LECTURE NOTES

DAVID WHITE

1. Motivation for Spectra 1

It is VERY hard to compute homotopy groups. We want to put as much algebraic structure as possible in order to make computation easier. You can't add maps in $HoTop$ but you can in Spectra (i.e. Spectra is an Ab-category). The motivation to go to $\mathbf{S} - \mathbf{Alg} = \mathbf{Ho}\mathcal{S}$ then becomes that you want an abelian category

We want to study the ring-like objects that arise in this category. "Ring-like" means ring-object, i.e. using the lens of category theory. They have no points, so you can't do traditional algebra. To measure complexity of these we'll use dimension.

2. Algebraic Motivation: why we care about dimension

Moral: Algebra $\subseteq$ Homological Algebra $\subseteq$ Stable Homotopy Theory

The simplest rings are fields, which clearly have Krull dim zero because no ideals. Dimension is telling us about complexity. For us, Krull dim fails because no points or ideals.

Note: Krull dim is the max length of a chain of prime ideals $P_0 \subset P_1 \subset \dots$. Zero ideal not prime.

Dimension gives amazing theorems in algebra:

R is **semisimple** iff all modules over R are projective iff R is a direct sum of simple submodules. My favorite way to define such a ring as one with **global dimension zero**

Semisimple implies Artinian and Noetherian.

Theorem 1 (Artin-Wedderburn Theorem)**.** *R is semisimple iff $R = R_1 \times \cdots \times R_n$ where $R_i = M_n(D)$ for D a division algebra*

Maschke's Theorem says k[G] is semisimple, so it sufficies to study irreducible representations

Theorem 2 (Serre's Theorem)**.** *If commutative R has finite global dimension then R is regular, i.e. for all prime $\mathcal{P}$, the min number of generators for $\mathcal{M} \subset R_{\mathcal{P}}$ is Krull dim($R_{\mathcal{P}}$).*

Cool fact: Commutative Noetherian local R is a regular local iff gl.dim(R) =Krull dim(R) $< \infty$.

Date: March 2, 2011.

3. Global Dimension

We say module P is **projective** if:

$$\begin{tikzcd} & P \arrow[d] \arrow[dl, dashed, "\exists"'] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

A module M is **flat** if the functor $- \otimes_R M$ is exact.

A **projective resolution** of M is $\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$, with all the P_i's projective.

Definition 1. ***Projective dimension*** $= pd(M) =$ *min. length of a projective resolution.*

Ex: If P is projective, $\text{pd}(P) = 0$ since $\cdots \to 0 \to 0 \to P \to P \to 0$ is a projective resolution.

Ex: For $R = \mathbb{Z}$, $\text{pd}(\mathbb{Z}/n) = 1$ since $\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ is minimal projective resolution, where the first map is mult by n and the second is quotient. THIS SHOWS $\text{pd}(\mathbb{Z}/n) \leq 1$. TO SHOW IT'S NOT 0, NOTE THAT IT CAN'T BE A SUMMAND OF A FREE MODULE.

Definition 2 (Right Global Dimension)**.** *r.gl.dim(R)* $= \sup\{pd(M) \mid M \in R - mod\}$

Ex: $\text{r.gl.dim}(k[x_1, \ldots, x_n]) = n$ because of the module $(x_1, \ldots, x_n)$

Ex: $\text{r.gl.dim}(k[x]/(x^2)) = \infty$ because k is an R-module and the minimal projective resolution is an infinite chain $\cdots \to k[x]/(x^2) \to k[x]/(x^2) \to k \to 0$, where each map takes $x \to 0$ and $1 \to x$.

Definition 3 (Weak Dimension)**.** *r.w.dim*$(R) = \sup\{fd(M) \mid M \in R - mod\}$

Projective $\Rightarrow$ Flat, so $\text{r.w.dim}(R) \leq \text{r.gl.dim}(R)$. If R is Noetherian then $\text{w.dim}(R) = \text{r.gl.dim}(R)$ because $fd(M) = pd(M)$ for all M.

R is **Von Neumann Regular** iff $\text{w.dim}(R) = 0$ iff all modules over R are flat.

$R = \prod_{i=0}^{\infty} \mathbb{F}_2$ is Von Neumann Regular but not Semisimple.

$\text{r.gl.dim}(R) = 1$ implies submodules of projective modules are projective. This is the next simplest ring after a semisimple ring. Ex: all PIDs.

$\text{w.dim}(R) = 1$ implies submodules of flat modules are flat.

