SOME NEW RESULTS IN HOMOTOPICAL ALGEBRA

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1. Abstract

Abstract: In my last GSS talk I discussed monoidal model categories, operads, and placing model structures on algebras over an operad. This lets you do algebra in a model category in a very general way. One thing people in algebra like to do is localization. Of course, localization of a category is already intrinsically tied to the story of model categories, since the whole point of a model category is to allow you to invert the weak equivalences. But what if you want to invert a map which is not a weak equivalence? Can you find a new model structure where this map is a weak equivalence? The answer to this question (and the process of moving from the old model structure to the new one) is called Bousfield Localization. I'll motivate this concept and then discuss my recent work on when Bousfield Localization preserves these model structures on algebras over operads discussed last time. Most likely I'll restrict attention entirely to commutative monoids and try to give lots of examples and motivation.

2. Review of last GSS talk

A **model category** $\mathcal{M}$ is the most general place one can do homotopy theory. They solve the following problem: given a category and a class of maps $\mathcal{W}$ to invert (e.g. homotopy equivalences), when can we find a new category $Ho(\mathcal{M})$ and a universal map $\mathcal{M} \to Ho(\mathcal{M})$ taking $\mathcal{W}$ into the isomorphisms? Note that universality implies the objects of $Ho(\mathcal{M})$ are the same as the objects of $\mathcal{M}$, but if we pass to isomorphism classes more objects will be identified than by isomorphisms in $\mathcal{M}$.

To actually construct $Ho(\mathcal{M})$ requires two other classes of maps: cofibrations Q are like gluing on cells, and it lets you build complicated objects from simple ones. If you think of Q as monomorphisms that's fine. Fibrations $\mathcal{F}$ are like covering spaces, or fiber bundles. It lets you take quotients.

Examples of model categories:

(1) Top
(2) sSet (like simplicial complexes)
(3) Spectra (X_n) used in stable homotopy theory. There are many structures here, e.g. Symmetric Spectra, S-modules, orthogonal spectra, G-spectra
(4) Ch(R) - this leads to Andre-Quillen cohomology, for which Quillen won the Fields
(5) DGA (graded algebra equipped with a map $d\colon A \to A$ which is degree -1 and has $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$)
(6) StMod - Daniel researched this
(7) EnlargedSchemes - used in Voevodsky's proof of Milnor Conjecture, won Fields

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Quillen coined the phrase "homotopical algebra" and used it to mean "the study of model categories," though in his case this usually entailed Ch(R) and the derived category D(R).

A **monoidal category** is one with a bifunctor $\otimes : C \times C \to C$ which is associative and has a unit S. A **monoid** $R \in C$ has $\mu : R \otimes R \to R$ associative and $\eta : S \to R$ with

$$\begin{tikzcd} R \otimes S \arrow[r, "\eta"] \arrow[dr, "\rho"'] & R \otimes R \arrow[d, "\mu"] & S \otimes R \arrow[l, "\eta"'] \arrow[dl, "\lambda"] \\ & R & \end{tikzcd}$$

Think about the category of rings. We have $\eta : \mathbb{Z} \to R$ for any R, and that picks out the identity element in R via the image of 1. Think of μ as taking $(a, b) \mapsto a \cdot b$. A **commutative monoid** is a monoid along with a twist isomorphism $\tau : R \otimes R \to R \otimes R$ which commutes with μ, i.e. $a \cdot b = b \cdot a$.

We can restrict focus to the **subcategory of monoids** in C. Call this $Mon(C)$. It's objects are monoids and its morphisms are monoid homomorphisms, i.e. $f : R \to R'$ such that $f \circ \mu = \mu \circ f$. Passage from C to $Mon(C)$ is just like passage from Set to $Group$. It's a passage that we have to make in order to "do algebra" in C. Just having products on the category is not enough without having morphisms containing the information of multiplication on monoids R. Similarly, let $CMon(C)$ be commutative monoids in C. You can get much more general algebraic structure via operads.

3. Bousfield Localization

So that's one way to "do algebra" in a model category, and we studied it last time. What if we want to "do localization" in the (2-) category of model categories? The localization which comes for free for model categories ($\mathcal{M} \to Ho(\mathcal{M})$) is unsatisfactory because it takes us out of the category of model categories. Also, the class of maps you're allowed to invert is fixed at the start. What if I want to invert some map $f \notin \mathcal{W}$? Because the homotopy category is nice (admits a calculus of fractions), we can do:

$$\begin{tikzcd} \mathcal{M} \arrow[r, dotted] \arrow[d] & ????? \arrow[d, dotted] \\ \mathrm{Ho}(\mathcal{M}) \arrow[r] & \mathrm{Ho}(\mathcal{M})[f^{-1}] \end{tikzcd}$$

We'd like a model category $L_f\mathcal{M}$ which actually sits above $\mathrm{Ho}(\mathcal{M})[f^{-1}]$. Because all three categories above have the same objects, its objects are determined. It's morphisms will be the same as those in $\mathcal{M}$, but we want f to become an isomorphism in $\mathrm{Ho}(\mathcal{M})[f^{-1}]$ so we need it to be a weak equivalences in $L_f\mathcal{M}$. So this category must have a **different model structure**, where $\mathcal{W}' = \langle f \cup \mathcal{W} \rangle$ and clearly $\mathcal{W} \subset \mathcal{W}'$. You can't change only $\mathcal{W}$ because it'll screw up the axioms. We want to keep the cofibrations fixed so we can build things out of them and have the two model structures related, so we have to shrink the fibrations: $\mathcal{F} \supset \mathcal{F}'$. **Bousfield's Theorem** (1978) says you can do this and you still get a model structure, but you have to be careful with how you generate $\mathcal{W}'$ from f.

