AN OVERVIEW OF LOCALIZATION IN STABLE HOMOTOPY THEORY

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THANK THE ORGANIZER, talk about keeping the exchange going (good for CV, practice in foreign environment), this is colloquium-style so questions are encouraged, but super specific things may get pushed off to the end.

I start by telling a bit about stable homotopy theory, to give a flavor of the world we'll be working in today. One of the key ideas in algebraic topology is to use algebraic invariants to distinguish topological spaces. This leads to H_*, H^*, and π_*. The first two are relatively easy to compute, but they are coarse invariants: they don't capture enough of the information of the space. The last is really too difficult to compute in practice. So Stable Homotopy Theory aims to compute something a bit simpler, namely π_*^s the stable homotopy groups. To define this we use the key theorem of stable homotopy theory.

1. Motivation for Stable Homotopy Theory

Homotopy can be thought of as "**throwing mud** at X and seeing what sticks" because $\pi_n(X) = [S^n, X]$. Stable homotopy is harder to explain.

Recall: The reduced suspension Σ functor (**DRAW THE PICTURE**). A space is n-**connected** if $\pi_k(X) = 0$ for all $1 \le k \le n$ (call it 0-connected if it's path connected).

Theorem 1 (Freudenthal Suspension Theorem (1937))**.** *Let X be an n-connected pointed space (a pointed CW-complex or pointed simplicial set). The inclusion $X \to \Sigma X$ induces a map*

$$\pi_k(X) \to \pi_{k+1}(\Sigma X)$$

which is an isomorphism if $k < 2n+1$ and an epimorphism if $k = 2n+1$. Note: the first half of these groups are zero, but they can be nonzero after $\pi_{n+1}(X) \to \pi_{n+2}(\Sigma X)$.

We see immediately that if X is n-connected, then ΣX is $n+1$-connected, since $\pi_{i+1}(\Sigma X) \cong \pi_i(X) = 0$ for all $i \le n$, i.e. for all $i + 1 \le n + 1$ (on the first subscript).

Now take X any path-connected space (same as 0-connected) and fix k. Starting with $n = 1$ and increasing n at each step gives:

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \pi_{k+3}(\Sigma^3 X) \dots$$

For $n > k + 1$ we have $\pi_{k+n}(\Sigma^n X) \cong \pi_{k+n+1}(\Sigma^{n+1} X)$, i.e. this chain of maps stabilizes. We can thus define: $\pi_k^s(X) = \lim_n \pi_{k+n}(\Sigma^n X)$. If X is the sphere S^0, i.e. 2 points, then this is called the stable k-stem, $\pi_k^s(S^0)$, and is MUCH easier to compute than the unstable groups because all that matters is the difference between n and k. Yet still the general pattern eludes us. Example: the identity on S^1 generates $\mathbb{Z} = \pi_1(S^1) \cong \pi_2(S^2) \cong \pi_3(S^3) \cong \pi_4(S^4) \dots$.

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Most of the modern theory looks at all the $\pi_k^s(S^0)$ simultaneously, because they form a graded ring (under composition). We now have an **algebraic object without a corresponding topological object** which gives it. The solution to this is to move from the category of topological spaces to a new category where the objects are "stable" and where the homotopy functor spits out this graded ring. This category is called **Spectra** (Lima, 1958), and its objects are sequences of spaces (X_n) with structure maps $\Sigma X_n \to X_{n+1}$. We'll talk more about categories later; for now, cool facts about Spectra!

Spaces sit inside Spectra as follows: Given X, define $(\Sigma^\infty X)_n$ to be $\Sigma^n X$. It's a spectrum.

Rings sit inside Spectra, too: Given R and $n \in \mathbb{N}$, the Eilenberg-Maclane space $K(R, n)$ has $\pi_n = R$ and $\pi_k = 0$ for all $k \neq n$. Define a spectrum $(HR)_n = (K(R, n))$.

Cohomology is a collection of functors $H^n(-; R)$:Spaces→GradedRings (with the cup product), like looking at X through R-tinted glasses. These satisfy a bunch of nice properties, and we can use these properties as Axioms (**Eilenberg-Steenrod Axioms (1945)**) to define when a functor is as good as H^n. Call such functors (h^n) generalized cohomology theories. It turns out these functors are representable (this is the **Brown Representability Theorem (1962)**), i.e. $h^n(X) \cong [X, E_n]$ for some space E_n and (E_n) forms a spectrum. For example, $H^n(-; R)$ is represented by the $K(R, n)$ above!

So Spectra is a category which contains all my favorite things: spaces, rings, and cohomology theories. My research is often of the flavor: take some **cool idea in algebra** and see how well it holds in this more general arena. You **need the right proof in algebra**, too, and then it takes work to push the proof forward. So instead of studying rings we study ring objects in this category ("ring spectra"), which act like rings (they are defined by commutative diagrams in the category). It's a fact that HR and h_n are ring spectra. We may define this term at the end. This is the subject of the area of research called Brave New Rings. Today's goal will be to talk about how to do localization here: in particular, can we localize one spectrum with respect to another? What does it mean to localize with respect to a homology theory?

