APPLYING LOCALIZATION TO CATEGORIES

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THANK THE ORGANIZER, this is colloquium-style so questions are encouraged at all times.

Theme: Stacking algebra on topology stacked on algebra....

1. Difference between Theory and Computation

Modern stable homotopy theory is split into 2 types. Gabriel is on the computation side, which is also what is generally taught in first year algebraic topology. He's an expert with spaces, vector bundles, H^* computations, spectral sequences, π_* computations, and now extraordinary cohomology theories like K-theory and MU. The algebra here consists of algebraic invariants we wish to compute.

I work on the opposite side–the theory side. I used to want to study algebraic geometry, and a lot of what I do has the same flavor. Categories and commutative diagrams are everywhere. To solve a hard problem we abstract away all the details and then prove some diagram in a category commutes. Now the algebra sits in the front seat. It's not just the goal; it's also tied up in the objects we study. For example, there is a category Spectra whose objects are sequences of spaces (X_n) with structure maps $\Sigma X_n \to X_{n+1}$ DRAW THE PICTURE. This is truly an example. It's not necessary for the rest of the talk.

Spaces sit inside Spectra as follows: Given X, define $(\Sigma^\infty X)_n$ to be $\Sigma^n X$. It's a spectrum.

Rings sit inside Spectra, too: Given R and $n \in \mathbb{N}$, the **Eilenberg-Maclane space** $K(R, n)$ has $\pi_n = R$ and $\pi_k = 0$ for all $k \neq n$. Define a spectrum $(HR)_n = (K(R, n))$.

Extraordinary Cohomology Theories are also spectra, but this requires more machinery to understand. For usual cohomology it's just **Brown Representability**: $H^n(X; R) \cong [X, K(R, n)]$.

So Spectra is a category which contains all my favorite things: spaces, rings, and cohomology theories. My research is often of the flavor: take some **cool idea in algebra** and see how well it holds in this more general arena. You **need the right proof in algebra**, too, and then it takes work to push the proof forward. The talk has been all topology so far...let's do some algebra

We study ring objects in this category ("ring spectra"), which act like rings (they are **defined by commutative diagrams** in the category). It's a fact that HR and h_n are ring spectra. We may define this term at the end. This is the subject of the area of research called Brave New Rings. Today's goal will be to talk about how to do localization here: in particular, can we localize one spectrum with respect to another? What does it mean to localize with respect to a homology theory?

2. The right way to think about localization in algebra

To answer this, let's recall localization. It's a **systematic way of adding multiplicative inverses to a ring**, i.e. given commutative R and multiplicative $S \subset R$ (i.e. contains 1, closed under $*$,

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doesn't contain 0), localization constructs a ring $S^{-1}R$ and a ring homomorphism $j : R \to S^{-1}R$ that takes elements in S to units in $S^{-1}R$. It's universal w.r.t. this property, i.e. for any $f : R \to T$ taking S to units we have a unique g:

$$\begin{tikzcd} R \arrow[r, "j"] \arrow[d, "f"'] & S^{-1}R \arrow[dl, dashed, "g"] \\ T & \end{tikzcd}$$

Recall that $S^{-1}R$ is just $R \times S/\sim$ where (r, s) is really r/s and $\sim$ says you can reduce to lowest terms without leaving the equivalence class. Ring just as $\mathbb{Q}$ is. The map j takes $r \mapsto r/1$, and given f you can set $g(r/s) = f(r)f(s)^{-1}$.

Examples: $(\mathbb{Z}, \mathbb{Z} - \{0\}) \mapsto S^{-1}R = \mathbb{Q}$. $(\mathbb{Z}, \langle 2 \rangle) \mapsto \mathbb{Z}[\frac{1}{2}]$. $(\mathbb{Z}, \mathbb{Z} - p\mathbb{Z}) \mapsto \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \not| b\}$

This is NOT the right definition to a category theorist (no operation, so "multiplicative inverses?"). Better: **systematic way of formally inverting maps**. We can't do this for all maps, but we can do it for maps of the form $\mu_s : R \to R$ which take $r \mapsto s \cdot r$ for an element s.

