EILENBERG-MOORE MODEL CATEGORIES AND BOUSFIELD LOCALIZATION

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Thanks for your hospitality. I'm here on an NSF-AAS grant and will be here till August 20. I'm happy to talk to anyone about basically any research topic. My research program can be found on my website: dwhite03.web.wesleyan.edu

1. Outline

Talk 1:

Big Goal of Alg Top, operads and model categories, fix notation for model categories, remarks about how difficult it is to verify model category axioms.

Motivation from equivariant spectra, and discussion of Kervaire.

Monoidal model categories, define the *inherited* model structure on the category of algebras over an operad.

Basic facts about Bousfield localization.

Preservation theorem and proof, a word about semi-model categories.

Talk 2:

Preliminary results about why semi-model categories are not so bad.

General transfer principles for putting (semi) model structures on T-algebras.

Review of Schwede-Shipley proof for monoids. Connection to tame polynomial monads.

Version for commutative monoids. Examples of model categories satisfying all our axioms so far.

Notion of cofibrancy for operads. John Harper's filtration and resulting axioms to get semi-model structures on algebras over operads.

Examples of localization preserving structure: truncation in spaces, homological algebra, stable localizations in spectra, bringing it back to the example of equivariant spectra.

2. Talk 1 Introduction

A big goal of algebraic topology is to understand how homotopy theory interacts with algebraic structure. For today we use model categories as our language for homotopy theory and operads as our language for algebra. We'll work on the subgoal of understanding when Bousfield localization preserves the structure of algebras over an operad.

Many proofs in recent years have demonstrated the value of working on the point-set level rather than in the homotopy category, so that's why we use model categories. Recall that model categories were invented in 1967 by Dan Quillen. It's often difficult to verify that a given category together with a class of weak equivalences, fibrations, and cofibrations forms a model category, but when you can then you can apply homotopy

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theory in various disciplines. Quillen won the Fields Medal for computations in algebraic K-theory which made use of homotopical methods in homological algebra (because Ch(R) is a model category). Voevodsky won the Fields Medal for resolving the Milnor Conjecture using homotopical methods in the category of Schemes, again by constructing an appropriate model structure. Steve Lack put a model structure on 2Cat.

Because we need to be doing algebra, we need a monoidal structure. This compatibility is encoded in the definition of a monoidal model category (explored in Hovey's book). We'll be investigating how Bousfield localization (explored by Hirschhorn's book) interacts with algebras over various monads in that context (mostly, monads given by operads).

3. Kervaire Invariant Problem and Equivariant Spectra

This project was motivated by a step in the proof of the Kervaire Invariant One Theorem. For a framed smooth manifold of dimension $4k + 2$, the Kervaire invariant $Ker(X) \in \mathbb{Z}/2$ is the Arf invariant? of the skew-quadratic form on the middle dimensional homology group.

A **framing** on a manifold M is like a trivialization of the tangent bundle. Formally, it's a trivialization of the normal bundle if the manifold is understood embedded in some Cartesian space, along with a trivialization of the stable tangent bundle. A manifold that admits a framing is also called a parallelizable manifold. A manifold equipped with a framing is also called a parallelized manifold.

Theorem. The n-spheres that admit a framing are precisely S^0, S^1, S^3, S^7.

Kervaire-Milnor 'Groups of Homotopy Spheres' proves: An odd dimensional framed manifold is framed cobordant to a homotopy sphere (if $n > 5$). In dimensions $8k+2$, **a framed manifold is framed cobordant to a homotopy sphere precisely when the Kervaire invariant is zero**. Kervaire constructed a manifold of Kervaire invariant 1 in dimension 10, so this manifold cannot admit a smooth structure.

A framing on M induces an isomorphism between the total space of the normal bundle ν and $M \times \mathbb{R}^k$, and so a homeomorphism $Th(\nu) \cong \Sigma^r(M_+)$. Viewing S^{n+r} as a one-point compactification of $\mathbb{R}^{n+r}$ and collapsing the boundary of $M \times S^r$ to a point yields a map $S^{n+r} \to \Sigma^r(M_+)$, so M gives an element of $\pi_{n+r}S^r$. This induces a map from the graded ring of framed cobordism classes of framed manifolds to $\pi_* S$, which Pontryagin proved was an isomorphism of graded rings.

Recall that **certain spheres can have non-diffeomorphic smooth structures**, e.g. Milnor's famous example on S^7. The answer to the question of which dimensions n allow this is contained in the stable homotopy groups of spheres. **The group of diffeomorphism structures can be constructed and understood up to a quotient term which depends on framed bordism**. The monoid of smooth structures on S^n is isomorphic to the group Θ_n of h-cobordism classes of oriented homotopy n-spheres. This group can be understood via a quotient and the J-homomorphism. The quotient depends on whether or not there are **framed manifolds** of non-zero Kervaire invariant. The connection:

$\pi_{n+k}(S^n) \cong \{\text{framed } M^k \subset \mathbb{R}^{n+k}\}/\text{bordism}$

This can be used to compute the image of $\Theta_n/bP_{n+1} \to \pi_n S^0/J$, where bP_{n+1} is the cyclic subgroups of n-spheres that **bound a parallelizable manifold** of dimension $n+1$, and J is the image of the J-homomorphism $J : \pi_r(SO(q)) \to \pi_{r+q}(S^q)$. Note that the order of these cyclic groups is related to the Bernoulli numbers. So from this point of view the Kervaire Invariant is a possible discrepancy stopping you from understanding which dimensions have exotic spheres.

In the 60s Browder showed the Kervaire Invariant can be non-zero only occur if $n = 2^k - 2$. They can exist for $k < 7$. Hill-Hopkins-Ravenel 2009 showed that they cannot exist for $k > 7$. Their proof came down to a computation in the stable homotopy groups of spheres which relied on extra structure brought in from equivariant stable homotopy groups (with a $\mathbb{Z}/2$-action).

The authors needed a 256-periodic $\Omega = D^{-1}MU^{(4)}$ for some D. They were working with MU considered as a commutative equivariant spectrum and **needed Ω to be commutative, i.e. have multiplicative norms**, i.e. to have $\pi_\star$ forming a Tambara functor. This was needed for reasons related to the spectral sequence computations which occupy the technical details of the proof.

Let G be a compact Lie group and let $\mathscr{S}^G$ be the model category of G-spectra. Sadly, not every localization of an equivariant commutative ring spectrum is commutative.

Example (Hill, Oberwolfach). Let G be a (non-trivial) finite group.

Consider the reduced real regular representation $\overline{\rho}$ obtained by taking the quotient of the real regular representation ρ by the trivial representation, which is a summand. We write $\overline{\rho}_G = \rho_G - 1$ where 1 means the trivial representation $\mathbb{R}[e]$. Taking the one-point compactification of this representation yields a representation sphere $S^{\overline{\rho}}$. There is a natural inclusion $a_{\overline{\rho}} : S^0 \to S^{\overline{\rho}}$ induced by the inclusion of the trivial representation into $\overline{\rho}$. Consider the spectrum $E = S[a_{\overline{\rho}}^{-1}]$ obtained from the unit S (certainly a commutative algebra in $\mathscr{S}^G$) by localization with respect to $a_{\overline{\rho}}$. We will show that this spectrum does not admit maps from the norms of its restrictions, and hence cannot be commutative.

