1. Outline

(1) Basic Graph Theory and graph coloring

(2) Pigeonhole Principle

(3) Definition of Ramsey Numbers

(4) Some bounds on Ramsey numbers

(5) Ramsey's Theorem

(6) Erdos's famous lower bound on $R(n)$

2. Basic Graph Theory

Definition 1. *A* ***graph*** *G is a pair (V, E) where V is a set of points, called* ***vertices****, and E is a set of pairs of points (v_i, v_j) called* ***edges****.*

For us $|V|$ will always be finite.

The complete graph on n vertices has n vertices and edges between all pairs of vertices.

Graphs are useful all over mathematics and computer science. Much of the world can be modeled through graphs. For example, a group of people can make a friendship graph where there is an edge if two people are friends. Or the internet is a graph with computers as nodes and edges if they're on the same network (or visiting the same websites, or whatever).

3. Graph Coloring

Definition 2. *A* ***coloring*** *of a graph is an assignment of colors (living in some finite set $\{c_1, \ldots, c_r\}$) to the edges of the graph.*

You may have seen this defined before and probably with the nodes being colored such that no edge connects two nodes of the same color. But for this talk I will need to be coloring the edges of graphs and I will not insist that two adjacent edges are of different colors.

Ramsey Graph picture showing a 2-coloring of a graph which is almost a K_6

4. Pigeonhole Principle

Proposition 1 (Pigeonhole Principle)**.** *If you are placing $n+1$ pigeons into n holes, then some hole will end up containing at least two pigeons (obviously this holds for placing m pigeons into n holes whenever $m > n$).*

If you are placing $2n-1$ pigeons into 2 holes then some hole will end up containing at least n balls. So if you have $2n-1$ people at a party then at least n are of the same gender.

Because combinatorics studies discrete objects (mostly finite), this is crucial to the field and many proofs rely on it. The notion of placing pigeons into 2 holes is exactly the same as 2-coloring the pigeons.

5. Ramsey

Ramsey Theory generalizes the Pigeonhole Principle and solves the party problem above more generally. The question is: what is the minimum number of guests that must be invited so that at least n will know each other?

Definition 3. *$R(n)$ is the smallest integer m such that in any 2-coloring of K_m there is a monochromatic K_n.*

In the most general sense, Ramsey Theory asks how many elements are necessary to insure that some property is met.

6. Known Ramsey numbers

Theorem 1 (Friendship Theorem)**.** *$R(3) = 6$, i.e. 6 is the smallest number such that any 2-coloring of K_6 has a monochromatic K_3 (i.e. a monochromatic triangle)*

This means at any party with at least six people, there are either three people who are all mutual acquaintances (each one knows the other two) or mutual strangers (each one does not know either of the other two).

Proof. First, $R(3) \geq 6$ because here is a 2-coloring of K_5 with no monochromatic triangle. It's always easier to find lower bounds on Ramsey numbers because it's constructive.

Image here

Second, $R(3) \leq 6$ because if a is a vertex of K_6 then a has 5 edges touching it. By the Pigeonhole Principle, three of them are the same color (e.g. as shown above) without loss of generality red. Consider the three vertices b, c, d those edges connect to. If any edges between them are red (say (b, c) is) then we're done because $\triangle abc$ is red. So none of these edges are red and they must all be blue. This means $\triangle bcd$ is blue. So we have a monochromatic triangle. $\square$

7. Known and Unknown Ramsey Numbers

So we know $R(3) = 6$ and it's trivial that $R(1) = 1$ and $R(2) = 2$ since any coloring of a K_2 (i.e. 1 edge) has a monochromatic K_2.

Fact: $R(4) = 18$. You can prove this at home. To do so you'll need to find a 2-coloring of a K_{17} without a monochromatic K_4 and then prove that any 2-coloring of K_{18} has a monochromatic K_4.

Fact: $43 \leq R(5) \leq 49$ and $102 \leq R(6) \leq 165$. If you find better bounds you will have a very nice paper. If you nail either one down you'll probably get a PhD immediately!

Paraphrased from Paul Erdös: Aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

Erdös was one of the most prolific publishers of papers in mathematical history, second only to Leonhard Euler; Erdös published more papers, while Euler published more pages.

