Reference: Mark Hovey book   45 min

Talk 2: Model Categories

- Motivation: Given $\mathcal{C}$ & $W \subseteq \mathrm{Mor}\,\mathcal{C}$, I want to send those maps to isomorphisms.

Define $\mathcal{C}[W^{-1}](A,B) = A \xleftarrow{W} \cdot \rightarrow \cdot \xleftarrow{W} \cdot \rightarrow \cdots \xleftarrow{W} \cdot \rightarrow B$

But need to identify zigzags with id's & composables

Problem... $\mathcal{C}[W^{-1}](A,B)$ is not a set, so can't mod out by equiv. relation. Can pass to larger Grothendieck Universe, but still have little control over the maps

- It works for $\mathcal{C} = \mathrm{Top}$ & $W = \{f \mid \pi_* f \text{ is iso}\}$

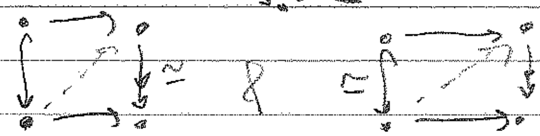
Get $\mathcal{C}[W^{-1}] = \mathrm{Ho}\,\mathcal{C}$. Same obj as Top, but morphisms are homotopy classes of maps.

- Reasons why it works: Top has Whitehead Thm, CW, Path Obj, Cylinders. So can define homotopies nicely & lift ho-equiv's. And nice projections ("fibrations")

- Quillen 1967 figured out the properties to generalize

$\mathcal{C}$ is a model category if it has (W, Q, F), cocomplete, W 2/3, All closed under retracts, functorial factorization

and lifting



Term: "Trivial" / "Acyclic" W

Covering Homotopy Prop   Homotopy Extension Prop

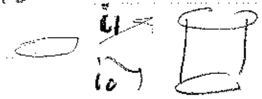
- Any 2 of the 3 classes determine the third

AND Mor Cat's

- Ex: Top, sSet, Ch(R), S-mod, SymSpec, OrthSpec, Motivic, Equivar, Fields Medals), DAG, Tors, Voev, Comodules over Hopf algs, sheaves over X, 9 on Set

Constructing the Ho-Cat of a model Cat

- $g, f : B \to X$. A <u>cylinder obj</u> for B is B' s.t. $\begin{tikzcd} B \amalg B \arrow[r, "\text{fold}"] \arrow[d, "i_0, i_1"'] & B \\ B' \arrow[ur, "\simeq"'] \end{tikzcd}$



A <u>path obj</u> for X is $\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\simeq"'] & X \times X \\ X' \arrow[ur, "p_0, p_1"'] \end{tikzcd}$

A left homotopy is $H : B' \to X$ w/ $H i_0 = f$, $H i_1 = g$
A rt homotopy is $K : B \to X'$ w/ $p_0 K = f$, $p_1 K = g$

- In gen these need not agree, but key idea is that it's ok to work up to homotopy. So
  replace B by cofib obj $(\emptyset \hookrightarrow QB \twoheadrightarrow B)$
  replace X by fibrant $(X \hookrightarrow RX \twoheadrightarrow *)$

Now it's an equiv. relation on $\mathcal{C}_{cf}$ & $f \overset{l}{\sim} g \Leftrightarrow f \overset{r}{\sim} g$

$\mathcal{C} \xrightarrow{\gamma} \mathrm{Ho}\,\mathcal{C}$

$$\mathcal{C}(QRX, QRY)/\sim \;\cong\; \mathrm{Ho}\,\mathcal{C}(\gamma X, \gamma Y) \cong \mathcal{C}(RQX, RQY)/\sim$$

<u>Thm</u>: $\mathrm{Ho}\,\mathcal{C} \cong \mathcal{C}_{cf}/\sim$ as categories
f is w.e. in $\mathcal{C}_{cf}$ iff $\gamma(f)$ is Ho Equiv.

How To Build Them

- In Top, $\mathcal{W}$ = weak ho equiv, $\mathcal{F}$ = Serre Fibrations i.e

$$\begin{tikzcd} D^n \arrow[r] \arrow[d, "\simeq"'] & E \arrow[d, two heads] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

So somehow $D^n \to D^n \times I$ generates $\mathcal{Q} \cap \mathcal{W}$
Also all $\mathcal{Q}$ gen by $S^{n-1} \to D^n$ b/c "build cells"

- $\mathcal{M}$ is <u>cofibrantly generated</u> if $\exists$ sets of maps I, J, s.t.
  $\mathcal{F} = J\text{-inj} = RLP(J)$, $\mathcal{W} \cap \mathcal{F} = I\text{-inj}$, $\mathrm{dom}(I)$ small rel I-cell,
  $\mathrm{dom}(J)$ small rel J-cell

- A is (λ)-small rel class of maps D if $\mathcal{C}(A, -)$ comm w/ $X_0 \to X_1 \to \cdots \to X_\lambda$
  where maps in D. rel I-cell = transf comp of push of maps in I
  retracts