First, $\rho_G|_H = [G:H]\rho_H$, so $\overline{\rho}_G|_H = [G:H]\rho_H - 1 = [G:H](\overline{\rho}_H + 1) - 1 = [G:H]\overline{\rho}_H + ([G:H]1 - 1)$. We will use this to prove that **for all proper $H < G$, $res_H(E)$ is contractible**. Because $[G:H] - 1$ is a number k greater than 0 we have $res_H S^{\overline{\rho}_G} = (S^{\overline{\rho}_H})^{\#[G:H]} \wedge S^k$. This means that as an H-spectrum it is contractible, because there is enough space in the S^k part to deform it to a point. Note, however, that E itself is not locally trivial. Thinking of S^0 as $\{0, \infty\}$ we see that the only fixed points of $a_{\overline{\rho}}$ are 0 and ∞, so the map $a_{\overline{\rho}}$ is not equivariantly trivial.

Now, a key property of commutative equivariant ring spectra is the **existence of multiplicative norms**. These functors $N_H^G : \mathscr{S}^H \to \mathscr{S}^G$ are left adjoint to the restriction res_H on the category of commutative ring spectra. If E were a commutative algebra in $\mathscr{S}^G$ then the counit of the norm-restriction adjunction would provide a ring homomorphism $N_H^G res_H(E) \to E$. But the domain is contractible for every proper subgroup H because $res_H(E)$ is contractible. This cannot be a ring map unless E to be contractible, and we know E **is not contractible** because $a_{\overline{\rho}}$ fixes 0 and ∞.

Here is an equivalent approach, which Hill presented at Oberwolfach. Let $\mathscr{F}$ be the family of proper subgroups of G and let $\widetilde{E\mathscr{F}}$ be the cofiber of the natural map from the classifying space $E\mathscr{F}_+$ to S^0. This $\widetilde{E\mathscr{F}}$ is a localization of S^0 obtained by killing all maps from induced cells. If G is finite then it is our E. It's not contractible because $E\mathscr{F}_+$ is not homotopy equivalent to S^0, because $\mathscr{F}$ doesn't contain G. So while any restriction to a proper subgroup views them to be homotopy equivalent, they are not homotopy equivalent in $\mathscr{S}^G$.

In this second approach it becomes clear that this example generalizes to other families of subgroups, proving that in any family model structures (other than $\mathscr{F} = \{e\}$, which recovers naive spectra) one can similarly disprove the preservation of commutativity by localization.

□

The take-away message from this example is that **we need a hypothesis on the maps being localized** so that equivariant commutativity is preserved. Viewed a certain way, what is failing above is the ability of the localization functor to commute with equivariant suspension with respect to certain representation spheres (namely, those which don't see all the information in G, but only see subgroup information). **When localizations kill representation spheres bad things happen**. A similar example, due to Carles Casacuberta, proves that not all localizations of spectra preserve ring structure. This is the example of the Postnikov Section:

The n^{th} Postnikov section functor P_n is a homotopical localization for all n but does not commute with suspension. Furthermore, if R is nonconnective, then $P_1 R$ does not admit a ring spectrum structure (not even the structure of a ring in the homotopy category). The reason is that if it were a ring then multiplication by the unit S would need to be a homotopy equivalence. But the unit map $\nu : S \to P_1 R$ is null since $\pi_0(P_1 R) = 0$. The real issue here is that suspension and localization do not commute, and nonconnective ring spectra can feel the difference. We've chopped off the dimension where the unit is supposed to live.

Casacuberta gets around this by placing hypotheses on the localization (he calls the well-behaved localizations "closed") and similarly Hill and Hopkins get around Hill's example by placing hypotheses on the maps:

Theorem 3.1 (Hill-Hopkins)**.** *If for all L-acyclics Z and for all subgroups H, $N_H^G Z$ is L-acyclic, then for all commutative G-ring spectra R, $L(R)$ is a commutative G-ring spectrum.*

Here commutativity can mean either strict commutativity (algebras over the operad Com) or E_∞-structure where E_∞ is the linear isometries operad (a model can be taken with $E_\infty[n] = E_G \Sigma_n$), because in $\mathscr{S}^G$ there is rectification between these operads, as recently proven by Blumberg and Hill in the appendix of their 2013 paper. The hypothesis in the theorem is precisely what is needed to make the EKMM proof (via the skeletal filtration) go through.

This example and theorem open a more general question: **find conditions on a general model category $\mathcal{M}$ and on a set of maps C so that the Bousfield localization L_C preserves commutativity**. We will answer this question, and when we specialize our machinery to $\mathcal{M} = \mathscr{S}^G$ we'll in fact characterize localizations which preserve commutative structure. This will show us yet another reason why Hill's example is failing. In particular, it will fail because for the set of maps being inverted we have that $C \otimes (G/H)_+$ is not contained in the C-local equivalences (because it contains the zero map, because proper H sees C to be trivial even though it is not). This makes it easy to see the correct condition on C so that L_C viewed in the family model structures preserves commutativity. The condition will be that $C \otimes (G/H)_+ \subset C$-local equivalences for all $H \in \mathscr{F}$.

4. Background: monoidal model categories, Eilenberg-Moore categories, Bousfield localization

Recall that we care about model categories $\mathcal{M}$ because the passage to Ho($\mathcal{M}$) works (this functor inverts the weak equivalences $\mathscr{W}$) and we have some control over the resulting maps because of cofibrant and fibrant replacement. Let $\mathscr{Q}$ and $\mathscr{F}$ be the cofibrations and fibrations. If we're going to talk about commutative monoids then we need to have a monoidal structure on $\mathcal{M}$. It turns out that we also need a compatibility hypothesis between $\mathcal{M}$ and the monoidal structure, as explained in chapter 4 of Hovey's book. Let $\otimes$ denote the monoidal product.

Given $f : A \to B$ and $g : X \to Y$, define the *pushout product* $f \square g$ to be the corner map in

$$\begin{tikzcd}
A \otimes X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Searrow"] & A \otimes Y \arrow[d] \arrow[ddr, bend left] & \\
B \otimes X \arrow[r] \arrow[drr, bend right] & Q_2 \arrow[dr] & \\
& & B \otimes Y
\end{tikzcd}$$

This is simply the Day convolution. A *monoidal model category* is a model category which is also a monoidal category and satisfies:

Pushout product axiom: if $f, g \in \mathscr{Q}$ then $f \square g \in \mathscr{Q}$. Additionally, if either is in $\mathscr{W}$ then $f \square g \in \mathscr{W}$.

Unit Axiom: If Z is cofibrant then $QS \otimes Z \to S \otimes Z \cong Z$ is a weak equivalence.

These axioms assure you that $Ho(\mathcal{M})$ is a monoidal category. We'll be studying objects in $\mathcal{M}$ which carry the additional algebraic structure encoded by a monad. Many times this monad will arise from a symmetric operad, e.g. monoids, commutative monoids, A_∞ or E_∞ algebras, Lie algebras, etc. All operads today are symmetric.

For us, an operad in $\mathcal{M}$ is a symmetric sequence $P = (P(n))_{n\in\mathbb{N}}$ of objects in $\mathcal{M}$ (i.e. each $P(n)$ is in $\mathcal{M}^{\Sigma_n}$ i.e. has an action of the symmetric group Σ_n) satisfying some axioms. The object $P(n)$ can be thought of as parameterizing maps of arity n. There is a notion for cofibrancy of an operad which comes down to requiring the left lifting property of $\emptyset \to P$ with respect to maps which are levelwise trivial fibrations in $\prod_{n\in\mathbb{N}} \mathcal{M}^{\Sigma_n}$ where $\mathcal{M}^{\Sigma_n}$ is the model category of objects in $\mathcal{M}$ with a Σ_n action.