2. Localization

To answer this, let's recall localization. It's a **systematic way of adding multiplicative inverses to a ring**, i.e. given commutative R and multiplicative $S \subset R$, localization constructs a ring $S^{-1}R$ and a ring homomorphism $j : R \to S^{-1}R$ that takes elements in S to units in $S^{-1}R$. It's universal w.r.t. this property, i.e. for any $f : R \to T$ taking S to units we have a unique g:

$$\begin{tikzcd} R \arrow[r, "j"] \arrow[d, "f"'] & S^{-1}R \arrow[dl, dashed, "g"] \\ T & \end{tikzcd}$$

Recall that $S^{-1}R$ is just $R \times S/\sim$ where (r, s) is really r/s and $\sim$ says you can reduce to lowest terms without leaving the equivalence class. Ring just as $\mathbb{Q}$ is. The map j takes $r \mapsto r/1$, and given f you can set $g(r/s) = f(r)f(s)^{-1}$.

Examples: $(\mathbb{Z}, \mathbb{Z} - \{0\}) \mapsto S^{-1}R = \mathbb{Q}$. $(\mathbb{Z}, \langle 2 \rangle) \mapsto \mathbb{Z}[\frac{1}{2}]$. $(\mathbb{Z}, \mathbb{Z} - p\mathbb{Z}) \mapsto \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \not| b\}$

This is NOT the right definition to a category theorist (no operation, so "multiplicative inverses?"). Better: **systematic way of formally inverting maps**. Let's focus on one map first. To invert s you take the ring generated by R and s^{-1}. It's equivalent to require that multiplication by s map $\mu_s : R \to R$ is invertible. Thus, define R_* to be a ring containing s such that μ_s is an isomorphism and $i : R \to R_*$ is universal. This is an example of finding the right proof in algebra to generalize to category theory. It's diagrammatic.

Proposition 1. $R_* \cong s^{-1}R$

Proof. Certainly $s \in s^{-1}R$ as $(s, 1)$. Also, μ_s is an isomorphism with inverse $\mu_{s^{-1}}$. So by the universal property of R_*, the map $j : R \to s^{-1}R$ gives $g : R_* \to s^{-1}R$ s.t. $g \circ i = j$

Next, the element s has an inverse in R_* because it's $\mu_s^{-1}(1)$ as $\mu_s^{-1}(1) \cdot s = \mu_s^{-1}(1) \cdot \mu_s(1) = (\mu_s^{-1} \circ \mu_s)(1) = 1$. So the universal property of $s^{-1}R$ gives $h : s^{-1}R \to R_*$ and

$$\begin{tikzcd} & R \arrow[dl, "i"'] \arrow[d, "j"] \arrow[dr, "i"] & \\ R_* \arrow[r, "g"'] & s^{-1}R \arrow[r, "h"'] & R_* \end{tikzcd}$$

The bottom is the identity because the two triangles are the same. So $h \circ g = id_{R_*}$. Same idea gets $g \circ h$. □

3. Localizing Categories and Model Categories

Recall that a category $\mathcal{C}$ is a class of objects and a class of morphisms which preserve the structure of the objects. Invented by Eilenberg and Maclane (1945). Maclane was an undergrad at Yale and from Connecticut. The **yoga of category theory** is that one must study maps between objects to study the objects. Applying this to categories themselves leads you to functors $F : \mathcal{C} \to \mathcal{D}$, i.e. maps from objects to objects and morphisms to morphisms compatible with id_A and $f \circ g$.

Thinking of localization as "formally inverting maps" then we want to pick a set S of morphisms and create a universal functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ where those morphisms land in the class of isomorphisms, i.e. $F(f)$ is an iso for all $f \in S$.

Example: If $\mathcal{C}$ is Top, and we want to study it "up to homotopy" (i.e. when X h.e. Y we say they are isomorphic), then we get the homotopy category and it's easy to see it's a category. What we've done is send the set of maps $\{f \mid \pi_n(f) \text{ is an iso} \forall\ n\}$ to isomorphisms in a universal way. Universal because we added the smallest numbers of isomorphisms possible.

To do this in general, note that given $f : X \to Y$ in S and $g : X \to Z$ I now have $g \circ f^{-1} : Y \to Z$, i.e. I have to generate new morphisms based on the inverses I added. So what are the morphisms $\mathcal{C}[S^{-1}]$ between X and Y? You can get there by any **zig-zag (DRAW IT)**, so you want to define $\mathcal{C}[S^{-1}](X, Y) = \{X \leftarrow \bullet \to \bullet \cdots \bullet \to Y\}/\sim$ where this relation at least allows us to add in pairs of identities or compose two when it's allowed. PROBLEM: the collection of zigzags $X \leftarrow \bullet \to \bullet \cdots \bullet \to Y$ is not a set, even just in the category Set you have a proper class worth of choices.