4. Definitions

Definition 4 (Spectrum)**.** *A spectrum X is a sequence (X_i) of topological spaces (path conn. CW-complexes) with maps from $\Sigma X_i \to X_{i+1}$ where Σ is reduced suspension. ΣX is shift.*

Example: For any space X, $Z = \Sigma^\infty X$ is the spectrum with $Z_i = \Sigma^i X$, and ϵ_i homeomorphism for all i. So we recover SPACES INSIDE RING SPECTRA

Example: the sphere spectrum $S = (S^n) = \Sigma^\infty S^0$. NOTE: We've erased dimension, but we have no points.

For spectrum X, $X_* = \pi_*(X) = [\Sigma^* S, X]$. S_* is homotopy groups of spheres.

$$\begin{tikzcd} \Sigma X_i \arrow[r] \arrow[d] & X_{i+1} \arrow[d] \\ \Sigma Y_i \arrow[r] & Y_{i+1} \end{tikzcd} \qquad \begin{tikzcd} S \wedge X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[d] \\ S \wedge Y_n \arrow[r] & Y_{n+1} \end{tikzcd}$$

i.e. maps in the category f must play nicely with ϵ_n and $S \wedge -$.

π_* is a functor to the category of graded rings. Smash product because of $S^n \wedge S^m \cong S^{n+m}$

Example: Singular cohomology theory $H^n(-)$ is a spectrum. $H^n(X;\Lambda) \cong [X, K(\Lambda, n)]$

$$\pi_m(K(\Lambda, n)) = \begin{cases} \Lambda & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

For all R, the Eilenberg-MacLane spectrum HR has $(HR)_n = K(R,n)$. Well-known: $K(R,n-1) \xrightarrow{\simeq} \Omega K(R,n)$. This gives $\Sigma K(R, n-1) \to K(R,n)$.

Example: Any cohomology theory is a ring spectrum, e.g. $H\mathbb{Q}$.

$(HR)_* = [S, HR] \cong R$, so we recover RINGS INSIDE RING SPECTRA

For RINGS R with identity e and mult μ:

$$\begin{tikzcd} R \times R \times R \arrow[r, "\mu \times 1_R"'] \arrow[d, "1_R \times \mu"] & R \times R \arrow[d, "\mu"] \\ R \times R \arrow[r, "\mu"] & R \end{tikzcd} \qquad \begin{tikzcd} (a,b,c) \arrow[r, mapsto, "\mu"] \arrow[d, mapsto, "\mu"] & (ab, c) \arrow[d, mapsto, "\mu"] \\ (a, bc) \arrow[r, mapsto, "\mu"] & abc \end{tikzcd}$$

$$\begin{tikzcd} \{e\} \times R \arrow[r, hook, "u \times 1"'] \arrow[dr, "proj"'] & R \times R \arrow[d, "\mu"] & R \times \{e\} \arrow[l, hook', "1 \times u"] \arrow[dl, "proj"] \\ & R & \end{tikzcd}$$

Definition 5 (Ring Spectrum)**.** *A* ***ring spectrum*** *E is a generalized cohomology theory with a cup product that is associative up to infinitely coherent homotopy. E comes with $\wedge : E \times E \to E$ and $u : S \to E$.*

$$\begin{tikzcd} E \times E \times E \arrow[r, "\wedge \times 1"'] \arrow[d, "1 \times \wedge"] & E \times E \arrow[d, "\wedge"] \\ E \times E \arrow[r, "\wedge"] & E \end{tikzcd} \qquad \begin{tikzcd} S \times E \arrow[r, "u \times 1"'] \arrow[dr, "proj"'] & E \times E \arrow[d, "\wedge"] & E \times S \arrow[l, "1 \times u"] \arrow[dl, "proj"] \\ & E & \end{tikzcd}$$

An S-algebra E is an S-module because we have $S \wedge E \to E$. In particular, $S^i \wedge (S^j \wedge E) \cong (S^i \wedge S^j) \wedge E \cong S^{i+j} \wedge E$.

KRULL DIM FAILS HERE BECAUSE NO POINTS. SO NEED HOMOLOGICAL DIM.

An E-module X has $E \wedge X \to X$ satisfying the usual action rule.

5. Derived Category

The correct category to study modules over an S-algebra E is $\mathcal{D}(E)$. Objects are E-modules, maps from M_1 to M_2 are $\{S-\text{algebra morphisms:} M_1 \to M_2\}/\sim$ where $f \sim g$ if $f = g \circ s^{-1}$ and s_* is an isomorphism.

CORRECT CATEGORY because triangulated. It's also compactly generated and has derived tensor products and derived Hom objects.