Examples:

(1) Ass is the operad encoding associativity. $Ass[n] = \Sigma_n$

(2) Com is the operad encoding strict commutativity. $Com[n] = *$

(3) $\mathscr{L}$ is the linear isometries operad. If we fix a universe U then the n^{th} space of $\mathscr{L}$ is $\mathscr{L}(U^n, U)$, the space of linear isometries from U^n to U.

(4) An E_∞ operad has $P(n)$ contractible and Σ_n acts freely. So the linear isometries operad and little cubes operad are both E_∞.

An *algebra over an operad* is an object $A \in C$ equipped with coherent maps $P(n) \times A^n \to A$. These objects form a category, with morphisms P-algebra homomorphisms (maps which respect this structure).

Let T be a monad. If I want to do homotopy theory with T-algebras then I'll want them to inherit a model structure. We will transfer it along the adjunction $T : \mathcal{M} \leftrightarrow T - alg(\mathcal{M}) : U$. Here U is the forgetful functor and T is the free algebra functor. In the case $T = P$ for an operad P, there is a nice formula: $P(X) = \coprod_n (P(n) \otimes X^{\otimes n})$.

If we wish to place a model structure on T-alg we will want it to be compatible with the model structure on $\mathcal{M}$. In particular, we want the forgetful functor to be right Quillen. So we need the model structure on T-alg to have weak equivalences and fibrations maps which are such as maps in $\mathcal{M}$. Cofibrations are therefore determined by the lifting property. In the second talk we'll discuss when this sort of transfer may be accomplished. For now we turn to the localization question.

5. Bousfield localization and preservation of T-algebra structure

Let's first consider the model category theoretic version of localization, which generalizes the localization in Hill's example and the Hill-Hopkins theorem. This all goes back to work of Bousfield on inverting maps f (of spaces or spectra) seen to be weak equivalences by a homology theory E.

What if I want to invert some maps $C \not\subset \mathscr{W}$? Because the homotopy category is nice (admits a calculus of fractions), we can do:

$$\begin{tikzcd} \mathcal{M} \arrow[r, dotted] \arrow[d] & ????? \arrow[d, dotted] \\ \mathrm{Ho}(\mathcal{M}) \arrow[r] & \mathrm{Ho}(\mathcal{M})[C^{-1}] \end{tikzcd}$$

We'd like a model category $L_C\mathcal{M}$ which actually sits above $\mathrm{Ho}(\mathcal{M})[C^{-1}]$. Because all three categories above have the same objects, its objects are determined. It's morphisms will be the same as those in $\mathcal{M}$, but we want maps in C to become isomorphisms in $\mathrm{Ho}(\mathcal{M})[C^{-1}]$ so we need them to be weak equivalences in $L_C\mathcal{M}$. So this category must have a different model structure, where $\mathcal{W}' = \langle C \cup \mathcal{W}\rangle$ and clearly $\mathcal{W} \subset \mathcal{W}'$. You can't change only $\mathcal{W}$ because it'll screw up the axioms. We want to keep the cofibrations fixed so we can build things out of them and have the two model structures related, so we have to shrink the fibrations: $\mathcal{F} \supset \mathcal{F}'$. **Bousfield's Theorem** (1978) says you can do this and you still get a model structure, but you have to be careful with how you generate $\mathcal{W}'$ from C. Details are in Hirschhorn's book.

Formally, define $X \in \mathcal{M}$ to be C-local if X is fibrant and $f^* : Map(B, X) \to Map(A, X)$ is a weak equivalence, for all $f : A \to B$ in C. These objects X look trivial to the eyes of C. Define $g : D \to E$ to be a C-local equivalence if for all C-local X, $Map(E, X) \to Map(D, X)$ is a weak equivalence. This follows the idea in algebra, where a module M is S-local if μ_s is an isomorphism for all $s \in S$. A map is an S-equivalence if applying $Hom(-, M)$ gives an isomorphism for all S-local M. It turns out $R \to R[S^{-1}]$ is an S-equivalence to an S-local object. We'd call that fibrant replacement in $L_C(\mathcal{M})$. Proving this object exists is the major technical difficulty faced by Bousfield, and is the reason hypotheses on $\mathcal{M}$ are needed.

This story works when $\mathcal{M}$ is left proper and either cellular or combinatorial. Left proper means the pushout of a weak equivalence by a cofibration is a weak equivalence. It makes the model category act more like Top. Combinatorial means all objects are small. Cellular means it's cofibrantly generated, the (co)domains of I are compact, the domains of J are small relative to I, and the cofibrations are contained in the effective monomorphisms (i.e. maps $f : X \to Y$ such that $X \to Y \rightrightarrows Y \coprod_X Y$ is an equalizer). We will assume $\mathcal{M}$ is left proper, but we need not assume cellular or combinatorial; only that the Bousfield localization in question exists.

The identity maps $\mathcal{M} \overset{\rightarrow}{\leftarrow} L_C\mathcal{M}$ are a Quillen adjoint pair and prove that $L_C\mathcal{M}$ satisfies a universal property as the "closest" model category to $\mathcal{M}$ in which C is contained in the weak equivalences. The fibrant objects in $L_C\mathcal{M}$ are the C-local objects, and local equivalences between local objects are weak equivalences in the original model category. Bousfield localization gives a Quillen pair (L_C, U_C), which are both the identity functors on objects and morphisms, and these induce (L_C^H, U_C^H) on the homotopy level.

Our goal is to find conditions on $\mathcal{M}$ and C under which Bousfield localization *preserves* $\mathbb{P}$*-algebra structure*, so let's define this notion. On the model category level the functor the Bousfield localization is the identity functor. So when we write L_C as a functor we shall mean the composition of derived functors $\mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(L_C(\mathcal{M})) \to \mathrm{Ho}(\mathcal{M})$, i.e. $E \to L_C(E)$ is the unit map of the adjunction $\mathrm{Ho}(\mathcal{M}) \leftrightarrows \mathrm{Ho}(L_C(\mathcal{M}))$. In particular, for any E in $\mathcal{M}$, $L_C(E)$ is weakly equivalent to $R_C QE$ where R_C is a choice of fibrant replacement in $L_C(\mathcal{M})$ and Q is a cofibrant replacement in $\mathcal{M}$.

Let T be a monad on $\mathcal{M}$. Because the objects of $L_C(\mathcal{M})$ are the same as the objects of $\mathcal{M}$, T is also valued in $L_C(\mathcal{M})$. Thus, we may consider T-algebras in both categories and these classes of objects agree (because the T-algebra action is independent of the model structure). We denote the categories of T-algebras by T-alg$(\mathcal{M})$ and T-alg$(L_C(\mathcal{M}))$. These are identical as categories, but in a moment they will receive different model structures.

Definition 5.1. Assume that $\mathcal{M}$ and $L_C(\mathcal{M})$ are monoidal model categories. Then L_C is said to *preserve T-algebras* if

(1) When E is a T-algebra there is some T-algebra $\widetilde{E}$ which is weakly equivalent in $\mathcal{M}$ to $L_C(E)$.

(2) In addition, when E is a cofibrant T-algebra, then there is a choice of $\widetilde{E}$ and a lift of the localization map $E \to L_C(E)$ to a T-algebra homomorphism $E \to \widetilde{E}$.

Cofibrancy as a T-algebra means it is constructed via transfinite composition and pushout of cells $T(I)$.

