BOUSFIELD LOCALIZATION OF MONOIDAL MODEL CATEGORIES

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1. Outline

(1) Localization
(2) Categories
(3) Localization of Categories

- My research in the context of model categories, sometimes the category Spectra
- apply algebraic ideas and methods, replacing algebraic concepts by their categorical analogs.
- need the right definition and proofs in algebra, then lift the proofs.

Alg. Top	ring-objects	derived category	qual talk	thesis	today
Algebra	Rings	R-mod	homological dimension	ideals	localization

2. Localization

Studying an algebraic object **"at" a prime**, b/c it's easier, e.g. solving eqn's mod p

Piecing these together (**"local-to-global question"**), e.g. Chinese Remainder Thm. HARD (obstruction theory)

In algebra, localization is a systematic method of **adding multiplicative inverses** to a ring. Given commutative R and $S \subset R$, construct some ring $S^{-1}R$ and ring homomorphism $j : R \to S^{-1}R$, such that the image of S consists of units (invertible elements) in $S^{-1}R$. Make $S^{-1}R$ the 'smallest' ring with this property, i.e. make it satisfy a **universal property**: the ring homomorphism $j : R \to S^{-1}R$ maps every element of S to a unit in $S^{-1}R$, and if $f : R \to T$ is some other ring homomorphism which maps every element of S to a unit in T, then there exists a unique ring homomorphism $g : S^{-1}R \to T$ such that $f = g \circ j$.

Example: If $R = \mathbb{Z}$ and $S = \mathbb{Z} - \{0\}$ then $S^{-1}R = \mathbb{Q}$

Suppose $S \subset R$ is a **multiplicative set**, i.e. 1 is in S and for s and t in S we also have $st \in S$. On $R \times S$ define an equivalence relation $\sim$ by setting $(r1, s1) \sim (r2, s2)$ iff there exists $t \in S$ such that $t(r1s2 - r2s1) = 0$. Think of the equivalence class of (r, s) as the "fraction" r/s and, using this intuition, the set of equivalence classes $S^{-1}R$ can be turned into a ring with operations that look identical to those of elementary algebra: $r1/s1 + r2/s2 = (r1s2 + r2s1)/s1s2$ and $(r1/s1)(r2/s2) = r1r2/s1s2$. The map $j : R \to S^{-1}R$ which maps r to the equivalence class of (r,1) is then a ring homomorphism.

$S^{-1}R$ satisfies the universal property because given f define $g(r/s) = f(r)f(s)^{-1}$. This is well defined because $r1/s1 = r2/s2$ implies $x(s2r1 - r2s1) = 0$ so $f(s2r1) = f(r2s1)$ so $f(r1)f(s1)^{-1} =$

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$f(r2)f(s2)^{-1}$. This g is a ring homomorphism because the operations on $S^{-1}R$ were defined with this in mind (they work just like in $\mathbb{Q}$). Also, $g(j(r)) = g(r/1) = f(r)f(1) = f(r)$.

Examples:

(1) If $R = k$ is a field and $0 \notin S$ then $S^{-1}R = k$ because all elements in S were already invertible. If $0 \in S$ then 0 becomes a unit, so $0 * 0^{-1} = 1$, proving $1 = 0$ and so $S^{-1}R = \{0\}$.

(2) Let p a prime ideal, so $R - p$ is a multiplicative system and the corresponding localization is denoted R_p. The unique maximal ideal is then p, so R_p is a local ring. From the point of view of the **spectrum of a ring**, the primes are the points of a ring, and thus localization studies a ring at just one point.

(3) If $R = K[X]$ is the polynomial ring and $S = \{X\}$ then the localization produces the ring of Laurent polynomials $K[X, X^{-1}]$. In this case, localization corresponds to the embedding $U \to \mathbb{A}^1$, where $\mathbb{A}^1$ is the affine line and U is its Zariski open subset which is the complement of 0.

The ring homomorphism $R \to S^{-1}R$ is injective if and only if S does not contain any zero divisors. We see examples above where this holds and where it fails.

I'll define categories later, but before that I need to let you in on a secret. This is NOT the right definition of localization (to a category theorist). Categories don't have an operation, so what does "multiplicative inverse" mean?

Another way to think about localization of rings is as formally inverting *maps*. In particular, to invert $s \in S$ you take the ring generated by R and s^{-1}. Equivalently, simply insist that the multiplication by s map $\mu_s : R \to R$ be invertible.

