45 min
9/7/13

Talk 5: Stable model categories are cat's of modules

- Yesterday Jonathan gave us

Thm 0.1: If $\mathcal{A}$ has a small projective generator P
Then $\mathcal{A} \simeq R\text{-mod}$ for $R = \mathcal{A}(P,P)$

Thm 0.2: $R\text{-Mod} \underset{G}{\overset{F}{\rightleftarrows}} T\text{-mod}$ is an equivalence

iff $\exists$ T-R-bimodule P s.t. ${}_TP$, P_R are small projective gen's.
and $F \simeq P \otimes_R (-)$, $G \simeq \mathrm{Hom}_T(P,-)$

Thm 0.3: $D(R) \rightleftarrows D(T)$ is derived equivalence $\Leftrightarrow$ $\exists$ Tilting complex

Today:

Thm 1: Let $\mathcal{C}$ be a stable model cat w/ compact gen. P
Then $\exists$ chain of simplicial Quillen Equivalences
$\mathcal{C} \simeq_Q R\text{-mod}$ for $R = \mathrm{End}(P) = $ Ring Spectrum
$\subseteq$ SymSpec

- **Thm 2**: $\mathcal{C}$ is equiv. to R-mod via chain of simp. Q.E.'s
(R is cofibrant) iff $\exists M \in R$-$\mathcal{C}$-bimod s.t. $R\text{-mod} \underset{\longleftarrow}{\xrightarrow{-\wedge_R M}} \mathcal{C}$ is Q.E

iff $\mathcal{C}$ has compact, bifibrant gen. M & $R \simeq \mathrm{End}(M)$ as stable ring spectra

Thm 3: $\mathrm{Ho}\,\mathcal{C} \simeq D(R)$ is triangulated equivalence iff $\exists$ set of tilting

$\mathcal{T} = \Delta$-cat
$X, P \in \mathcal{T}$

- **Defn**: X is compact if $\bigoplus_i [X, A_i] \longrightarrow [X, \coprod_i A_i]$ is iso

This is Δ-cat version of "small" in model cat
Paul worked w/ combinatorial model cat's where all obj small
so defn of cat gen easier & Thms above weaker conditions

- **Defn**: P is a generator if $\mathrm{loc}(P) = \mathcal{T}$ = smallest full triangulated subcat (w/ shifts & Δs from $\mathcal{T}$) containing P & closed under coprod
- P is gen iff "$X \simeq 0 \Leftrightarrow [P,X] = 0$" $\{P_i\}$ gen $\Rightarrow \coprod P_i$ is single gen but not nec. compact

Examples

- Spectra... S is compact gen. (fact about SHC so model indep)
- Let R = ring spectrum, i.e. $\mu: R \wedge R \to R$ & $\eta: S \to R$

  M is an <u>R-module</u> (left) if $\exists\ a: R \wedge M \to M$ s.t.

$$\begin{array}{ccc} R \wedge R \wedge M & \xrightarrow{\mu} & R \wedge M \\ {\scriptstyle a}\downarrow & \circlearrowleft & \downarrow{\scriptstyle a} \\ R \wedge M & \xrightarrow{a} & M \end{array} \quad \text{etc}$$

  R-mod is compactly gen by R

- Bous loc preserves gen. but not nec. compactness (I dealt w/ this in my thesis a bit)

- $Ch(R)$... compact $\Leftrightarrow$ quasi-iso to bdd complex of f.g. projectives
- $K(A)$ from when chain homotopy equiv's used — no set of comp gen

- StMod... compact = f.g.
- qcsheaves... compact = perfect complexes = locally quasi-iso to bdd complex of VB's

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- $End(P)$ = ring spec s.t. $\pi_* End(P) \cong [P, P]_*^{Ho\,\mathcal{C}}$

  If P gen. $\mathcal{C}$ do cof. rep then Σ^∞ then fib rep

  Get $\Sigma_f^\infty P \in Sp(\mathcal{C})$. Define the endomorphism ring as $R = Hom_{Spc}(\Sigma_f^\infty P, \Sigma_f^\infty P)$

- Note: If you have a set of compact objects you can do an analogous thing. You get a spectral cat worth of objects, aka a ringoid

- Sketch Pf of Thm 1 here

- The examples which justify the ringoid theory are ① equivariant & ② Motivic. Multiple gen. needed b/c of grading

- Note: Don't need "projective" P b/c can do QP
- Note $End(P)$ is homotopy invariant, i.e. $\mathcal{C} \simeq_Q \mathcal{D}$ & $P_{\mathcal{C}} \mapsto P_{\mathcal{D}}$ then $End(P_{\mathcal{C}}) \simeq End(P_{\mathcal{D}})$

Talk 5 (cont)

- Examples of Thm 1...

- Cor: A simplicial, cof gen, proper, stable $\mathcal{C}$ is simp Q.E. to SymSpec iff $\exists$ compact gen $P \in \mathcal{C}$ s.t. $\eta: S \to End(P)$ is stable equiv.
(can eliminate all conditions except stable)

- $A\text{-mod} \underset{Q}{\simeq} HA\text{-mod}$ (free A-mod of rank 1 in dim 0) "compact gen

- Smashing Bous Loc... E-local spectra $\underset{Q}{\simeq} L_E S^0\text{-mod}$
$L_E S^0 = End_{loc}(S^0)$ in $L_E(Sp)$

- Keller did for StMod, showing $\exists$ DGA A s.t. StMod $\simeq D(A)$

Sketch Pf of Thm 2

$X \in \mathcal{C}$ is R-$\mathcal{C}$-bimod if R acts thru $\mathcal{C}$-morphisms, i.e. if $\exists$ homomorphism $R \to End(X)$ of sym ring spec.
If $\mathcal{C} = T$-mod then X is a $R^{op} \wedge T$-module

- Partial Converse to Thm 1 b/c "if $\exists$ Q.E. cham then $\exists$ compact gen M s.t. $R \simeq End(M)$"

- Observe that Thm 2 lets you replace zigzag of Q.E's by just 1.

- Mention Thm 3, Tilting complexes, & "set of tiltors"

- Thm 1 + Cor show SymSpec is in some sense a universal stable model cat. Story gets even nicer in $(\infty,1)$-language.

(End of Talk)