Definition 1. *Given $s \in R$ a localization is a ring R^* containing s, such that*

(1) $\mu_s : R^* \to R^*$ is an isomorphism

(2) R^* is universal with respect to this property, i.e. there is a map $i : R \to R^*$ and any time a map $g : R \to T$ takes μ_s to an isomorphism,

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & R^* \arrow[dl, dotted] \\ & T & \end{tikzcd}$$

We prove these two notions of localization are the same, i.e. produce isomorphic rings. This is an example of finding the right proof in algebra to generalize to category theory. It's diagrammatic.

Proposition 1. $R_* \cong s^{-1}R$

Proof. Certainly $s \in s^{-1}R$ as $(s, 1)$. Also, μ_s is an isomorphism with inverse $\mu_{s^{-1}}$. So by the universal property of R_*, the map $j : R \to s^{-1}R$ gives $g : R_* \to s^{-1}R$ s.t. $g \circ i = j$

Next, the element s has an inverse in R_* because it's $\mu_s^{-1}(1)$ as $\mu_s^{-1}(1) \cdot s = \mu_s^{-1}(1) \cdot \mu_s(1) = (\mu_s^{-1} \circ \mu_s)(1) = 1$. So the universal property of $s^{-1}R$ gives $h : s^{-1}R \to R_*$ and

$$\begin{tikzcd} & R \arrow[dl, "i"'] \arrow[d, "j"] \arrow[dr, "i"] & \\ R_* \arrow[r, "g"'] & s^{-1}R \arrow[r, "h"'] & R_* \end{tikzcd}$$

The bottom is the identity because the two triangles are the same. So $h \circ g = id_{R_*}$. Same idea gets $g \circ h$. $\square$

3. Localizing Categories and Model Categories

Recall that a category $\mathcal{C}$ is a class of objects and a class of morphisms which preserve the structure of the objects. The **yoga of category theory** is that one must study maps between objects to study the objects. Applying this to categories themselves leads you to functors $F : \mathcal{C} \to \mathcal{D}$, i.e. maps from objects to objects and morphisms to morphisms compatible with id_A and $f \circ g$.

Thinking of localization as "formally inverting maps" then we want to pick a set T of morphisms and create a universal functor $\mathcal{C} \to \mathcal{C}[T^{-1}]$ where those morphisms land in the class of isomorphisms, i.e. $F(f)$ is an iso for all $f \in T$.

Example (to remind ourselves that this is still a topology talk): If $\mathcal{C}$ is Top, and we want to study it "up to homotopy" (i.e. when X h.e. Y we say they are isomorphic), then we get the homotopy category and it's easy to see it's a category. What we've done is send the set of maps $T = \{f \mid \pi_n(f)$ is an iso$\forall\ n\}$ to isomorphisms in a universal way. Universal because we added the smallest numbers of isomorphisms possible.

To do this in general, note that given $f : X \to Y$ in T and $g : X \to Z$, we get $g \circ f^{-1} : Y \to Z$, i.e. we have to generate new morphisms based on the inverses I added. So what are the morphisms $\mathcal{C}[T^{-1}]$ between X and Y? You can get there by any **zig-zag (DRAW IT)**, so you want to define $\mathcal{C}[T^{-1}](X, Y) = \{X \leftarrow \bullet \to \bullet \cdots \bullet \to Y\}/\sim$ where this relation at least allows us to add in pairs of identities or compose two when it's allowed. PROBLEM: the collection of zigzags $X \leftarrow \bullet \to \bullet \cdots \bullet \to Y$ is not a set; even just in the category Set you have a proper class worth of choices.

To get around this you are **forced into homotopy theory** again (but I tried to make it an algebra talk). You need restrictions on the types of T you can invert. It worked for Top, so let's generalize the properties we had there. This leads to the concept of a Model Category (Quillen 1967). The idea is you have a special class of maps $\mathcal{W}$ called the weak equivalences, and these generalize the homotopy equivalences above. But **algebraic topology is about more than just homotopy equivalences**.

For instance, recall from Gabriel's talk that vector bundles $E \to X$ (where the fibers are vector spaces) are very important. Algebraic geometers also study fiber bundles $F \to E \to X$, which generalize this concept. We say $E \to X$ is a **fibration**. For example, $O(n) \to O(n)/O(n-1)$. More generally, the quotient of any two Lie groups.