Proposition 1. *Suppose R_* is a ring containing s on which μ_s is an isomorphism. Further, suppose R_* is* ***universal with respect to this property****, i.e. there is a unique ring homomorphism $i : R \to R_*$ and for any $f : R \to T$ with $\mu_s : T \to T$ an isomorphism, there exists a unique $g : R_* \to T$ such that $g \circ i = f$. Then $R_* \cong s^{-1}R$.*

Proof. First, $s^{-1}R$ contains s and has μ_s an isomorphism (it's inverse if $\mu_{s^{-1}}$. Thus, the map $j : R \to s^{-1}R$ yields a unique map $g : R_* \to s^{-1}R$ such that $g \circ i = j$. Next, R_* is a ring where s is invertible because $\mu_s^{-1}(1) \cdot s = \mu_s^{-1}(1) \cdot \mu_s(1) = (\mu_s^{-1} \circ \mu_s)(1) = 1$. So the universal property of localization implies there's a unique map $h : s^{-1}R \to R_*$ such that $h \circ j = i$:

$$\begin{tikzcd}
 & R \arrow[dl, "i"'] \arrow[d, "j"] \arrow[dr, "i"] & \\
R_* \arrow[r, "g"] & S^{-1}R \arrow[r, "h"] & R_*
\end{tikzcd}$$

The bottom composition must be the identity on R_* because the two triangles are the same. This proves $h \circ g = id_{R_*}$. Draw a similar picture to prove $g \circ h$ is the identity on $S^{-1}R$. □

We can use this alternate characterization of localization to generalize localization to modules, since μ_s acts on any R-module M.

Going off this idea, let's say M is S**-local** if $\mu_s : M \to M$ is an isomorphism for all $s \in S$.

A map $f : M \to N$ is an S**-local equivalence** if $f^* : \mathrm{Hom}(N, T) \to \mathrm{Hom}(M, T)$ is an isomorphism for all S-local T.

Note: f^* is like the S-localization of f, so an S-equivalence is one which goes to an iso after S-localization

Proposition 2. *$j : R \to S^{-1}R$ is an S-equivalence and $S^{-1}R$ is S-local.*

Proof. We already proved $S^{-1}R$ is S-local. Let T be S-local. Then $\mathrm{Hom}(S^{-1}R, T) \to \mathrm{Hom}(R, T) \cong T$ sends f to $f \circ j$ to $(f \circ j)(1)$, i.e. to $f(1/1)$. This is 1-1 because if $f(1_{S^{-1}R}) = 0$ then f must be the zero map. This is a homomorphism because $(f+g)(1) = f(1)+g(1)$ and $(f \times g)(1) = f(1)g(1)$. This is onto because for any $t \in T$ we simply define f to take $1/1$ to t. We needed T to be S-local to even form $\mathrm{Hom}(S^{-1}R, T)$, i.e. for these maps to be well-defined. □

R-Mod: Let S a multiplicatively closed subset of R. Then $S^{-1}M = M \times S/\sim$ where $(m, s) \sim (n, t)$ if there is $u \in S$ such that $u(sn - tm) = 0$.

Universal Property: There is a module homomorphism $j : M \to S^{-1}M$ s.t. for any S-local T with $M \to T$ there exists a unique module homomorphism $S^{-1}M \to T$ making the triangle commute. Note: $S^{-1}M = M \otimes_R S^{-1}R$, by the very definition of "extension of scalars."

3. Categories

Category comprises "objects" linked by "arrows". Two basic properties: the ability to compose arrows and the existence of an identity arrow for each object. The objects and arrows may be **abstract entities of any kind**, so category theory is a fundamental and abstract way to **describe all kinds of mathematical entities and their relationships**, independent of what the objects and arrows represent.

A **category** C consists of

- a class ob(C) of objects
- a class hom(C) of morphisms (arrows) between the objects. Each f has a unique source object a and target object b in ob(C). Write $f : a \to b$. Write $C(a, b)$ to denote the class of all morphisms from a to b.
- for every three objects a, b and c, a binary operation $C(a, b) \times C(b, c) \to C(a, c)$ called composition of morphisms: $(f, g) \to g \circ f$

Satisfying the axioms:

- (associativity) $ho(gof) = (hog)of$
- (identity) for every object x, there exists a morphism $1_x : x \to x$ called the identity morphism for x, such that for every morphism $f : a \to b$, we have $1_b of = f = fo1_a$.