We will use the fact that Bousfield localization works via the derived functors of the identity, so $L_C(E)$ is $R_C QE$ where R_C be fibrant replacement in $L_C(M)$.

Results related to localization have appeared as **arXiv:1404.5197**

Theorem 5.2. *Let $\mathcal{M}$ be a monoidal model category and let T be a monad valued in $\mathcal{M}$. If T-algebras in $\mathcal{M}$ and in $L_C(\mathcal{M})$ inherit (semi) model structures from $\mathcal{M}$ and $L_C(\mathcal{M})$, then L_C preserves T-algebras up to weak equivalence.*

Recall: inherit means weak equivalences and fibrations in T-alg$(\mathcal{M})$ come from $\mathcal{M}$.

Proof. Fact: $L_C(E) \simeq R_C QE$. We will show $R_C QE \simeq R_{C,T} Q_T E$ where subscript means replacement in T-alg. We need a map between them:

$$\begin{tikzcd}
QE \arrow[rr, "\simeq"] \arrow[dd] & & Q_T E \arrow[dl] \arrow[dd] \\
& R_C Q_T E \arrow[dr, "\simeq_C"] & \\
R_C QE \arrow[ur, "\therefore\simeq_C"] \arrow[rr, "\therefore\simeq_C"] & & R_{C,T} Q_T E
\end{tikzcd}$$

To start, we need $QE \to Q_T E$. We'll use lifting:

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, hook] & Q_T E \arrow[d, two heads, "\simeq"] \\
QE \arrow[r, "\simeq"'] \arrow[ur, dotted] & E
\end{tikzcd}$$

The lift is a weak equivalence in $\mathcal{M}$ by the 2 out of 3 property. When we apply R_C to this map we get a C-local equivalence. Now we construct the last map:

$$\begin{tikzcd}
Q_T E \arrow[r, "\simeq_C"] \arrow[d, hook, "\simeq_C"'] & R_{C,T} Q_T E \arrow[d, two heads] \\
R_C Q_T E \arrow[r] \arrow[ur, dotted, "\therefore\simeq_C"] & *
\end{tikzcd}$$

In this diagram the left vertical map is a weak equivalence in $L_C\mathcal{M}$ and the top map is a weak equivalence in $P - alg(L_C\mathcal{M})$. Because this model category $T - alg(L_C\mathcal{M})$ inherits weak equivalences from $L_C\mathcal{M}$ this map is a weak equivalence in $L_C\mathcal{M}$. Therefore, by the 2 out of 3 property, the lift is a weak equivalence in $L_C\mathcal{M}$.

By the 2 out of 3 property, $R_C QE \to R_{C,T} Q_T E$ is a weak equivalence in $L_C\mathcal{M}$. All the objects in the triangle are fibrant in $L_C\mathcal{M}$ so these C-local equivalences are actually weak equivalences in $\mathcal{M}$. ◻

This proof also holds if T-algebras only form a semi-model category. In a semi-model category all objects admit cofibrant replacement, but only cofibrant objects admit fibrant replacement. Lifting of a trivial cofibration against a fibration only holds if the domain is cofibrant. Everywhere we've applied fibrant replacement it's been to an object which is cofibrant in the underlying category, so that's no problem. The lifting argument is for a map which has cofibrant domain, so that's fine too. Thus, even if the monoid axiom fails to hold in $L_C\mathcal{M}$ we can still say T-alg is a semi-model category.

We define semi-model categories following Spitzweck's thesis. Formally, a **semi-model category** is a bicomplete category $\mathcal{D}$, an adjunction $F : \mathcal{M} \leftrightarrows \mathcal{D} : U$ where $\mathcal{M}$ is a model category, and subcategories of weak equivalences, fibrations, and cofibrations in $\mathcal{D}$ satisfying the following axioms:

(1) U preserves fibrations and trivial fibrations.

(2) $\mathcal{D}$ satisfies the two out of three axiom and the retract axiom.

(3) Cofibrations in $\mathcal{D}$ have the left lifting property with respect to trivial fibrations. Trivial cofibrations in $\mathcal{D}$ whose domain is cofibrant have the left lifting property with respect to fibrations.

(4) Every map in $\mathcal{D}$ can be functorially factored into a cofibration followed by a trivial fibration. Every map in $\mathcal{D}$ whose domain is cofibrant can be functorially factored into a trivial cofibration followed by a fibration.

(5) The initial object in $\mathcal{D}$ is cofibrant.

(6) Fibrations and trivial fibrations are closed under pullback.

Note: all objects admit cofibrant replacement, but only cofibrant objects admit fibrant replacement. So that's why the proof above works. It's a bit unfair to just assume P-algebras form a semi model category. After all, it can be very difficult to get your hands on $L_C\mathcal{M}$. We'd rather have hypotheses on $\mathcal{M}$ and C to make sure this situation happens. That's what we'll discuss in the next talk, along with applications.

6. Talk 2

We begin with some pre-theorems about Semi-Model Categories (i.e. these are in-progress and should be taken with a grain of salt).

Theorem 6.1. *If $\mathcal{M}$ is locally presentable and has a left properness condition that cofibrations are contained in the h-cofibrations, then every cofibrantly generated semi-model category is Quillen equivalent to a model category.*

Conjecture: making it work without left properness via Ching-Riehl

Theorem 6.2. *If $\mathcal{M}$ is a combinatorial semi-model category then you can do Bousfield localization without left properness and still get a semi-model category.*

Current work: getting the universal property to hold.

Conjecture: making it work for cellular instead of combinatorial.

Current work: re-do Michael's work relating T-alg($L_C M$) and L_{TC}(T-alg(M)) in semi-model category fashion, i.e. only assuming these categories carry semi-model structures prove they are Quillen equivalent as semi-model categories.

7. Putting (semi) model structures on Eilenberg-Moore categories

Recall that $\mathcal{M}$ is a monoidal model category and T is a monad on $\mathcal{M}$. We need a (semi) model structure on T-alg and we're going to try to transfer it along $T : \mathcal{M} \leftrightarrows T - alg(\mathcal{M}) : U$. It's not always true that the model structure on $\mathcal{M}$ can be passed across this adjunction. Sometimes it can. At the bare minimum we need $\mathcal{M}$ to be cofibrantly generated, and if the generating maps are I and J then the generators for T-alg are $T(I)$ and $T(J)$. Let's work through an example to see what kind of hypotheses are needed on $\mathcal{M}$ and T for this to work. Consider the following general lemma from Schwede-Shipley (related of course to Kan's principle for recognizing cofibrantly generated model structures, and to Crans's work on transfer):

Lemma 7.1. *Suppose $\mathcal{M}$ is cofibrantly generated and T is a monad which commutes with filtered direct limits. If the domains of $T(I)$ and $T(J)$ are small relative to $T(I)$-cell and $T(J)$-cell respectively and EITHER*

(1) $T(J)-cell \subset \mathscr{W}$, or

(2) All objects are fibrant and every T-algebra has a path object (factoring $\delta : X \to X \otimes X$ into $\overset{\simeq}{\hookrightarrow}\twoheadrightarrow$)

then T-alg inherits a cofibrantly generated model structure with fibrations and weak equivalences created by the forgetful functor to $\mathcal{M}$.

One half of lifting comes for free, 2 out of 3 and retracts are inherited from $\mathcal{M}$, so only factorization must be proven. If T preserves smallness then the small object argument is used to get the generators above and to get cofibration-trivial fibration factorization. For the other factorization axiom we need to know that when every homomorphism p which is a transfinite composition of pushouts of coproducts of maps of the form $T(f)$ where f is a trivial cofibration in $\mathcal{M}$ has p being a weak equivalence in $\mathcal{M}$ (hence in $T - alg(\mathcal{M})$). Once you have this you get the other half of lifting by the retract argument.