Definition 6. *A map $f : X \to Y$ in $\mathcal{D}(E)$ is **ghost** if $f_* = 0$*

Such maps CANNOT BE SEEN BY π_*

EXAMPLE: any map from $HR \to \Sigma^k HR$ is ghost if $k > 0$ because $\pi_n(HR) = R$ iff $n = 0$, so $\pi_n(\Sigma^k HR) = \pi_{n-k}(HR) = 0$

We have a categorical equivalence: $\mathcal{D}(HR) \cong \mathcal{D}(R)$

$X \in \mathcal{D}(E)$ is **projective** iff X_* is a projective E_*-module. Define $\mathrm{pd}(X) = 1$. Projective E_*-modules are realizable.

Definition 7. $\mathrm{pd}(X) \leq n+1$ *iff $Y \to P \to \widetilde{X} \to \Sigma Y$ with P projective,* $\mathrm{pd}(Y) \leq n$*, and X a retract of $\widetilde{X}$.*

6. Dimensions of Ring Spectra

Definition 8. *$pd(X) \leq n+1$ iff $Y \to P \to \widetilde{X} \to \Sigma Y$ with P projective, $pd(Y) \leq n$, X a retract of $\widetilde{X}$*

Definition 9. *$r.gl.dim(E) = \sup\{pd(X) \mid X \in \mathcal{D}(E)\}$*

Definition 10 (Ghost Dimension)**.** *$gh.dim(E) = \sup\{pd(X) \mid X \in \mathcal{D}(E)$ is compact$\}$*

Proposition 1. *$X \in \mathcal{D}(E)$ is projective iff the natural map $\mathcal{D}(E)(X, Y) \to \mathrm{Hom}_{E_*}(X_*, Y_*)$ is iso for all Y*

We use this in practice all the time, especially to show when ghosts are null.

Proposition 2. *$pd(X) \leq n$ iff every composite of $n+1$ ghosts $f_{n+1} \circ \cdots \circ f_1$ is null where $\mathrm{Dom}(f_1) = X$. This holds iff $E_2^{s,t} = \mathrm{Ext}^{s,t}_{E_*}(X_*, Y_*) \Rightarrow \mathcal{D}(E)(X, Y)_{t-s}$ has $E_\infty^{s,*} = 0\ \forall\ s > n$*

Here we have algebra on the E_2 term converging to topology on the E_∞ term.

EXAMPLE: $\mathrm{gh.dim}(S) = \infty$. Suppose it's $n < \infty$. Then you need an S-module X with $\mathrm{pd}(X) \geq n+1$, i.e. find a chain of n ghosts out of X which is non-null. Any spectrum is an S-module. Turns out you can take $X = \Sigma^\infty \mathbb{R}P^k$ for large k and use the Steenrod Squares, which are well-studied maps that turn out to be ghost.

7. Analogy to Ring Theory Holds

Recall: $\mathbf{depth}(R)$ = length of the longest regular sequence $((x_1, \ldots, x_n)$ s.t. $\sum x_i R \neq R$ and x_i not a zero-divisor in $R/(x_1 R + \cdots + x_{i-1} R))$

Theorem 3. *If E is a commutative S-algebra then $depth(E_*) \leq gh.dim(E) \leq \min\{w.dim(E_*), r.gl.dim(E) \leq r.gl.dim(E_*)$*

Fact: gh.dim(E) $\leq$ r.gl.dim(E)

Fact: r.gl.dim(E) =r.gl.dim(E_*). Also, gh.dim(E) =gh.dim(E_*).

Fact: If $E = HR$ then r.gl.dim(E) =r.gl.dim(R). Same for weak dim...Example of E of gl.dim n is HR for $R = k[x_1, \ldots, x_n]$.

We have everything except the first inequality. It's an induction on n. Given a regular sequence, realize $R/(x_1, \ldots, x_n)$, then use the Universal Coefficient Spectral Sequence and an Ext computation to get some $E_\infty^{s,t} \neq 0$ for $s = n$, so gh.dim(E) $\geq n$

E_* semisimple $\Rightarrow E$ semisimple. Converse fails because of a DGA with homology $k[x]/(x^2)$. DGA is equiv to an $H\mathbb{Z}$-algebra and these are types of spectra. $HoDGA \leftrightarrow \mathcal{D}(H\mathbb{Z})$

Theorem 4. *If E is a commutative S-algebra and E_* is Noetherian with gl.dim(E_*) $< \infty$ then gh.dim(E) =r.gl.dim(E) = r.gl.dim(E_*)*

Proof: depth = gl.dim so the chain of inequalities collapses to equalities. Let $R = E_*$. Then $\mathrm{gl.}\dim(R) < \infty \Rightarrow R$ is regular (all localizations $R_{\mathcal{P}}$ are regular local rings). For regular rings, Krull dim = depth (Regular $\Rightarrow$ Cohen-Macaulay) and Krull dim = gl.dim

We need to have the Noetherian condition on E_* because without IDEALS we have no definition for E to be Noetherian.

