains K of $I \cup J$, localization preserves the property of cofibrant objects being flat.*

Proof. Let X be a cofibrant object in $L_f\mathcal{M}$. Let $g : A \to B$ be an f-local equivalence. To prove $-\otimes X$ preserves f-local equivalences, it suffices to show that it takes $L_f\mathcal{M}$ trivial cofibrations between cofibrant objects to weak equivalences. This is because we can always do cofibrant replacement on g to get $Qg : QA \to QB$. While Qg need not be a cofibration in general, we can always factor it into $QA \hookrightarrow Z \overset{\simeq}{\twoheadrightarrow} QB$. We then abuse notation to treat Z as QB and rename the cofibration $QA \to Z$ as Qg since Z is cofibrant and maps via a trivial fibration to B. Smashing with X gives:

$$\begin{tikzcd} QA \otimes X \arrow[r] \arrow[d] & QB \otimes X \arrow[d] \\ A \otimes X \arrow[r] & B \otimes X \end{tikzcd}$$

If we prove that $Qg \otimes X$ is an f-local equivalence, then $g \otimes X$ must also be by the 2-out-of-3 property, since the vertical maps are weak equivalences in $\mathcal{M}$ due to X being cofibrant and cofibrant objects being flat in M. So we assume that g is an $L_f\mathcal{M}$ trivial cofibration between cofibrant objects. Since X is built as a transfinite composition of pushouts of maps in I, we proceed by transfinite induction. For the rest of the proof, let K, K_1, and K_2 denote domains/codomains of maps in I. These objects are cofibrant in $\mathcal{M}$ by hypothesis, so they are also cofibrant in $L_f\mathcal{M}$.

For the base case $X = K$ we appeal to Theorem 3.1.6 in Hirschhorn's book. Because the functor $F = -\otimes K$ is a left-adjoint in a pair of functors from $\mathcal{M}$ to itself, F takes f-local equivalences between cofibrant objects to f-local equivalences if and only if the total left derived functor $\mathbf{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(L_f\mathcal{M})$ takes $[f]$ to an isomorphism. This occurs because $f \otimes K$ is an f-local equivalence and so $[f \otimes K]$ is an isomorphism in $\mathrm{Ho}(L_f\mathcal{M})$. Thus, $g \otimes K$ is a weak equivalence in $L_f\mathcal{M}$.

For the successor case, suppose X_α is built from K as above and is flat in $L_f\mathcal{M}$. Suppose $X_{\alpha+1}$ is built from X_α and a map in I via a pushout diagram:

$$\begin{tikzcd}
K_1 \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\Searrow"] & K_2 \arrow[d] \\
X_\alpha \arrow[r] & X_{\alpha+1}
\end{tikzcd}$$

We smash this diagram with $g : A \to B$ and note that smashing a pushout square with an object yields a pushout square.

$$\begin{tikzcd}
A \otimes K_1 \arrow[rr, "A\otimes i"] \arrow[dr, "g\otimes K_1"] \arrow[dd] & & A \otimes K_2 \arrow[dr, "g\otimes K_2"] \arrow[dd] & \\
& B \otimes K_1 \arrow[rr] \arrow[dd] & & B \otimes K_2 \arrow[dd] \\
A \otimes X_\alpha \arrow[rr, "B\otimes i"] \arrow[dr, "g\otimes X_\alpha"'] & & A \otimes X_{\alpha+1} \arrow[dr, "g\otimes X_{\alpha+1}"] & \\
& B \otimes X_\alpha \arrow[rr] & & B \otimes X_{\alpha+1}
\end{tikzcd}$$

Because g is a cofibration of cofibrant objects, A and B are cofibrant. Because pushouts of cofibrations are cofibrations, $X_\alpha \hookrightarrow X_{\alpha+1}$ for all α. Because X_0 is cofibrant, X_α is cofibrant for all α. So all objects above are cofibrant. Furthermore, $g \otimes K_i = g\square(0 \hookrightarrow K_i)$. Thus, by the Pushout Product axiom on $\mathcal{M}$ and the fact that cofibrations in $\mathcal{M}$ match those in $L_f\mathcal{M}$, these maps are cofibrations.

Finally, the maps $g \otimes K_i$ are weak equivalences in $L_f\mathcal{M}$ by the base case above, while $g \otimes X_\alpha$ is a weak equivalence in $L_f\mathcal{M}$ by the inductive hypothesis. Thus, by Dan Kan's Cube Lemma (Lemma 5.2.6 in Hovey's book), the map $g \otimes X_{\alpha+1}$ is a weak equivalence in $L_f\mathcal{M}$.

