GENERAL THEORY OF LOCALIZATION

DAVID WHITE

- Localization in Algebra
- Localization in Category Theory
- Bousfield localization

Thank them for the invitation. Last section contains some of my PhD research, under Mark Hovey at Wesleyan University. For more, please see my website: dwhite03.web.wesleyan.edu

1. The right way to think about localization in algebra

Localization is a **systematic way of adding multiplicative inverses to a ring**, i.e. given a commutative ring R with unity and a multiplicative subset $S \subset R$ (i.e. contains 1, closed under product), localization constructs a ring $S^{-1}R$ and a ring homomorphism $j : R \to S^{-1}R$ that takes elements in S to units in $S^{-1}R$. We want to do this in the best way possible, and we formalize that via a universal property, i.e. for any $f : R \to T$ taking S to units we have a unique g:

$$\begin{tikzcd} R \arrow[r, "j"] \arrow[d, "f"'] & S^{-1}R \arrow[dl, dashed, "g"] \\ T & \end{tikzcd}$$

Recall that $S^{-1}R$ is just $R \times S/\sim$ where (r, s) is really r/s and $r/s \sim r'/s'$ iff $t(rs' - sr') = 0$ for some t (i.e. fractions are reduced to lowest terms). The ring structure can be verified just as for $\mathbb{Q}$. The map j takes $r \mapsto r/1$, and given f you can set $g(r/s) = f(r)f(s)^{-1}$. Demonstrate commutativity of the triangle here.

The universal property is saying that $S^{-1}R$ is the **closest ring to R with the property** that all $s \in S$ are units. A category theorist uses the universal property to *define* the object, then uses $R \times S/\sim$ as a *construction* to prove it exists. An algebraist might define the localization to be $R \times S/\sim$ and then prove the universal property as a corollary. It's a philosophical difference.

Examples:

- $\mathbb{Z}, \mathbb{Z} - \{0\}) \mapsto S^{-1}R = \mathbb{Q}$. More generally: $Frac(R)$
- $(\mathbb{Z}, \langle 2\rangle) \mapsto \mathbb{Z}[\frac{1}{2}]$ the dyadic rationals.
- $(\mathbb{Z}, \mathbb{Z} - p\mathbb{Z}) \mapsto \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \nmid b\}$
- $R = \mathbb{Q} \times \mathbb{Q}$, $S = (1, 0)$ has $S^{-1}R = \mathbb{Q} \times \{0\} \cong \mathbb{Q}$, smaller than R. Indeed, $R \to S^{-1}R$ is an injection iff S does not contain any zero divisors.

This is NOT the right definition to a category theorist (no operation, so "multiplicative inverses?"). Better: **systematic way of formally inverting maps**. We can't do this for all maps, but we can do it for maps of the form $\mu_s : R \to R$ which take $r \mapsto s \cdot r$ for an element s.

Date: October 21, 2013.

Definition 1. *Given $s \in R$ a localization is a ring R_* containing s, such that*

(1) $\mu_s : R_* \to R_*$ *is an isomorphism*

(2) R_* *is universal with respect to this property, i.e. there is a map* $i : R \to R_*$ *and any time a map* $g : R \to T$ *takes* μ_s *to an isomorphism,*

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & R_* \arrow[dl, dotted] \\ & T & \end{tikzcd}$$

We prove these two notions of localization are the same, i.e. produce isomorphic rings. This is an example of finding the right proof in algebra to generalize to category theory. It's diagrammatic.

Proposition 2. $R_* \cong s^{-1}R$

Proof. We need maps $g : R_* \to s^{-1}R$ and $h : s^{-1}R \to R_*$ with $hg = id$ and $gh = id$. We'll get them by proving first $R \to s^{-1}R$ satisfies the universal property for R_* and second that $R \to R_*$ satisfies the universal property for $s^{-1}R$

Certainly $s \in s^{-1}R$ as $(s, 1)$. Also, μ_s is an isomorphism with inverse $\mu_{s^{-1}}$. So by the universal property of R_*, the map $j : R \to s^{-1}R$ gives $g : R_* \to s^{-1}R$ s.t. $g \circ i = j$