There are also other general transfer principles more recent than Schwede-Shipley's work:

Theorem 7.2 (Fresse's Transfer)**.** *Suppose we have an adjunction $F : X \leftrightarrows A : U$ where A is bicomplete and X is cofibrantly generated. Suppose*

(1) U preserves colimits over non-empty ordinals

(2) Any pushout of $A \leftarrow F(K) \to F(L)$ for A an X-cofibrant $F(cof)$-cell complex has $U(f)$ a trivial cofibration whenever i is a trivial cofibration.

(3) $UF(\varnothing)$ is cofibrant.

then A inherits a cofibrantly generated semi-model structure.

Theorem 7.3 (Johnson-Yau Transfer)**.** *Suppose $\mathcal{M}$ is strongly cofibrantly generated and T is a monad such that*

(1) Alg_T is bicomplete and U preserves filtered colimits.

(2) $\mathcal{M}$ has a fibrant replacement functor which is compatible with T (i.e. there's a natural transformation $\tau : TR \to RT$ compatible with the unit so $\tau \circ \eta_{RA} = R\eta_A$ and multiplication so $\tau \circ \mu_{RA} = Q\mu_A \circ \tau_{TA} \circ T\tau_A$).

(3) $\mathcal{M}$ has a path object functor compatible with T (so $s : Id \to Path, d_i : Path \to Id$, and Path(A) factors the diagonal).

Then T-alg is cofibrantly generated and the Eilenberg-Moore adjunction is a strong Quillen pair.

We won't have need of these principles in this talk, but we believe they may be useful to work in the generality of monads not arising from operads.

Schwede-Shipley prove that the extra condition in their lemma can be deduced if every object of $\mathcal{M}$ is fibrant and if every T-algebra has a path object (using the retract argument). A great deal of this theory has been worked out in the case where all objects are fibrant by Berger and Moerdijk. Of course, this fails in sSet and all the categories of spectra so I'm more interested in the other approach. Let's work out an example:

The simplest P is Ass. In that case the free algebra functor is $T(X) = S \wedge X \wedge X^2 \wedge \dots$. If we have a trivial cofibration $f : K \to L$ then applying this functor gives $T(K) \to T(L)$ and we need to look at pushouts of this map in the category of monoids: $X \leftarrow T(K) \to T(L)$. Call the pushout P

The trick is to factor $X \to P$ as $X = P_0 \to P_1 \to \dots$. Because of the structure of T we can define each map $P_{n-1} \to P_n$ inductively. Let Q_n denote the colimit of the punctured n-dimensional cube with vertices $X \wedge K \wedge X \wedge K \wedge \dots X$ and with varying numbers and placements of L's. Then we have

$$\begin{tikzcd} Q_n \arrow[r] \arrow[d] & (X \wedge L)^n \wedge X \arrow[d] \\ P_{n-1} \arrow[r] & P_n \end{tikzcd}$$

We can then shuffle the X's to the side and we see that exactly the condition needed on $\mathcal{M}$ for this argument to work is the following: $(\mathcal{M} \wedge TrCof)-cell \subset \mathscr{W}$. The elements in this collection of maps are $Z \wedge f$ where Z is an object of $\mathcal{M}$ and f is a trivial cofibration. Applying cell means taking transfinite compositions of pushouts. Indeed, only countable transfinite compositions are necessary.

8. The case of commutative monoids

Let's talk about when commutative monoids inherit a model structure. For monoids this is done by Schwede-Shipley and the hypothesis needed on $\mathcal{M}$ is the *monoid axiom*, which says that for all objects X, $(id_X \otimes (\mathscr{Q} \cap \mathscr{W}))-cell \subset \mathscr{W}$. Here applying cell to a class of maps means taking its closure under transfinite compositions and pushouts. For commutative monoids the correct hypothesis is the *commutative monoid axiom*: If g is a (trivial) cofibration then $g^{\square n}/\Sigma_n$ is a (trivial) cofibration. The results of this and the next section are covered in my paper on commutative monoids: arXiv 1403.6759.

Theorem 8.1. *If a monoidal model category satisfies the monoid axiom and the commutative monoid axiom then commutative monoids form a model category and the forgetful functor is right Quillen.*

Proof. This goes basically the same way as the SS00 result. Now we use the functor $Sym(X) = S \wedge X \wedge X^2/\Sigma_2 \wedge \dots$. Again we take a pushout of $Sym(K) \to Sym(L)$ in the category of commutative monoids and again we factor $X \to P$ into a transfinite composition. Letting $\mathrm{Sym}^n(L;K)$ denote the colimit of the punctured cube defined by n-length products of L and K, we see that the pushout in question is $X = P_0 \to P_1 \to \cdots \to P$ where $P_{n-1} \to P_n$ is defined by

$$\begin{tikzcd}
X \otimes \mathrm{Sym}^n(L;K) \arrow[r] \arrow[d] & X \otimes \mathrm{Sym}^n(L) \arrow[d] \\
P_{n-1} \arrow[r] \arrow[ur, phantom, "\Searrow"] & P_n
\end{tikzcd}$$

The commutative monoid axiom ensures us that the part of this map after the $X \otimes -$ is a trivial cofibration. The monoid axiom ensures us that taking transfinite compositions and pushouts do not ruin this. □

This result generalizes a theorem of Lurie's from DAGIII, i.e. my hypothesis is weaker. Lurie's hypothesis is that for all (trivial) cofibrations f, $f^{\square n}$ is a (trivial) cofibration in the projective model structure on $\mathcal{M}^{\Sigma_n}$, i.e. objects and morphisms of $\mathcal{M}$ together with a Σ_n action.

Examples:

(1) Ch(k) where $char(k) = 0$. Lurie had this too. More generally, can get any $\mathbb{Q}$-algebra

(2) sSet - this fails Lurie's hypothesis. My proof uses the fact that cofibrations are monomorphisms to get the bit about cofibrations. For the weak equivalences part we rely on a clever trick of Dror Farjoun.

(3) Positive (Flat) model structure on symmetric spectra. Lurie doesn't apply here. He acknowledges his error in DAGIII 4.3.25 in Math Overflow post 146438. My proof needed a technical lemma that it was sufficient to check the commutative monoid axiom on the generators. Luis Pereira proved the same for Lurie's hypothesis

(4) Top - this fails for Lurie. It works for me because the proof of Farjoun generalizes to any Cartesian concrete category, and with a bit more care we don't need cofibrations to be monomorphisms either, because we have our hands on the generators.

(5) Positive orthogonal (equivariant) spectra - using again that it's sufficient to check it on the generators

(6) Positive motivic symmetric spectra - I'm developing this category with Markus Spitzweck.

If we drop the monoid axiom we only get a semi-model structure on *Com*-alg, but that is enough for preservation by localization.

9. Generalizing to other operads

The model structure on the category of operads is obtained via the transfer principle applied to the adjunction $F : Coll(\mathcal{M}) \leftrightarrow Op(\mathcal{M}) : U$ where $Coll(\mathcal{M}) = \Pi\, \mathcal{M}^{\Sigma_n}$ is the category of collections. This transfer doesn't always work, but even if operads don't form a model category you can still talk about Σ-cofibrant operads as operads which are cofibrant as collections. Even more generally you can talk about operads whose underlying collection is cofibrant. Even more generally there are levelwise cofibrant operads.