Another thing topology studies is when one space X can be built from another A by adjoining cells. We use this for example to write $H_n(X, A) \cong H_n(X/A)$. Call such a map $A \to X$ a **cofibration**.

Quillen's brilliant idea was to focus on just these three types of maps, pick out their most important properties, and use these properties to make a definition. A **Model Category** is a category $\mathcal{M}$ with distinguished classes $\mathcal{W}, \mathcal{F}, \mathcal{Q}$ satisfying those properties. The localization described above for spaces works on any model category, i.e. you get a concrete way to make a <u>universal</u> functor $\mathcal{M} \to \operatorname{Ho}\mathcal{M}$ taking $\mathcal{W}$ to isomorphisms. So functors $\mathcal{M} \to \mathcal{C}$ which do this induce functors $\operatorname{Ho}\mathcal{M} \to \mathcal{C}$

Model categories are the **most general place you can do homotopy theory**, and this transforms algebraic topology from the study of topological spaces into a general tool useful in many areas of mathematics. This viewpoint let's you do homotopy theory in algebraic geometry, e.g. on the category of Schemes. Voevodsky won a fields medal in 2002 by creating the **motivic stable homotopy category** from a model category structure on an enlargement of **Schemes** to resolve the Milnor Conjecture.

Spaces and Spectra are model categories, with homotopy categories HoTop and the **stable homotopy category SHC**. $Ch(R)$ is also a model category with homotopy category = the **derived category** $\mathcal{D}(R)$, which is studied in algebraic geometry and elsewhere. Proving it's triangulated uses the model category structure. Given F, the model category structure helps you **construct from an induced functor between derived categories**, e.g. the left derived functor of an abelianization functor gives **Quillen homology**.

The **∞-categories** of Joyal, much studied by Lurie's group at Harvard and MIT, are a way to study categories of categories. Homotopy plays a motivating role, but ∞-categories are not used in stable homotopy theory. Still, model categories serve as a way to do computations in $(\infty, 1)$-categories, so results in model categories are prized because they suggest things which should be true for (∞, n)-categories. Also, computations and constructions are much easier on the model category level than on the ∞-level or on the homotopy level. These ∞ categories are the basis of **Derived Algebraic Geometry**, a hot new research area.

4. Localization on Monoidal Model Categories

The localization above always lands in a homotopy category and always takes exactly the zig-zags of weak equivalences to isomorphisms. What if we want to invert some map which is not a weak equivalence? Let T be a set of maps in $\mathcal{M}$. Because the homotopy category is nice (admits a calculus of fractions), we can do:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, dotted] \arrow[d] & ????? \arrow[d, dotted] \\
\mathrm{Ho}(\mathcal{M}) \arrow[r] & \mathrm{Ho}(\mathcal{M})[T^{-1}]
\end{tikzcd}$$

We'd like a model category $L_T\mathcal{M}$ which actually sits above $\mathrm{Ho}(\mathcal{M})[T^{-1}]$. Because all three categories above have the same objects, its objects are determined. It's morphisms will be the same as those in $\mathcal{M}$, but we want the maps in T to become isomorphisms in $\mathrm{Ho}(\mathcal{M})[T^{-1}]$ so we need them to be weak equivalences in $L_T\mathcal{M}$. So this category must have a **different model category structure**, where $\mathcal{W}' = \langle T \cup \mathcal{W} \rangle$ and clearly $\mathcal{W} \subset \mathcal{W}'$. You can't change only $\mathcal{W}$ because it'll screw up the axioms. We want to keep the cofibrations fixed so we can build things out of them and have the two model structures related, so we have to shrink the fibrations: $\mathcal{F} \supset \mathcal{F}'$. **Bousfield's Theorem** (1978) says you can do this and you still get a model category structure, but you have to be careful with how you generate $\mathcal{W}'$ from T.