Next, the element s has an inverse in R_* because it's $\mu_s^{-1}(1)$ as $\mu_s^{-1}(1) \cdot s = \mu_s^{-1}(1) \cdot \mu_s(1) = (\mu_s^{-1} \circ \mu_s)(1) = 1$. So the universal property of $s^{-1}R$ gives $h : s^{-1}R \to R_*$ and

$$\begin{tikzcd} & R \arrow[dl, "i"'] \arrow[d, "j"] \arrow[dr, "i"] & \\ R_* \arrow[r, "g"'] & s^{-1}R \arrow[r, "h"'] & R_* \end{tikzcd}$$

The bottom is the identity because the two triangles are the same. So $h \circ g = id_{R_*}$. Same idea gets $g \circ h$. ∎

Punchline: Localization should be thought of as inverting maps

Second punchline: Universal properties are cool

2. Categories

A category $\mathcal{C}$ is a **class of objects** linked by arrows which preserve the structure of the objects. Formally, between any two objects X, Y we have a **set of morphisms** $\mathcal{C}(X, Y)$ such that there is always $1_X \in \mathcal{C}(X, X)$ and if $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ then $g \circ f \in \mathcal{C}(X, Z)$. The composition is associative and unital ($1_Y \circ f = f = f \circ 1_X$).

An **isomorphism** is a morphism $f : X \to Y$ such that there is some $h : Y \to X$ with $f \circ h = 1_Y$ and $h \circ f = 1_X$.

The notion of category provides a fundamental and abstract way to **describe mathematical entities and their relationships**. **Virtually every branch of modern mathematics** can be described in terms of categories. Thus, if we can phrase a concept in terms of categories, it will have versions in most fields of mathematics. Because of the power of the results, proofs can be technical, but life can be made much easier by finding the right definitions and proofs classically first, as we did above.

DECIDE WHETHER OR NOT TO INCLUDE THIS, BASED ON WHO COMES

Examples:

- Set, the category of sets and set functions. Isomorphisms are bijections
- Grp, the category of groups and group homomorphisms. Isomorphisms.
- Ab, the category of abelian groups and group homomorphisms. Isomorphisms.
- Graphs and graph homomorphisms $f : V(G) \to V(H)$ such that $f(v)f(u)$ is an edge of H whenever vu is an edge of G.
- Lie Groups, group homomorphisms which are also maps of smooth manifolds. Lie Algebras with algebra morphisms preserving the Jacobi
- Dynamical Systems and finitary maps.
- Top, the category of topological spaces and continuous maps. Point-set topology deals with spaces up to homeomorphism (the notion of isomophism here), whereas homotopy theory is less discriminating and views spaces X and Y as the same if they are homotopy equivalent (i.e. if X can be gradually deformed to match Y).
- HoTop, the category of topological spaces and homotopy classes of maps. Isomorphisms are now homotopy equivalences. Formally: view two spaces as isomorphic if there are maps $f : X \to Y$ and $g : Y \to X$ with $fg \sim id_Y$ and $gf \sim id_X$. For example, a circle and a cylinder (or donut) are now the same but they are not homeomorphic because the circle has cut vertices. Or the letters P and O.

The **yoga of category theory** is that one must study maps between objects to study the objects. Applying this to categories themselves leads you to functors $F : \mathcal{C} \to \mathcal{D}$, i.e. maps from objects to objects and morphisms to morphisms compatible with id_A and $f \circ g$ (i.e. they preserve structure). Formally, $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Examples:

- Forgetful functor: Grp $\to$ Set
- Free functor: Set $\to$ Grp. Or free abelian functor: Set $\to$ Ab
- Abelianization functor: Grp $\to$ Ab
- Cayley graph functor: Grp $\to$ Graph
- Inclusion: R-Mod $\to$ Ch(R), including the module in at level 0.

3. Localization for Categories

Thinking of localization as "formally inverting maps" then we want to pick a set W of morphisms and create a universal functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ where those morphisms land in the class of isomorphisms, i.e. $F(f)$ is an iso for all $f \in W$. Universal means if there is $\mathcal{C} \to \mathcal{D}$ taking W to isomorphisms then we have $\mathcal{C}[W^{-1}] \to \mathcal{D}$ making the triangle commute. This universal property forces ob($\mathcal{C}[W^{-1}]$) =ob($\mathcal{C}$).

Group theory - identify two groups A and B if A/B is abelian (more generally, if it lives in a **Serre $\mathcal{C}$-class**). In this way we can study group theory by removing some of the weird bits, e.g. metabelian groups, composition series, etc.