Examples:

- Set, the category of sets and set functions. Isomorphisms are bijections.
- Grp, the category of groups and group homomorphisms. Isomorphisms.
- Ab, the category of abelian groups and group homomorphisms. Isomorphisms.
- CRing, the category of commutative rings and ring homomorphisms
- R-Mod, the category of R-modules and module homomorphisms
- Top, category of topological spaces and continuous maps. Homeomorphisms
- Top$_*$, category of topological spaces with a distinguished choice of basepoint and continuous basepoint-preserving maps. Homeomorphisms

- Banach spaces and bounded linear maps...but no localization here because no operation.
- Graphs and graph homomorphisms.

An **isomorphism** is a morphism $f : a \to b$ s.t. there exists $g : b \to a$ and $f \circ g = 1_b$, $g \circ f = 1_a$.

Morphisms preserve structure of objects. The fundamental observation is **need to understand maps to understand objects**. The "maps between categories" are **functors**, i.e. $F : C \to D$ which associates to each object $X \in C$ an object $F(X) \in D$ and associates to each morphism $f : X \to Y \in C$ a morphism $F(f) : F(X) \to F(Y) \in D$ such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object $X \in C$
- $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$. Note that a contravariant functor has $F(g \circ f) = F(f) \circ F(g)$.

Functors preserve isomorphisms because $Ff \circ Fg = F(f \circ g) = F(1_b) = 1_{Fb}$ and $Fg \circ Ff = F(g \circ f) = F(1_a) = 1_{Fa}$.

Examples

- Forgetful functor: Grp$\to$ Set.
- Free functor: Set$\to$ Grp. Or free-abelian functor from Set$\to$ Ab
- Abelianization: Grp$\to$ Ab.

4. Localization of Categories

Think of localization as "formally inverting maps," sending a class of morphisms into isomophisms. Example: $\mathcal{C}$ is Top$_*$ and $\mathcal{S}$ is the class of homotopy equivalences (i.e. $f : X \to Y$ s.t. there exists $g : Y \to X$ and $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$). Then $\mathcal{C}[\mathcal{S}^{-1}]$ is $Ho\mathcal{C}$ the category of topological spaces up to homotopy equivalence. Another example is the definition of a derived category, inverting quasi-isomorphisms.

Universal Property: Localization in categories gives a functor $F : \mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ **universal** w.r.t. the property that it takes $s \in \mathcal{S}$ to an isomorphism.

Just like you have to **generate** using R and s^{-1} in rings, you need to formally add all composites which use the new morphisms. So now you could have an isomorphism $\{a \leftarrow \bullet \to \bullet \leftarrow \bullet \cdots \bullet \to b\}$, call it a **zigzag**.

To construct $\mathcal{C}[\mathcal{S}^{-1}]$ we want to allow morphisms to be equivalence classes of zigzags, i.e. $\mathcal{C}[\mathcal{S}^{-1}](a, b) = \{a \leftarrow \bullet \to \bullet \leftarrow \bullet \cdots \bullet \to b\}/\sim$. Sadly, this is a **proper class** NOT A SET so we can't do equivalence relations. It worked above with Top$_*$, and it will work again if we simply generalize that example. This leads to the notion of a **Model Category**, i.e. a category $\mathcal{M}$ along with three classes of morphisms called weak equivalences ($\mathcal{W}$), fibrations ($\mathcal{F}$), and cofibrations ($\mathcal{C}$), satisfying some complicated axioms. It's the **most general place you can do homotopy theory**. What model categories are good for is doing this process and getting homotopy categories, plus doing constructions at the point-set level which you know will carry over to the homotopy level.

For topological spaces, F is Serre fibrations, W is weak homotopy equivalences, and C is harder to describe, but is determined by the other two. It's not easy to get a flavor for what F and C are in general, but W is the class you're going to invert, so it's always your **choice of "homotopy equivalence."** On the category Set there are 9 valid choices for W, F, C which give different model category structures. In one, C are injections and F are surjections. In another that's switched. The point is: it gets pretty crazy.