Model categories are fun, but there's not enough algebra in the picture for my tastes. Let's remedy this by looking at ring objects in $\mathcal{M}$. The functor $\mathcal{M} \to L_T\mathcal{M}$ preserves a lot of properties. For many years everyone assumed it preserved ring objects and commutative ring objects (now you need a monoidal structure on $\mathcal{M}$) because it does for Spectra. Mike Hill (2011) showed that for the model category of G-equivariant spectra it **does NOT preserve commutative ring objects**. Now I need to define terms.

A category $\mathcal{C}$ is a **strict monoidal category** if it comes equipped with a product bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative (i.e. $\otimes \circ (\otimes \times 1) = \otimes \circ (1 \times \otimes)$) and has a unit object S (i.e. $S \otimes E = E = E \otimes S$). The word strict is there because in general these $=$ signs could be made into natural isomorphisms. That's a **monoidal category**

In a monoidal category, a monoid (or **ring object**) is E with maps $\mu : E \times E \to E$ and $\eta : S \to E$ satisfying some diagrams. Note that every ring has a map $\mathbb{Z} \to R$ and every function $f : X \to X$ has a map from the identity to that function by applying 1 on domain and f on codomain. Anyway, diagrams for associativity and unit:

$$\begin{tikzcd}
E \otimes (E \otimes E) \arrow[r, "\cong"] \arrow[d, "1\otimes\mu"] & (E \otimes E) \otimes E \arrow[r, "\mu\otimes 1"] & E \otimes E \arrow[d, "\mu"] \\
E \otimes E \arrow[rr, "\mu"] & & E
\end{tikzcd}
\qquad
\begin{tikzcd}
S \otimes E \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & E \otimes E \arrow[d, "\mu"] & E \otimes S \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\
& E &
\end{tikzcd}$$

A **commutative ring object** is one which also has a twist isomorphism $\tau : E \otimes E \to E \otimes E$ and more diagrams which say the twist commutes with multiplication.

If $\mathcal{M}$ is a model category and a monoidal category (coherently), then we can ask whether or not $L_T(E)$ is also a commutative ring object. Note that there is a question of whether the diagrams should commute up to homotopy or up to isomorphism. It's been studied in the former case (A_∞ and E_∞) but not the latter (some would call this a strict ring object). There are standard hypotheses on a model category $\mathcal{M}$ when one is working with Bousfield localization (cocomplete, cofibrantly generated, left proper, almost finitely generated, can choose domains and codomains of generating (trivial) cofibrations to be cofibrant) and also when one is in a monoidal situation (pushout product axiom, cofibrant objects flat, monoid axiom). I found that even under all these hypotheses, $L_T\mathcal{M}$ could fail to be a monoidal model category, though it is always a model category and monoidal. The coherence fails. If we place an assumption on the maps T to be inverted (just like Quillen had to do), we can get around this.

Theorem 1. *Under the standing hypotheses above, if for all domains and codomains K of $I \cup J$, maps in $T \otimes id_K$ are T-local equivalences, then $L_T\mathcal{M}$ is a monoidal model category.*

Similarly, the monoid axiom can fail, but if we add a hypothesis about how the cofibrations behave (which makes $\mathcal{M}$ a little bit more like Top) then we have

Theorem 2. *Under the standing hypotheses on $\mathcal{M}$ and T, $L_T\mathcal{M}$ satisfies the monoid axiom.*

This gives a model category structure on $\mathrm{Mon}(\mathcal{M})$. We can thus ask the question of when $L_T(E)$ is a (commutative) ring object.

Theorem 3. *Under the hypotheses above, L_T preserves ring objects (of course, this really means L_T then U_T going back to $\mathcal{M}$)*

Currently we are trying to weaken the hypotheses. Furthermore, we are trying to find a model category structure on $\mathrm{CommMon}(\mathcal{M})$ and on $\mathrm{CommMon}(L_T\mathcal{M})$. If we can get this, then the proof above will generalize and tell us that L_T preserves commutative ring objects.

This problem is a good example of the sort of things a modern stable homotopy theorist thinks about. Constructions and theorems are done on the model category level or the ∞-category level so they hold for all sorts of categories classically studied (e.g. spaces, spectra, schemes) as well as for new categories of interest in DAG. I hope this gave you a flavor for the field.