Example (from homotopy theory): If $\mathcal{C}$ is Top, and we **want to study it "up to homotopy"** (i.e. when X h.e. Y we say they are isomorphic), then we get the homotopy category. This functor zooms in on homotopy theoretic information by formally setting the homotopy equivalences to be isomorphisms (think of it as putting on a different pair of glasses). In $HoTop$ some spaces are declared to be "the same" up to isomorphism even if they are not homeomorphic. This process is universal because we added the smallest numbers of isomorphisms possible.

Example (homological algebra): Consider $Ch(R)$, the category of chain complexes over R (descending sequence A_n with $d^2 = 0$) with chain maps (f_n) with $f \circ \partial = \partial \circ f$.

$$\begin{tikzcd}
\dots \arrow[r, "d_A^{n-2}"] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f^{n-1}"'] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"'] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, "f^{n+1}"'] & \dots \\
\dots \arrow[r, "d_B^{n-2}"] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r, "d_B^{n+1}"] & \dots
\end{tikzcd}$$

Isomorphisms are levelwise. Works bounded or unbounded

Having $d^2 = 0$ means $\mathrm{im}(f_\bullet) \subset \ker(f_\bullet)$. If we think of the image as things which f knows how to construct and the kernel as the way of testing our knowledge of f then this says whatever we can construct can be tested. Obviously, we'd like the converse too, i.e. we want to find constructions for objects satisfying prescribed properties. So what we really want is $\mathrm{im}(f_\bullet) = \ker(f_\bullet)$. We can measure how much this fails by looking at homology $H_n(A_\bullet) = \mathrm{im}(f_n)/\ker(f_{n-1})$ at every level n.

$\mathcal{D}(R)$ the derived category of R, with objects as above and morphisms roofs $A \leftarrow X \to B$ where there is a map $f : A \to X$ which is a quasi-isomorphism, i.e. $H_*(f)$ is an isomorphism of homology groups. The passage from $Ch(R)$ to $\mathcal{D}(R)$ is localization at the quasi-isomorphisms. Again, this localization is putting on a different pair of glasses.

It is not true that for every choice of category and class of maps $\mathcal{C}, W$ there is a localized category $\mathcal{C}[W^{-1}]$. If we try to construct the category $\mathcal{C}[W^{-1}]$ in the only reasonable way, we see that the universal property forces it to have the same class of objects as $\mathcal{C}$. The morphisms are trickier. Given $f : X \to Y$ in W and $g : X \to Z$, we get $g \circ f^{-1} : Y \to Z$, i.e. we have to generate new morphisms based on the inverses I added. You can get there by any **zig-zag (DRAW IT)**, so you want to define $\mathcal{C}[T^{-1}](X, Y) = \{X \leftarrow \bullet \to \bullet \cdots \bullet \to Y\}/\sim$ where this relation at least allows us to add in pairs of identities or compose two when it's allowed. PROBLEM: the collection of zigzags $X \leftarrow \bullet \to \bullet \cdots \bullet \to Y$ is not a set; even just in the category Set you have a proper class worth of choices.

Attempting to get around these set-theoretic issues **leads you to model categories** (invented by Quillen in 1967). So you make some assumptions so that $\mathcal{C}$ behaves more like Top and W behaves like the (weak) homotopy equivalences. The idea is you have a special class of maps $\mathcal{W}$ called the weak equivalences, and these generalize the homotopy equivalences above. Quillen was able to define a very general notion of homotopy, via cylinder objects and path objects. To prove it's an equivalence relation is hard.

Quillen's clever observation was that in the examples of interest you can always replace your object by a nicer one which is homotopy equivalent. For spaces this can be CW approximation. For R-Mod and Ch(R) it can be **projective and injective resolution**. These are all cases where you can build more complicated object from simpler ones. Borrowing terminology from topology, we call a map a **cofibration** if we build it via colimits, wedges, and retracts of spheres and disks. In algebra this means building it from chains of free modules, so applying this process to an object $M_\bullet$ gives a projective resolution $P_\bullet \to M_\bullet$ called the cofibrant replacement of $M_\bullet$ (since the domain is cofibrant and the map is a quasi-isomorphism).