Ok, so we know why we care about model categories. The problem is that the localization above takes you out of the category $\mathcal{M}$ you'd been working in. It's somehow not of the same flavor as the ring localizations above, mainly because $S^{-1}R$ was still a ring. What if we want to study a space or a spectrum localized at p? How can we reduce studying π_* to a single prime using the localization above? The answer is: you can't. This is why we have **Bousfield Localization**. The idea here is to **add to the class of weak equivalences** in a model category, knowing that these will BECOME isomorphisms in the homotopy category. So suppose we have a class $\mathcal{S}$ of maps we'd like to turn into weak equivalences.

Following Prop 2, define $M \in \mathcal{M}$ to be **$\mathcal{S}$-local** if M is fibrant and for all $s : X \to Y \in \mathcal{S}$, $s^* : \mathcal{M}(Y, M) \to \mathcal{M}(X, M)$ is a weak equivalence. This s^* is just like mult by s.

An object M is fibrant if the map $M \to *$ is a fibration (here $*$ is the terminal object).

A map $f : A \to B$ is an **$\mathcal{S}$-local equivalence** if for all $\mathcal{S}$-local M, $f^* : \mathcal{M}(B, M) \to \mathcal{M}(A, M)$ is a weak-equivalence.

The **(left) Bousfield localization** of $\mathcal{M}$ w.r.t. $\mathcal{S}$ is a new model category structure on $\mathcal{M}$ with the same cofibrations as $\mathcal{M}$ and with weak equivalences equal to $\mathcal{S}$-local equivalences. Denote this model structure M_L. Note that weak equivalences of $\mathcal{M}$ are still weak equivalences, but now there are more of them.

w.e.$(M) \subset$ w.e.(M_L)

$\text{cof}(M) = \text{cof}(M_L)$

$\text{fib}(M) \supset \text{fib}(M_L)$

The identity functor gives a Quillen adjoint pair (i.e. a functor which preserves the model category structure):

$1 : \mathcal{M} \rightleftarrows M_L : 1$. Fibrant X go via weak equivalence to LX, the L-locals (analogs of $S^{-1}R$.

$F : Ho\mathcal{M} \rightleftarrows HoM_L : U$ and F takes the images in $Ho\mathcal{M}$ of maps in $\mathcal{S}$ into isomorphisms in HoM_L, and M_L is the smallest model category with this property, i.e. if there's another $\mathcal{N}$ then we get a unique left Quillen functor: $M_L \to \mathcal{N}$.

It's not that surprising that localization in algebra is a special case of this, since we defined it in complete analogy. What is amazing is that **completion** in algebra is also a special case. Algebraic geometers often need to localize and then complete, and they are unrelated operations. In algebraic topology the situation is often far more complicated than that for algebra, but in this one case it's simpler. Completion often takes the form in algebra of an inverse limit. There has to be a topology running around in order for complete to make sense (Cauchy sequences converge).

5. My Work

To really generalize algebra, you need a notion of a ring in a category. A category is said to be **monoidal** if there is a bifunctor $\otimes : C \times C \to C$ which is **associative** ($\otimes(\otimes \times 1) = \otimes(1 \times \otimes)$ and has a **unit object** e along with $\lambda_a : e \otimes a \to a$ and $\rho_a : a \otimes e \to a$ s.t.

$$\begin{tikzcd}
a \otimes (e \otimes c) \arrow[rr, "\alpha"] \arrow[dr, "\lambda"'] & & (a \otimes e) \otimes c \arrow[dl, "\rho"] \\
& a \otimes c &
\end{tikzcd}$$

You also need coherence diagrams for 4-fold associativity. A **ring object** $R \in C$ **has** $\mu : R \otimes R \to R$ **which is associative and** $\eta : e \to R$ s.t.

$$\begin{tikzcd} e \otimes c \arrow[r, "\eta"] \arrow[dr, "\lambda"'] & c \otimes c & c \otimes e \arrow[l, "\eta"'] \arrow[dl, "\rho"] \\ & c & \end{tikzcd}$$

My task: find conditions on $\mathcal{M}$ and on the functor L (equiv: on the class $\mathcal{S}$) such that a commutative ring object $R \in \mathcal{M}$ goes to a commutative ring object in $LR \in L_{\mathcal{S}}\mathcal{M}$. We know already that $Ho(R) \mapsto Ho(LR)$, but not on the level of model categories. This really comes down to understanding those functors F and U from before, and using the fact that they are derived functors of much nicer functors. It also involves proving $L_{\mathcal{S}}\mathcal{M}$ is monoidal, and that the category of (commutative) ring objects forms a model category.