Examples:

Ass is Σ-cofibrant, A_∞ is cofibrant

Com is not Σ-cofibrant, though it is levelwise cofibrant if the monoidal unit is cofibrant. Any E_∞ operad is a Σ-cofibrant replacement. Morally this is good enough to be a "cofibrant replacement" for *Com*. If you want an honestly cofibrant operad you need to use the Fulton MacPherson operad. The algebras over all E_∞ operads are Quillen equivalent because any two homotopy equivalent Σ-cofibrant operads have Quillen equivalent categories of algebras.

A recurring theme in this talk will be that **there is a cofibrancy price to pay** in order to pass this model structure across this adjunction. For example, consider the following theorem of Spitzweck:

Theorem 9.1. *Suppose P is a Σ-cofibrant operad and $\mathcal{M}$ is a monoidal model category. Then P-alg is a semi-model category which is a model category if P is cofibrant and $\mathcal{M}$ satisfies the monoid axiom.*

This is proven in his thesis, using a certain filtration based on trees which I would love to understand better (does anyone want to read this paper with me?). As usual, you analyze the pushout $X \leftarrow P(K) \rightarrow P(L)$ in the category of P-algebras by breaking it down into a transfinite composition $X = X_0 \rightarrow X_1 \rightarrow \dots$ and observing that the cofibrancy hypothesis on P lets you control the form the maps $X_i \rightarrow X_{i+1}$ take. So if X is cofibrant to start then each such map is a trivial cofibration and so the composite is as well. Hence, you get a semi-model structure (not a full model structure because you needed X to be cofibrant).

A (possibly) different filtration may be found in John Harper's work. The proof proceeds in the same way, but now the filtration involves a new object (factoring the information of both P and the algebra X) which we must define.

Proposition 9.2. *Let $\mathcal{O}$ be a Σ-operad, $A \in \mathcal{O}-alg$, and $Y \in \mathcal{M}$. Consider any coproduct in $\mathcal{O}-alg$ of the form*

$$A \amalg (\mathcal{O} \circ Y). \tag{9.1}$$

There exists a symmetric sequence $\mathcal{O}_A$ and natural isomorphisms

$$A \amalg (\mathcal{O} \circ Y) \cong \mathcal{O}_A \circ Y = \coprod_{q \geq 0} \mathcal{O}_A[q] \otimes_{\Sigma_q} Y^{\otimes q}$$

in the underlying category $\mathcal{M}$. If $q \geq 0$, then $\mathcal{O}_A[q]$ is naturally isomorphic to a colimit of the form

$$\mathcal{O}_A[q] \cong \operatorname{colim}\Big(\coprod_{p\geq 0} \mathcal{O}[p+q] \otimes_{\Sigma_p} A^{\otimes p} \underset{d_1}{\overset{d_0}{\leftleftarrows}} \coprod_{p\geq 0} \mathcal{O}[p+q] \otimes_{\Sigma_p} (\mathcal{O} \circ A)^{\otimes p} \Big),$$

in $\mathcal{M}$, with d_0 induced by operad multiplication and d_1 induced by $m : \mathcal{O} \circ A \rightarrow A$.

It follows that

$$A \amalg (\mathcal{O} \circ Y) \cong \operatorname{colim}\Big((\mathcal{O} \circ A) \amalg (\mathcal{O} \circ Y) \underset{d_1}{\overset{d_0}{\leftleftarrows}} (\mathcal{O} \circ \mathcal{O} \circ A) \amalg (\mathcal{O} \circ Y) \Big).$$

With a bit more work, one can prove that the colimit in $\mathcal{O}$-alg:

$$\begin{tikzcd} \mathcal{O} \circ X \arrow[r, "f"] \arrow[d, "id \circ i"] & A \arrow[d, "j"] \\ \mathcal{O} \circ Y \arrow[r] & A \amalg_{(\mathcal{O} \circ X)} (\mathcal{O} \circ Y). \end{tikzcd} \tag{9.2}$$

is naturally isomorphic to a filtered colimit $A_0 \to A_1 \to \dots$ in the underlying category $\mathcal{M}$, with $A_0 := \mathcal{O}_A[0] \cong A$ and A_t defined inductively by pushout diagrams in $\mathcal{M}$ of the form

$$\begin{tikzcd} \mathcal{O}_A[t] \otimes_{\Sigma_t} Q^t_{t-1} \arrow[r, "f_*"] \arrow[d, "id \otimes_{\Sigma_t} i_*"] & A_{t-1} \arrow[d, "j_t"] \\ \mathcal{O}_A[t] \otimes_{\Sigma_t} Y^{\otimes t} \arrow[r, "\xi_t"] & A_t \end{tikzcd} \tag{9.3}$$

Harper uses this to prove:

Theorem 9.3. *Suppose $\mathcal{M}$ is a model category such that all symmetric sequences in $\mathcal{M}$ are projectively cofibrant. Assume the requisite smallness for domains of I and J so that the transfer principle can work. Then for any operad P, P-alg inherits a model structure from $\mathcal{M}$.*

As far as I can tell, only Ch(k) for char(k)=0 satisfies this hypothesis.

A careful reading of his arguments demonstrates the minimum hypothesis needed on $\mathcal{M}$ so that a given operad P is admissible. Let $\mathscr{J}$ denote the class of trivial cofibrations of $\mathcal{M}$ and let $\mathscr{J}^{\square n}$ denote the class of maps $j^{\square n}$ where $j \in \mathscr{J}$.

$\mathcal{M}$ satisfies the *P-algebra axiom* if for all P-algebras A and for all n, transfinite compositions of pushouts of maps in $P_A[n] \otimes_{\Sigma_n} \mathscr{J}^{\square n}$ are weak equivalences.

An easier condition to check in principle would be the condition that maps in $P_A[n] \otimes_{\Sigma_n} \mathscr{J}^{\square n}$ are trivial cofibrations, and this is what Harper's hypothesis in the theorem above implies.

We seek conditions to check on $\mathcal{M}$ which do not require indexing over all P-algebras or using the mysterious sequence P_A. The commutative monoid axiom generalizes to give such a family of axioms. We saw already that if P is cofibrant then basically no hypotheses are needed on $\mathcal{M}$ to get admissibility. Harper's theorem shows a strong cofibrancy hypothesis on $\mathcal{M}$ can imply that all operads are admissible. My result below shows that you can pay the cofibrancy price partially on $\mathcal{M}$ and partially on P, e.g. to get levelwise cofibrant P you need for all $X \in \mathcal{M}^{\Sigma_n}$ which are cofibrant in $\mathcal{M}$ one has $X \otimes_{\Sigma_n} f^{\square n}$ is a trivial cofibration. There is also a generalized version of the regular monoid axiom, which requires that applying cell to a certain class of maps results in a weak equivalence.

Theorem 9.4. *Let $\mathcal{M}$ be a cofibrantly generated monoidal model category. Let f run through the class of (trivial) cofibrations. In each row of the following table, placing the hypotheses in the first column on M gives a semi-model structure on P-algebras for all P satisfying the hypotheses in the second column.*

The hypotheses going down the first column are cumulative, e.g. the last row says that if $\mathcal{M}$ is cofibrantly generated, monoidal, satisfies the monoid axiom, and has the property that $\forall X \in \mathcal{M}^{\Sigma_n}$, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration, then all operads are admissible.