Dually, we can build an injective resolution for $M_\bullet$ by mapping to an injective object then taking the cokernel and mapping that to an injective and continuing in this way. Again we are building a chain complex up inductively from pieces we understand, but the pieces are dual to the above and now we get a map $M_\bullet \to I_\bullet$. Objects built in this way are called fibrant (because cocofibrant means fibrant) and the replacement of $M_\bullet$ by $I_\bullet$ is called fibrant replacement. (Remark for readers: there is a point here to be cautious. If you want projective resolution to be cofibrant replacement you need to be working with bounded below chain complexes, otherwise you only get DG projectives = cofibrants. Also, you can't have both cof rep = proj res and fib rep = inj res. There are two *different* model structures which provide these examples. See Hovey's book)

A **Model Category** is a category $\mathcal{M}$ with distinguished classes $\mathcal{W}, \mathcal{F}, \mathcal{Q}$ satisfying those properties. The localization described above for spaces works on any model category, i.e. you get a concrete way to make a universal functor $\mathcal{M} \to \operatorname{Ho}\mathcal{M}$ taking $\mathcal{W}$ to isomorphisms. So functors $\mathcal{M} \to \mathcal{C}$ which do this induce functors $\operatorname{Ho}\mathcal{M} \to \mathcal{C}$

Model categories are a general place you can do homotopy theory, and this transforms algebraic topology from the study of topological spaces into a general tool useful in many areas of mathematics.

Examples of model categories:

- Topological Spaces, with homotopy category HoTop.
- $Ch(R)$ is also a model category with homotopy category = the **derived category** $\mathcal{D}(R)$, which is studied in algebraic geometry and elsewhere. Proving it's triangulated uses the model category structure, which was a hard problem for unbounded chain complexes. Given F, the model category structure helps you **construct from an induced functor between derived categories**, e.g. the left derived functor of an abelianization functor gives **Quillen homology**. Quillen coined the phrase **homotopical algebra** for this type of study and it helped him do new **computations in algebraic K-theory**, for which he won a Fields Medal in 1978.
- Voevodsky won a Fields Medal in 2002 by creating the **motivic stable homotopy category** from a model category structure on an enlargement of **Schemes** to resolve the Milnor Conjecture. This is an application to algebraic geometry and number theory.
- Graphs. Two finite graphs are homotopy equivalent iff they have the same zeta series (reference Beth Malmskog).

4. Localization on Monoidal Model Categories

The localization above always lands in a homotopy category and always takes exactly the zig-zags of weak equivalences to isomorphisms. What if we want to invert some map which is not a weak equivalence? Let C be a set of maps in $\mathcal{M}$. Because the homotopy category is nice (admits a calculus of fractions), we can do:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, dotted] \arrow[d] & ????? \arrow[d, dotted] \\
\mathrm{Ho}(\mathcal{M}) \arrow[r] & \mathrm{Ho}(\mathcal{M})[C^{-1}]
\end{tikzcd}$$

We'd like a model category $L_C\mathcal{M}$ which actually sits above $\mathrm{Ho}(\mathcal{M})[C^{-1}]$, i.e. we'd like to know the functor $\mathcal{M} \to \mathrm{Ho}(\mathcal{M})[C^{-1}]$ factors through a model category. As above, its objects will be the same as those in $\mathcal{M}$. It's morphisms will be the same as those in $\mathcal{M}$, but we want the maps in C to become isomorphisms in $\mathrm{Ho}(\mathcal{M})[C^{-1}]$ so we need them to be weak equivalences in $L_C\mathcal{M}$. So this category must have a **different model category structure**, where $\mathcal{W}' = \langle C \cup \mathcal{W} \rangle$. For example the bottom arrow could be $K(R) \to \mathcal{D}(R)$ because you could first invert chain homotopy equivalences then quasi-isomorphisms.

5. Monoidal Model Categories

This functor L_C is known to preserve some types of structure and destroy others. My interest is in model categories where we can do algebra, i.e. in which we can study monoids, commutative rings, Lie algebras, etc. So my research is concerned with when L_C preserves this kind of structure. To do algebra we must assume our category is a **monoidal category**, i.e. it comes equipped with a product bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ which is associative (i.e. $\otimes \circ (\otimes \times 1) = \otimes \circ (1 \times \otimes)$) and has a unit object S (i.e. $S \otimes E = E = E \otimes S$). Here "=" means naturally isomorphic.

For a long time people assumed that L_C always preserved commutative structure, and in fact this led to a false claim in the proof of the Kervaire Invariant One problem. Mike Hill came up with a counterexample and was later able to patch the proof. That was the starting point of my thesis.