For the limit case, suppose we are given a cofibrant object $X = \mathrm{colim}_{\alpha<\beta} X_\alpha$ where each X_α is flat in $L_f\mathcal{M}$. Because each X_α is cofibrant, $g \otimes X_\alpha = g\square(0 \hookrightarrow X_\alpha)$ is still a cofibration. By the inductive hypothesis, each $g\otimes X_\alpha$ is also an f-local equivalence, hence a trivial cofibration in $L_f\mathcal{M}$. Since trivial cofibrations are always closed under transfinite composition, $g \otimes X = g \otimes \mathrm{colim}\, X_\alpha = \mathrm{colim}(g \otimes X_\alpha)$ is also a trivial cofibration in $L_f\mathcal{M}$. □

Next we prove the Pushout Product Axiom is preserved.

Lemma 10. *If h is a $L_f\mathcal{M}$ trivial cofibration then $h \otimes K$ is a $L_f M$ trivial cofibration.*

Proof. Using Proposition 4, we see that $h\otimes K$ is a weak equivalence in $L_f\mathcal{M}$. Because h and g are cofibrations in $\mathcal{M}$, $h\square g$ is a cofibration in $\mathcal{M}$ by the pushout product axiom on $\mathcal{M}$. Thus, $h \otimes K$ is both a cofibration and a weak equivalence in $L_f\mathcal{M}$. □

Proposition 11. *If h is an $L_f\mathcal{M}$ trivial cofibration and g is a generating cofibration in $L_f\mathcal{M}$ then $h\square g$ is an $L_f\mathcal{M}$ trivial cofibration*

Proof. Suppose $h : X \to Y$ and $g : K \to L$. By hypothesis, K and L are cofibrant. Because h is a cofibration, $K \otimes h$ and $L \otimes h$ are cofibrations. Because cofibrant objects are flat in $L_f M$, $K \otimes h$ and $L \otimes h$ are also weak equivalences. Consider the following diagram:

$$
\begin{tikzcd}
K \otimes X \arrow[r, hook, "\simeq"] \arrow[d] \arrow[dr, phantom, "\Searrow"] & K \otimes Y \arrow[d] \arrow[ddr, bend left] & \\
L \otimes X \arrow[r, "\simeq"] \arrow[drr, bend right, "\simeq"'] & (K \otimes Y) \coprod_{K \otimes X} (L \otimes X) \arrow[dr, "h \Box g"] & \\
 & & L \otimes Y
\end{tikzcd}
$$

The map $L \otimes X \to (K \otimes Y) \coprod_{K \otimes X} (L \otimes X)$ is a trivial cofibration because it is the pushout of a trivial cofibration. Thus, by the 2-out-of-3 property for the lower triangle, $h \Box g$ is a weak equivalence. Since we already knew it was a cofibration, this means it is a trivial cofibration.

□

Thus, we have characterized monoidal localizations and there are examples of localizations which fail to be monoidal, e.g. in $Ch(R[G])$ when a localization kills a representation sphere. For an example, we look to the Σ_n-equivariant world, where there are multiple spheres due to the different group actions. In this world one can suspend by representations of Σ_n, i.e. copies of $\mathbb{F}_2$ on which Σ_n acts. The 1-point compactification of such an object is a sphere S^n on which Σ_n acts. This means not all cofibrant objects are built from a single sphere as they normally would be in R-mod without a group action (which is a monogenic category). This can cause trouble on homology, since $H_*(X \otimes S^n_1)$ need not equal $H_*(X \otimes S^n_2)$. For instance, if we have two spheres and localization kills one, then it is possible for the localization functor to go from a category where all objects are cofibrant to one where this property fails.