8. Analogy to Ring Theory Almost Holds

Theorem 5. *A semisimple S-algebra E with E_* commutative has $E_* \cong R_1 \times \cdots \times R_n$ where each R_i is either a graded field k or an exterior algebra $k[x]/(x^2)$ over a graded field k*

Corollary: r.gl.dim(E) $= 0 \Rightarrow E_*$ is quasi-Frobenius, hence 0-Gorenstein, i.e. R is commutative Noetherian and has injective dimension 0 as an R-module.

Conjecture: $\mathrm{r.gl.dim}(E) = n \Rightarrow E_*$ is n-Gorenstein.

Theorem 6. *Suppose $E \to F$ in S-alg gives F_* free over E_*. Then* $\mathrm{gh.dim}(E) \leq \mathrm{gh.dim}(F)$

Proof: Because F_* is flat over E_* we have $F_* \otimes_{E_*} X_* \to (F \wedge_E X)_*$ is an iso. Thus, $F \wedge_E (-)$ preserves ghosts. Let g be a composite of n ghosts. Because F_* is free over E_* we know $F \wedge_E X$ is a coproduct of copies of X as an E-module, so g is a restriction of $F \wedge_E g$. This means we can't have $F \wedge_E g = 0$ unless $g \neq 0$.

9. Analogy to Ring Theory Fails

KO is 2-local periodic real K-theory

$KO_* = \mathbb{Z}_{(2)}[\eta, w, v, v^{-1}]/(\eta^3, 2\eta, w\eta, w^2 - 4v)$ where $\langle \eta \rangle = \pi_1(KO), \langle w \rangle = \pi_4(KO), \langle v \rangle = \pi_8(KO)$. Infinite global dim.

ko is 2-local connective real K-theory

$KO = v^{-1}ko$, specifically it's the direct limit of $ko \xrightarrow{\cdot v} \Sigma^{-8}ko \xrightarrow{\cdot v} \ldots$

$ko_* = \mathbb{Z}_{(2)}[\eta, w, v]/(\eta^3, 2\eta, w\eta, w^2 - 4v)$

Theorem 7. $1 \leq \mathrm{gl.dim}\, KO \leq 3$ *and* $4 \leq \mathrm{gl.dim}\, ko \leq 5$

Lower bound: $\mathrm{Sq}^2\mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 \neq 0$ for $ko-$module $ko \wedge A(1)$.

Upper bound: Follow Bousfield and view a $KO-$module as a CRT-module where CRT$= \{KO_*, K_*, KSC_*\}$ stands for complex, real, self-conjugate K-theory ($KSC = KT$)

Show that for any $KO-$module X, a composite of four ghosts out of X is null (uses fact that CRT-module is "built from KO" via $K = KO \wedge C(\eta)$ and $KSC = KO \wedge C(\eta^n)$). For any $ko-$module Z, a composite of 6 ghosts out of Z is null because too high filtration in $E_2^{s,t} = \mathrm{Ext}_{crt}^{s,t}(\pi_*^{crt}(X), \pi_*^{crt}(Y)) \Rightarrow \mathcal{D}(ko)(X,Y)_{t-s}$

Thus, gl.dim $E_* = \infty$ means we cannot apply our theorems

More general "build from" statement does exist (here we built ku from $ko \wedge C(\eta)$):

Theorem 8. *If E is an S-algebra and X is a spectrum s.t. r.gl.dim($E \wedge X$) $= m$, pd(X) $= k$, and S can be built from X in ℓ steps then gl.dim(E) $\leq (k+1)(\ell+1)(m+1) - 1$*

$H\mathbb{F}_2^*(KO) = [KO, H\mathbb{F}_2]^* =$ maps of spectra. This is an $\mathcal{A}-$module but not a ring. It's zero.

$(H\mathbb{F}_2)_*(KO) = \pi_*(H\mathbb{F}_2 \wedge KO) = [S^0, H\mathbb{F}_2 \wedge KO]$. It's zero.

Cor: $\mathrm{r.gl.dim}(E) = n \not\Rightarrow E_*$ is n-Gorenstein. A counterexample is KO because it's not n-Gorenstein for any n. We can see this because Gorenstein is a special case of Cohen-Macaulay, and KO_* is not Cohen-Macaulay: Krull dim = 1 (prime ideals are (η) and $(\eta, 2, w)$...maximal) but Depth = 0 (no non-zero divisors in the maximal ideal, so if x is any non-unit then x is a zero divisor, so no regular sequences at all).