DETAILS ON KERVAIRE...

Recall that **certain spheres can have non-diffeomorphic smooth structures**, e.g. Milnor's famous example on S^7. The answer to the question of which dimensions n allow this is contained in the stable homotopy groups of spheres. The group of diffeomorphism structures can be constructed and understood up to a quotient term which depends on framed bordism. The monoid of smooth structures on S^n is isomorphic to the group Θ_n of h-cobordism classes of oriented homotopy n-spheres. This group can be understood via a quotient and the J-homomorphism. The quotient depends on whether or not there are **framed manifolds** of non-zero Kervaire invariant. The connection:

$\pi_{n+k}(S^n) \cong \{\text{framed } M^k \subset \mathbb{R}^{n+k}\}/\text{bordism}$

In the 60s Browder showed this could only occur if $n = 2^k - 2$. They can exist for $k < 7$. Hill-Hopkins-Ravenel 2009 showed that they cannot exist for $k > 7$. Their proof came down to a computation in the stable homotopy groups of spheres which relied on extra structure brought in from equivariant stable homotopy groups (with a $\mathbb{Z}/2$-action). In 2011 a small error was pointed out and a counterexample to a particular claim was given. Namely, the Bousfield localization of a commutative monoid need not be commutative. This was fixed in unpublished work of Hill and Hopkins. My advisor wanted a more general answer and that's where my thesis kicks off.

Some terminology...in a monoidal category, a monoid is E with maps $\mu : E \times E \to E$ and $\eta : S \to E$ satisfying some diagrams which encode the usual monoid laws for associativity and unit:

$$\begin{tikzcd} E\otimes(E\otimes E) \arrow[r, "\cong"] \arrow[d, "1\otimes\mu"] & (E\otimes E)\otimes E \arrow[r, "\mu\otimes 1"] & E\otimes E \arrow[d, "\mu"] \\ E\otimes E \arrow[rr, "\mu"] & & E \end{tikzcd} \qquad \begin{tikzcd} S\otimes E \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & E\otimes E \arrow[d, "\mu"] & E\otimes S \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\ & E & \end{tikzcd}$$

A **commutative monoid** is one which also has a twist isomorphism $\tau : E\otimes E \to E\otimes E$ and more diagrams which say the twist commutes with multiplication.

A useful tool to encode algebraic structure is the notion of an **operad**. This is a gadget for **universal algebra**, i.e. every single operad corresponds to a whole type of algebraic structure, e.g. monoid structure, commutative monoid structure, Lie algebra structure, etc. The shadow of an operad P in a category $\mathcal{C}$ is the collection of objects with that algebraic structure (formally, E with $P \to End(E)$).

If $\mathcal{M}$ is a model category and a monoidal category (coherently), and E is a P-algebra then we can ask whether or not $L_C(E)$ is also a P-algebra. In my thesis I prove a general theorem for when this occurs.

Theorem 3. *If P-algebras in $\mathcal{M}$ and in $L_C(\mathcal{M})$ inherit model structures in the usual way then L_C preserves P-algebras.*

The standard hypotheses for this situation are that $\mathcal{M}$ has the pushout product axiom and the monoid axiom. Without those you have no hope of a good homotopy theory for P-algebras. I also assume other technical but non-restrictive hypotheses on $\mathcal{M}$, and then **I provide conditions on C so that the pushout product and monoid axioms are preserved** (indeed, I classify localization functors which accomplish this; my hypothesis is implied by $L(X\otimes Y) \simeq LX \otimes LY$ or by $L\Sigma \simeq \Sigma L$). For P cofibrant (in the model structure on operads) this is all you need, so I have a very general theorem which states that P-algebra structure is preserved for cofibrant operads. This has led to work recently, joint with Javier Gutierrez, about when an equivariant operad is cofibrant and when a localization is monoidal.

The **case of commutative monoids is harder**, because the operad Com is not cofibrant. So I needed a hypothesis on a model category which guarantees that commutative monoids inherit a model structure. I find a very general condition and it recovers the examples of interest (CDGA over characteristic 0, positive symmetric spectra, simplicial sets). I also find conditions on C to preserve this (one needs $Sym(-)$ to preserve L-equivalences) and in doing so **recover the theorem of Hill and Hopkins** which patched the Kervaire proof.