Talk 2 (cont)

- Defn of Cof Gen is exactly to make the small obj arg work. This is a transfinite construction of a functorial factorization $\xrightarrow{\text{I-cell}} \xrightarrow{\text{I-inj}}$ / $\xrightarrow{f}$

- You use that to build M from I & J.
  You also use it to construct Bous. Loc (basically)

- Ex: Top — all obj fib

Ex: SSet — Cofibs = injective (all obj cof) w/ $\partial\Delta[n] \to \Delta[n]$ & $\Lambda^r[n] \to \Delta[n]$ "Horn Complex"

J-cof = Anodyne Extensions

"Horn inclusion"

$\Lambda^r[n]$ = Closed star of r

Omit interior of $\Delta[n]$ & $n-1$ dim face opp r

Ex: Ch(R) = unbounded chain complexes

$I = S^{n-1} \to D^n$, $J = 0 \to D^n$

$S^n(M)$ is $\cdots 0 \to M \to 0 \cdots$ (M in degree n) $D^n(M)$ is $\cdots 0 \to M = M \to 0 \cdots$

Model Struct: W = homology iso's

F = Surjections

(Q = dim-wise split inj w/ proj coker)

Also have one w/ W & Q = injections

Ex: Diagram Cat's w/ Proj Model Struct

Quillen Pair

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

F pres Q & Q∩W

$f : FX \to Y$ is we iff $\phi(f) : X \to UY$

Quill Equiv if pass to equiv on HoCat's

Derived Fun

$$\mathrm{Ho}\,\mathcal{C} \underset{RU}{\overset{LF}{\rightleftarrows}} \mathrm{Ho}\,\mathcal{D}$$

LF is $\mathrm{Ho}\,\mathcal{C} \xrightarrow{\mathrm{Ho}\,Q} \mathrm{Ho}\,\mathcal{C}_c \xrightarrow{\mathrm{Ho}\,F} \mathrm{Ho}\,\mathcal{D}$

RF similar

Ex: F = U = id ⇒ LF = Q(−), RU = R(−)

Monoidal Model Cat's

- Defn: Monoidal cat which is model cat plus coherence b/t:

① $\boxed{\text{Pushout Product Axiom}}$ Let $f: A \to B$, $g: X \to Y$

$$\begin{tikzcd}
A \otimes X \arrow[r] \arrow[d] & A \otimes Y \arrow[d] \arrow[ddr, bend left] & \\
B \otimes X \arrow[r] \arrow[drr, bend right] & P \arrow[dr, "f \square g"] & \\
 & & B \otimes Y
\end{tikzcd}$$

If $f, g \in Q$ then $f \square g \in Q$

Further, if either f or $g \in W$ then $f \square g \in W$

(This guarantees $\mathrm{Ho}\mathcal{M}$ is monoidal $[X \otimes Y] := [X] \otimes [Y]$ well def)

② $\boxed{\text{Unit Axiom}}$. Let S be unit, X cofibrant.

Then $X \otimes QS \longrightarrow X \otimes S = X$ is w.e.

(This gets $[S]$ to be unit for $\mathrm{Ho}\mathcal{M}$)

- Monoidal functor needs $FX \otimes FY \xrightarrow{\sim} F(X \otimes Y)$ & $FS \xrightarrow{\sim} S$ plus coherence for $(FX \otimes FY) \otimes FZ$, etc
- Monoidal Quillen needs $FQS \xrightarrow{\sim} QS$
- It's "closed" if $\exists$ internal hom objects $\mathrm{Hom}(X, Y) \in \mathcal{M}$, plus Hom-$\otimes$ Adjunct.
- Dual & Equiv. Condition... $g: W \hookrightarrow X$, $p: Y \twoheadrightarrow Z$ give

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Z) \times_{\mathrm{Hom}(W,Z)} \mathrm{Hom}(W,Y) \text{ is fibration}$$

& triv. if either g or p is triv.

- Ex: $(sSet, \times)$ ← cartesian

$(Top, \times)$ if we use compactly gen. spaces so that $\mathrm{Hom}(X,Y)$ gets compact-open top
- Ex: $Ch(R)$ with $(X \otimes Y)_n = \bigoplus_k (X_k \otimes_R Y_{n-k})$

$$d(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes dy$$

- Closed Symmetric monoidal if $\exists$ nat iso $\tau(X \otimes Y) \cong Y \otimes X$ $\forall X, Y$
- Ex: $Ch(R)$ has $\tau(x \otimes y) = \underbrace{(-1)^{|x| \cdot |y|}}_{} y \otimes x$

Price to move x past y, kept track of by Σ_2
(Analogy to SymSpectra & $\Sigma_n \times_{\Sigma_p \times \Sigma_q} (-)$)

- In monoidal cat can define monoids & comm. monoids. If you want model cat of comm. mon. use Shipley's Positive Sym. Spectra (but then S not cofib). $E_\infty \underset{QE}{\simeq} \mathrm{Com}$