8. Erdös Probability Method

All the proofs in the last slide were upper bounds. There are many creative constructive lower bounds (e.g. giving polynomial lower bounds of any fixed degree), but nothing reaching c^n for any $c > 1$. This is achieved only by the Erdös Probability Method and the following two facts:

The probability of the union of events is at most the sum of their probabilities with equality iff the events are pairwise disjoint

Only the empty set has probability zero.

The key fact is that if the probability of an event occurring is strictly greater than 0, then there must exist some model in which the event occurs. This can be used to prove:

Theorem 2. *If* $\binom{m}{n} < 2^{\binom{n}{2}-1}$ *then* $m < R(n)$

Corollary 1. *For* $n \geq 3$, $2^{\frac{n}{2}} < R(n)$

The best asymptotics for m come easily from the corollary and we get

$$m = 2^{\frac{n}{2}} \cdot \frac{n}{e\sqrt{2}} < R(n)$$

The proof of the theorem is as follows. The probability space is the space of 2-coloring of the edges of K_m, which has $N = 2^{\binom{m}{2}}$ elements. A subset of n vertices in K_m yield a monochromatic K_n with probability $p = \frac{1}{2^{\binom{n}{2}-1}}$. Therefore the probability of having a monochromatic K_n is at most $p \cdot \binom{m}{n}$. This is strictly less than 1 from the hypothesis on m, so the probability of having no monochromatic K_n is positive. Therefore there must exist a 2-coloring with no monochromatic K_n which implies m is a lower bound on $R(n)$ without actually having to find the coloring!

9. Generalizing Ramsey numbers

Theorem 3 (Ramsey's Theorem)**.** *Given integers* $n_1, \ldots, n_r$ *there is a number* $m = R(n_1, \ldots, n_r)$ *such that for any* r*-coloring of the edges of* K_m *there exists an* i *such that* $1 \leq i \leq r$ *and there exists a complete* K_{n_i} *monochromatic in color* i*.*

That $R(n_1, \ldots, n_r)$ is the generalized Ramsey Number. For example, $R(3,4) = 9$, i.e. 9 is the minimal m for which any 2-coloring of K_m contains a red triangle or a blue K_4. Easy exercises: $R(3,5) = 14, R(3,6) = 18, R(3,7) = 23, R(3,8) = 28$

The only non-trivial case known exactly with $r = 3$ is $R(3,3,3) = 17$. This isn't that hard to show and it's the smallest non-trivial case because any time 2 appears we reduce to $r = 2$ since the third color can't be used at all.

10. Some Bounds

1. Easy to show: $(n-1)^2 < R(n) \leq 4^n$
2. Relatively easy to show and very useful: $R(s,t) \leq R(s-1,t) + R(s,t-1)$
3. $R(n_1, \ldots, n_c) \leq R(n_1, \ldots, n_{c-2}, R(n_{c-1}, n_c))$
4. Theorem of Erdös-Szekeres: $R(s,t) < \binom{s+t-2}{t-1}$. Corollary of (2).
5. $R(n) \leq 4R(n-2,n) + 2$

(6) $R(n) \leq R_3(6, n)$ is a non-trivial exercise due to Kiran Kedlaya.

(2) and (3) can be used to prove Ramsey's Theorem.

11. References

Advanced Combinatorics packet by András Gyárfás (from my class with him in the Budapest Semesters in Mathematics program)

12. Extensions of Ramsey Numbers

They extend to directed graphs: let $R(n)$ be the smallest number m such that any K_m with singly-directed arcs (also called a tournament) contains an acyclic (also called transitive) n-node subtournament. For directed graphs, $R(n)$ is computed up to $n = 6$ and there are sharper bounds above.

They extend to hypergraphs (where edges can have more than 2 nodes) and you get $R_t(n_1, \ldots, n_r)$ to be the smallest m such that if the edges of K_m^t are r-colored in any way, then for some i you have a monochromatic K_i^t in color i.

They extend to infinite graphs. Theorem: Let X be some countably infinite set and color the elements of $X(n)$ (the subsets of X of size n) in r different colors. Then there exists some infinite subset M of X such that the size n subsets of M all have the same color.

13. An alternate construction of Ramsey numbers

$R(s, t)$ is the smallest n such that for every graph G with $|G| = n$ either G contains K_s or $\overline{G}$ contains K_t. Here $\overline{G}$ has as vertices the edges of G and connects two if they were adjacent (shared a vertex) in G. Note that $\overline{\overline{G}} = G$ for all graphs. Ramsey used this construction and some set theory to originally prove his theorem.