Hypothesis on $\mathcal{M}$	Class of operad
$\forall X \in \mathcal{M}^{\Sigma_n}$ projectively cofibrant, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration (this follows from the pushout product axiom)	Σ-Cofibrant
$\forall X \in \mathcal{M}^{\Sigma_n}$ cofibrant in $\mathcal{M}$, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration Note: $X = *$ is the commutative monoid axiom	Levelwise cofibrant Special case: $P = \mathrm{Com}$
$\forall X \in \mathcal{M}^{\Sigma_n}$, $X \otimes_{\Sigma_n} f^{\square n}$ is a (trivial) cofibration	Arbitrary

The proof works the same as what we've seen, but now we break the pushout down into steps via $O_A[n] \otimes_{\Sigma_n} Q_n \to O_A[n] \otimes_{\Sigma_n} L^n$. These extra axioms ensure that this pushout works. They are satisfied by simplicial sets and Ch(k) at least and likely other places such as the positive flat model structures on spectra. For the positive flat model structure on symmetric spectra, all operads are admissible (i.e. their algebras form model categories).

10. Localization Results

For our applications T will be given by a (one-colored) operad which will be either Com or Σ-cofibrant. So we know the right axioms on $\mathcal{M}$ to ensure T-alg(M) inherits a model structure. We turn now to finding hypotheses on the maps C so that T-alg($L_C(\mathcal{M})$) also inherits a model structure.

For Σ-cofibrant operads, Spitzweck's theorem (applied to $L_C(\mathcal{M})$) implies our preservation result as soon as the pushout product axiom passes from $\mathcal{M}$ to $L_C(\mathcal{M})$. So we make a definition:

Definition 10.1. L_C is said to be a **monoidal Bousfield localization** if $L_C(\mathcal{M})$ satisfies the pushout product axiom, the unit axiom, and the axiom that cofibrant objects are flat.

We can characterize when this occurs. First, we need a new axiom on the model category:

A common strengthening of the unit axiom is the Resolution Axiom, which states that cofibrant objects are flat, i.e. whenever $f \in \mathscr{W}$ and X is cofibrant, then $X \otimes f \in \mathscr{W}$.

For simplicity we'll also assume $\mathcal{M}$ is tractable, meaning the domains of the generators I are cofibrant (and not meaning that $\mathcal{M}$ is combinatorial). There is also a version of our theorem without this hypothesis. We characterize monoidal localizations:

Theorem 10.2. *Suppose $\mathcal{M}$ is a tractable monoidal model category in which the Resolution Axiom holds. Let I denote the generating cofibrations of $\mathcal{M}$. Then L_C is a monoidal Bousfield localization if and only if every map of the form $f \otimes id_K$, where f is in C and K is a domain or codomain of a map in I, is a C-local equivalence.*

Note: given a set of maps C this theorem tells us what we must do to C in order to ensure that $L_C(\mathcal{M})$ is a monoidal model category. We must replace C by $C' = \{C \otimes K\}$. This is called the **smallest monoidal Bousfield localization** inverting C and was used in my joint paper with Hovey, which is at **arXiv:1312.3846**.

Already this is enough to resolve the question we began with for equivariant spectra, since G-equivariant commutativity is encoded by a cofibrant operad E_∞^G. Blumberg-Hill 2013 prove this operad rectifies to Com, so preservation of E_∞^G-algebras implies preservation of Com-algebras too. However, for completeness (and for applications to categories where this rectification fails such as sSet) we include results about the preservation of the commutative monoid axiom. Recall that commutative monoids are built via the functor $Sym(X) = S \wedge X \wedge X^2/\Sigma_2 \wedge \dots$. For monoidal structure we needed localization to play well with tensoring. Now we'll need it to work with Sym:

Theorem 10.3. *Suppose $\mathcal{M}$ is a simplicial model category satisfying the commutative monoid axiom. If $Sym(-)$ preserves weak equivalences in $L_C(\mathcal{M})$ then $L_C(\mathcal{M})$ satisfies the commutative monoid axiom.*

I am hoping to remove the hypothesis about 'simplicial' as soon as I can. Finally, we have a result about the monoid axiom on $L_C(\mathcal{M})$ even though we don't need it for the general preservation theorem to apply (because semi-model categories are fine for us):

Theorem 10.4. *Suppose $\mathcal{M}$ is a tractable, left proper, h-monoidal model category such that the (co)domains of I are finite relative to the h-cofibrations and cofibrant objects are flat. Then for any monoidal Bousfield localization L_C, the model category $L_C(\mathcal{M})$ satisfies the monoid axiom.*

You might think that all Bousfield localizations are monoidal. This is not true:

10.1. **Non-Example.** Thus, we have characterized monoidal localizations and there are examples of localizations which fail to be monoidal, e.g. in $Ch(R[G])$ when a localization kills a representation sphere.

Let $R = \mathbb{F}_2[\Sigma_3]$. An R module is simply an $\mathbb{F}_2$ vector space with an action of the symmetric group Σ_3. Because R is a Frobenius ring, we may pass from R-mod to the stable module category $StMod(R)$ by identifying any two morphisms whose difference factors through a projective module. Let $\mathcal{M}$ be the corresponding model category, discussed in Subsection **??**.

Proposition 4.2.15 of [**?**] proves that for $R = \mathbb{F}_2[\Sigma_3]$, this model category is a monoidal model category because R is a Hopf algebra over $\mathbb{F}_2$. The monoidal product of two R-modules is $M \otimes_{\mathbb{F}_2} N$ where R acts via its diagonal $R \to R \otimes_{\mathbb{F}_2} R$.

We now check that cofibrant objects are flat in $\mathcal{M}$. By the pushout product axiom, $X \otimes -$ is left Quillen. Since all objects are cofibrant, all weak equivalences are weak equivalences between cofibrant objects. So Ken Brown's lemma implies $X \otimes -$ preserves weak equivalences.

Let $f : 0 \to \mathbb{F}_2$, where the codomain has the trivial Σ_3 action. We'll show that the Bousfield localization with respect to f cannot be a monoidal Bousfield localization. First observe that being f-locally trivial is equivalent to having no Σ_3-fixed points, and this is equivalent to failing to admit Σ_3-equivariant maps from $\mathbb{F}_2$ (the non-identity element would need to be taken to a Σ_3-fixed point because the Σ_3-action on $\mathbb{F}_2$ is trivial).

If the pushout product axiom held in $L_f(\mathcal{M})$ then the pushout product of two f-locally trivial cofibrations g, h would have to be f-locally trivial. We will now demonstrate an f-locally trivial object N for which $N \otimes_{\mathbb{F}_2} N$ is not f-locally trivial, so $(\varnothing \to N) \square (\varnothing \to N)$ is not a trivial cofibration in $L_f(\mathcal{M})$.

Define $N \cong \mathbb{F}_2 \oplus \mathbb{F}_2$ where the element (12) sends $a = (1, 0)$ to $b = (0, 1)$ and the element (123) sends a to b and b to $c = a + b$. The reader can check that $(12)(123)$ acts the same as $(123)^2(12)$, so that this is a well-defined Σ_3-action. This object N is f-locally trivial. It does not admit any maps from $\mathbb{F}_2$ because it has no Σ_3-fixed points. However, $N \otimes_{\mathbb{F}_2} N$ is not f-locally trivial because $N \otimes_{\mathbb{F}_2} N$ does admit a map from $\mathbb{F}_2$ which takes the non-identity element of $\mathbb{F}_2$ to the Σ_3-invariant element $a \otimes a + b \otimes b + c \otimes c$. Thus, $L_f(\mathcal{M})$ is not a monoidal model category.