For concreteness, let $R = \mathbb{F}_2[\Sigma_3]$. Then an R module is simply an $\mathbb{F}_2$ vector space with an action of the symmetric group Σ_3. The category R-mod is a stable monoidal model category. Let f be the map $f : \mathbb{F}_2 \to \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$ taking 1 to $(1, 1, 1)$. We'll show that the Bousfield localization with respect to f cannot be monoidal. Bousfield localization with respect to f introduces a quotient $\mathrm{coker}(f) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ generated by $(1, 0, 0)$ and $(0, 1, 0)$. In this quotient, $(0, 0, 1) = (1, 1, 0)$. The localized category $L_f M$ is not monoidal because this cokernel cannot be built from the sphere due to the fact that the Σ_3 action is more complicated than it would be if $\mathrm{coker}(f)$ was just $\mathbb{F}_2 \wedge \mathbb{F}_2$. For instance, $(123) \cdot (0, 1, 0) = (0, 0, 1) = (1, 1, 0)$ but in $\mathbb{F}_2 \wedge \mathbb{F}_2$ it would be $(0, 1, 0)$. Similarly, $(12) \cdot (0, 1, 0) = (1, 0, 0)$, $(12) \cdot (1, 0, 0) = (0, 1, 0)$ and $(123) \cdot (1, 0, 0) = (0, 1, 0)$.

Corollary 12. *If $\mathscr{S}^G$ a Bousfield localization preserves genuine commutativity iff $C \otimes (G/H)_+$ is a C-local equivalence for all H.*

We see where our opening example failed: applying $C \otimes -$ killed all G/H and we would have ended up inverting the zero map if the condition of the corollary was satisfied.

What if you don't care about all the norms, but rather only some of them?

The recent paper of Blumberg-Hill proves that this type of algebraic structure is captured by the class of N_∞ operads. Independently, Javier and I were studying operads based on families. There is a collection $E_{\mathscr{F}}\Sigma_n$ whose n^{th} space is the total space of a universal $\mathscr{F}$-equivariant principle Σ_n-bundle. When using the family model structure on G-spaces this becomes a cofibrant collection, and it's equivalent as a collection to an Blumberg-Hill N_∞ operad because of the universal property of the n^{th} space. We visualize the interpolation between naive and genuine E_∞ as a tower, and notice that localization can drop structure as follows:

$$\begin{tikzcd}
E_\infty^G \arrow[r, "L"] \arrow[d, dash] \arrow[dr] \arrow[dddr] \arrow[ddddr] & E_\infty^G \arrow[d, dash] \\
E_\infty^{\mathscr{F}} \arrow[r] \arrow[d, dash] \arrow[dddr] & E_\infty^{\mathscr{F}} \arrow[d, dash] \\
\dots \arrow[d, dash] & \dots \arrow[d, dash] \\
E_\infty^{\mathscr{F}'} \arrow[r] \arrow[d, dash] \arrow[dr] & E_\infty^{\mathscr{F}'} \arrow[d, dash] \\
E_\infty \arrow[r] & E_\infty
\end{tikzcd}$$

Thus, the general preservation theorem specializes to tell us that Bousfield localization preserves this N_∞ structure iff $C \otimes (G/H)_+$ is a C-local equivalence for all $H \in \mathscr{F}$. Such localizations will drop structure down to $E_\infty^{\mathscr{F}}$. The proof is to work in the family model structure on spaces and the corresponding semi-model structure on $Op(Top^{\mathscr{F}})$. Mike's example can be generalized to a collection of examples demonstrating necessity of this hypothesis at each level in the tower of $E_\infty^{\mathscr{F}}$'s interpolating between E_∞ and E_∞^G. His original example is maximally bad, i.e. drops from any norm structure all the way down to E_∞, but there are examples which make any drop you like, e.g. from some $E_\infty^{\mathscr{F}}$ to some other $E_\infty^{\mathscr{F}'}$.

5. Strict Commutativity

Turning now to when localization preserves the commutative monoid axiom, recall that commutative monoids are built via the functor $Sym(X) = S \wedge X \wedge X^2/\Sigma_2 \wedge \dots$. For monoidal structure we needed localization to play well with tensoring. Now we'll need it to work with Sym:

Theorem 13. *Suppose $\mathcal{M}$ satisfies the commutative monoid axiom. If $Sym(-)$ preserves weak equivalences in $L_C(\mathcal{M})$ then $L_C(\mathcal{M})$ satisfies the commutative monoid axiom.*

Combining this with our general preservation result gives:

Corollary 14. *Truncations in sSet, Top, and Ch(k) all preserve strict commutative monoids. Via Farjoun's trick, any monoidal localization in sSet will also preserve, e.g. L_E for a homology theory E.*

I hope to investigate these L_E further and recover classical (unstable) theorems of Bousfield using this general machinery.

In the future I hope to study when localization preserves the generalizations of the commutative monoid axiom to other non-cofibrant P.