To get around this you are **forced into homotopy theory** again. You need restrictions on the types of S you can invert. It worked for Top, so let's generalize the properties we had there. This leads to the concept of a Model Category (Quillen 1967). The idea is you have a special class of maps $\mathcal{W}$ called the weak equivalences, and these generalize the homotopy equivalences above. But **algebraic topology is about more than just homotopy equivalences**.

For instance, it also studies vector bundles $E \to X$ where the fibers are vector spaces. This generalizes to a fiber bundle $F \to E \to X$, and we say $E \to X$ is a **fibration**. For example, $O(n) \to O(n)/O(n-1)$ the quotient of any two Lie groups.

It also studied when one space X can be built from another A by adjoining cells. We use this for example to write $H_n(X, A) \cong H_n(X/A)$. Call such a map $A \to X$ a **cofibration**.

Adding some axioms about how these classes of maps work together gives a **Model Category**, and for such $\mathcal{M}$ the localization described above works, i.e. you get a concrete way to make a universal functor $\mathcal{M} \to Ho\mathcal{M}$ taking $\mathcal{W}$ to isomorphisms. We now know the **most general place you can do homotopy theory**, and this transforms algebraic topology from the study of topological spaces into a general tool useful in many areas of mathematics.

Spaces and Spectra are model categories, with homotopy categories HoTop and the **stable homotopy category SHC**. $Ch(R)$ is also a model category with homotopy category = the **derived category** $\mathcal{D}(R)$, which is studied in algebraic geometry and elsewhere. Proving it's triangulated uses the model category structure. Given F, the model category structure helps you **construct from an induced functor between derived categories**, e.g. the left derived functor of an abelianization functor gives **Quillen homology**.

Voevodsky won a fields medal in 2002 by creating the **motivic stable homotopy category** from a model category structure on an enlargement of **Schemes** to resolve the Milnor Conjecture.

The **∞-categories** of Joyal, much studied by Lurie at Harvard and MIT, are a direct generalization of model categories, and so results in model categories are prized because they show the way for ∞-categories. Also, computations and constructions are much easier on the model category level than on the ∞-level or on the homotopy level. These ∞ categories relate to spectra because S plays the role of $\mathbb{Z}$ in the world of E_∞-algebra ("derived" commutative algebra).

4. Back to spectra and our original goal

Suppose we want to localize at $T = \{f \mid h^n(f) \text{ is an isomorphism } \forall\ n\}$. We can do this as follows:

$$\begin{tikzcd} Spectra \arrow[d] & \\ SHC \arrow[r] & SHC[T^{-1}] \end{tikzcd}$$

However, we'd like a model category which actually sits above $HoSpectra[T^{-1}]$. Its objects must be spectra again, and it's morphisms will be morphisms of spectra, but it'll have a **different model category structure**. In particular, $\mathcal{W}'$ will now be $\langle T \rangle$. We see that $\mathcal{W} \subset \mathcal{W}'$ because homotopy isomorphisms also give isomorphisms on cohomology.

But you can't just mess with $\mathcal{W}$ because it'll screw up the axioms. We want to keep the cofibrations fixed so we can build things out of them and have the two model structures related, so we have to shrink the fibrations: $\mathcal{F} \supset \mathcal{F}'$. **Bousfield's Theorem** (1978) says you can do this and you still get a model category structure, but you have to be careful with how you generate $\mathcal{W}'$ from T.

Thus, **for all h and for all spectra X there is a spectrum $L_h(X)$, which is h-local** (i.e. maps in from an h-acyclic are nullhomotopic, so it **doesn't fraternize with things h can't see**: the mud doesn't stick) the map $X \to L_h(X)$ **is an h equivalence**. Also, $L_h(L_h(X)) = L_h(X)$. Furthermore, L_h is characterized by these properties.

In case someone asks: $C_h(X) \to X \to L_h(X)$ and $C_h(X)$ is $\lim(h-\text{acyclics } Y \text{ which map to } X)$, except that's too big to make a limit so do homotopy limit and just for finite Y.

How to do this on a general model category? If we want to **force a map f to become a weak equivalence** and still preserve the cofibrations then the fibrations must shrink. It turns out that with mild conditions on $\mathcal{M}$ you can do this, even for a set of maps S:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, dashed] \arrow[d] & \mathcal{M}_L \arrow[d, dashed] \\
Ho\mathcal{M} \arrow[r] & Ho\mathcal{M}[S^{-1}]
\end{tikzcd}$$

Furthermore, this functor on top preserves a lot of nice properties of $\mathcal{M}$. For many years everyone assumed it preserved ring objects and commutative ring objects (now you need a monoidal structure) because it does for Spectra. Mike Hill (2011) showed that for the model category of G-equivariant spectra it **does NOT preserve commutative ring objects**. I'm now trying to find conditions under which it will preserve them.

This problem is a good example of the sort of things a modern stable homotopy theorist thinks about. Constructions and theorems are done on the model category level or the ∞-category level. I hope this gave you a flavor for the field.