11. Applications

In Ch(k) for a field k the only Bousfield localizations are the truncations (which are all nullifications in this case). So all of them are monoidal. If char(k)=0 then Quillen's homotopical algebra proves all preserve the commutative monoid axiom.

In sSet and Top all Bousfield localizations are monoidal. This was known previously for spaces with the homotopy type of CW complexes (by an argument of Farjoun), but now holds for k-spaces. Furthermore, all Bousfield localizations respect Sym and commutative monoids are preserved.

Combining this with our general preservation result gives:

Corollary 11.1. *Truncations in sSet, Top, and Ch(k) all preserve strict commutative monoids. Via Farjoun's trick, any monoidal localization in sSet will also preserve, e.g.* L_E *for a homology theory E.*

Connected commutative topological monoids are products of Eilenberg-Mac Lane spaces. So this bit about spaces may recover some classical unstable results of Bousfield.

In Spectra a localization is monoidal iff it is stable, i.e. $L \circ \Sigma \simeq \Sigma \circ L$. Let's turn now to commutative monoids in spectra...

Gaunce Lewis: if commutative monoids formed a model category then taking the zeroth space of the cofibrant replacement of the sphere in that category would give such a space, but this implies there are no homotopy operations present in the stable homotopy groups of spheres, contradiction.

Moving to the positive model structure fixes this (by breaking the cofibrancy of the unit). When two operads O and P have the property that their categories of algebras are Quillen equivalent then rectification is said to occur (e.g. P rectifies to O). Rectification occurs here because symmetric powers are weakly equivalent to homotopy symmetric powers, i.e. the smash product can't see the difference between the free algebra functors over these two operads. Different choices for E_∞ make no difference to the resulting algebras, so we'll choose the E_∞ operad with $E_\infty[n] = E\Sigma_n$.

Theorem 10.3 gives a condition to check so that the Bousfield localization respects this type of structure. Some applications are given in my thesis.

Let's finally return to G-spectra. Genuine E_∞ structure is encoded by any operad P where $P(n)$ is an $E_G\Sigma_n$, i.e. a space with a $G \times \Sigma_n$-action which is characterized up to $G \times \Sigma_n$-weak equivalence by the property that for $H < G \times \Sigma_n$, we have $(E_G\Sigma_n)^H = \emptyset$ if $H \cap \Sigma_n \neq \{e\}$ and $(E_G\Sigma_n)^H \simeq *$ otherwise. This space $E_G\Sigma_n$ can be defined as the total space of the universal G-equivariant principle Σ_n-bundle.

Theorem 11.2. *In* $\mathscr{S}^G$ *a Bousfield localization is monoidal iff* $C \otimes (G/H)_+$ *is a* C*-local equivalence for all* H.

This is a much easier to check hypothesis than the one in Hill's patching theorem, and it's also an iff so it's best possible. We see where our opening example failed: applying $C \otimes -$ killed all G/H and we would have ended up inverting the zero map if the condition of the corollary was satisfied.

Furthermore, we have developed a theory of what happens when C respects some but not all of the subgroups. A family of subgroups is closed under conjugation and passage to subgroup.

Recall that for every family of subgroups $\mathscr{F}$ of G there is a model structure on Top^G where weak equivalences and fibrations are maps such that $(-)^H$ is again such a map in Top for all $H \in \mathscr{F}$. For each family there is a universal $\mathscr{F}$-space EF which is a G-CW complex such that $(EF)^H$ is contractible for $H \in \mathscr{F}$ and empty otherwise. The family model structure on Top has generating cofibrations $(G/H \times S^{n-1})_+ \to (G/H \times D^n)_+$ for all n and all $H \in \mathscr{F}$, and the analogous generating trivial cofibrations. These family model structures are also present in $\mathscr{S}^G$, and can be defined by similarly changing the generating (trivial) cofibrations.

For any family $\mathscr{F}$ of subgroups of G there is a collection $E_{\mathscr{F}}\Sigma_n$ whose n^{th} space is the total space of a universal $\mathscr{F}$-equivariant principle Σ_n-bundle. When using the family model structure on G-spaces this becomes a cofibrant collection, and it's equivalent as a collection to an Blumberg-Hill N_∞ operad because of the universal property of the n^{th} space. We may thus introduce operads $E_\infty^{\mathscr{F}}$ which interpolate between naive E_∞ (i.e. if you forget the G-action) and genuine E_∞ (which HHR worked with). Such algebras have multiplicative 'up through $\mathscr{F}$' but not necessarily above.

Theorem 11.3. *The category of simplicial (resp. topological) G-operads can be given a model structure via transfer from the category of collections on G-spaces. Neither Com nor the naive E_∞ operads are cofibrant. Their Σ-cofibrant replacement E_∞^G can be described by $E_\infty^G[n] = E_G\Sigma_n$.*

The general preservation result says algebras over such operads are preserved precisely when $C \wedge (G/H)_+$ is contained in the C-local equivalences for all $H \in \mathscr{F}$. It is also possible for localization to reduce the amount of structure an algebra has (from some $E_\infty^{\mathscr{F}}$ to some other $E_\infty^{\mathscr{F}'}$) and one can generalize Hill's example to force this to occur. Pictorially:

$$
\begin{tikzcd}
E_\infty^G \arrow[r, "L"] \arrow[d, dash] \arrow[dr] \arrow[dddr] \arrow[ddddr] & E_\infty^G \arrow[d, dash] \\
E_\infty^{\mathscr{F}} \arrow[r] \arrow[d, dash] \arrow[dddr] & E_\infty^{\mathscr{F}} \arrow[d, dash] \\
\cdots \arrow[d, dash] & \cdots \arrow[d, dash] \\
E_\infty^{\mathscr{F}'} \arrow[r] \arrow[d, dash] \arrow[dr] & E_\infty^{\mathscr{F}'} \arrow[d, dash] \\
E_\infty \arrow[r] & E_\infty
\end{tikzcd}
$$

12. Future Work

I intend to study the properties of the operads $E_\infty^{\mathscr{F}}$ further in joint work with Javier Gutierrez.

I hope to study when localization preserves the generalizations of the commutative monoid axiom to other non-cofibrant P. I am in the process of generalizing the semi-model category table to colored operads in joint work with Donald Yau.

Michael's machinery lets you pass to P-algebras then apply localization. Mine is localization then passage to P-algebras (because the requisite axioms are passed to $L_C(\mathcal{M})$). We intend to link up these two machines.

$$
\begin{tikzcd}
& \mathcal{M} \arrow[dl] \arrow[dr] & \\
L_C(\mathcal{M}) \arrow[d] & & P - alg(\mathcal{M}) \arrow[d] \\
P - alg(L_C(\mathcal{M})) \arrow[rr, leftrightarrow, dotted, "??"] & & L_{PC}(P - alg(\mathcal{M}))
\end{tikzcd}
$$

We believe the commutative monoid axiom implies a model structure on the category of symmetric (non-reduced) operads, and can be generalized to the setting of non-polynomial monads. So we hope to extend the results of the Batanin-Berger paper a bit into the realm of non-polynomial monads (note that polynomial monads correspond to Σ-cofibrant colored operads, so this type of extension is very much in line with my thesis).

We hope to better understand the connection between assuming T-alg$(L_C(\mathcal{M}))$ inherits a model structure, assuming L_C preserves T-algebras, and assuming L_C lifts to a localization L_{TC} on the level of algebras.