Formally, define $X \in \mathcal{M}$ to be f-local if X is fibrant and $f^* : Map(B, X) \to Map(A, X)$ is a weak equivalence, where $f : A \to B$. These look like trivial objects to the eyes of f. Define $g : C \to D$ to be an f-local equivalence if for all f-local X, $Map(D, M) \to Map(C, M)$ is a weak equivalence. This follows the idea in algebra, where a module M is S-local if μ_s is an isomorphism for all $s \in S$. A map is an S-equivalence if applying $Hom(-, M)$ gives an isomorphism for all S-local M. It turns out $R \to R[S^{-1}]$ is an S-equivalence to an S-local object. We'd call that fibrant replacement in $L_f(\mathcal{M})$.

The identity maps $\mathcal{M} \rightleftarrows L_f\mathcal{M}$ are a Quillen adjoint pair and prove that $L_f\mathcal{M}$ satisfies a universal property as the "closest" model category to $\mathcal{M}$ in which f is a weak equivalence. The fibrant objects in $L_f\mathcal{M}$ are the

f-local objects. Bousfield localization gives a Quillen pair (L_f, U_f), which are both the identity functors on objects and morphisms, and these induce (L_f^H, U_f^H) on the homotopy level. The functor L_f preserves many nice properties, e.g. left properness, which is a standard hypothesis that lets you build things via pushouts (aka gluing) in a way which is compatible with the homotopy theory.

4. Motivating Question

Bringing back the monoidal structure, recall that the right notion of monoidal in the model category context involves a coherence condition between the monoidal structure on $\mathcal{M}$ and the model structure:

Given $f : A \to B$ and $g : X \to Y$, define the **pushout product** $f \Box g$ to be the corner map in

$$\begin{tikzcd}
A \otimes X \arrow[r] \arrow[d] & A \otimes Y \arrow[d] & \\
B \otimes X \arrow[r] & Q_2 \arrow[dr] & \\
& & B \otimes Y
\end{tikzcd}$$

Pushout product axiom: if $f, g \in \mathcal{Q}$ then $f \Box g \in \mathcal{Q}$. Additionally, if either is in $\mathcal{W}$ then $f \Box g \in \mathcal{W}$.

Unit Axiom: If Z is cofibrant then $QS \otimes Z \to S \times Z \cong Z$ is a weak equivalence.

These axioms assure you that $Ho(\mathcal{M})$ is a monoidal category. In a monoidal model category $\mathcal{M}$, a strict commutative ring object is an object $[E] \in \mathrm{Ho}(\mathcal{M})$ such that there exists a representative $E \in \mathcal{M}$ whose structure diagrams (which make it a commutative ring object) commutes on the nose rather than up to homotopy. For many years everyone assumed Bousfield localization preserved strict commutative ring objects, because it works in S-modules and because Bousfield localization preserves E_∞ algebras, A_∞ algebras, and monoids. Mike Hill (2011) showed that for the model category of G-equivariant spectra it **does NOT preserve strict commutative monoids**. My goal is to **find conditions on $\mathcal{M}$ and f under which Bousfield localization does preserve strict commutative monoids**. This means we're asking for $(U_f^H \circ L_f^H)([E])$ to be a strict commutative monoid.

Reason we care: in the Hill-Hopkins-Ravenel proof of the Kervaire Invariant One Theorem the authors implicitly assumed Bousfield localization in G-equivariant spectra does preserve strict commutative monoids. The error was pointed out by Justin Noel, and the authors then found a counterexample to the claim. Mike Hill was able to patch this by adding strong hypotheses which happened to be satisfied in his case and which only work in G-equivariant spectra.

The Kervaire invariant problem is 45 years old. It asks in which dimensions n there are n-dimensional framed manifolds of nonzero Kervaire invariant. The solution completes the work on 'exotic spheres begun by John Milnor in the 1950s which led to his Fields Medal. This is a central part of the classification of manifolds.

The Kervaire invariant of a manifold M is the Arf invariant of a particular quadratic form determined by the framing on a $\mathbb{Z}/2\mathbb{Z}$ homology of M. The quadratic form can be defined by algebraic topology using functional Steenrod squares, and geometrically via the self-intersections of immersions $S^{2m+1} \to M^{4m+2}$ determined by the framing, or by the triviality/non-triviality of the normal bundles of embeddings $S^{2m+1} \to M^{4m+2}$ (for $m \neq 0, 1, 3$) and the mod 2 Hopf invariant of maps $S^{4m+2+k} \to S^{2m+1+k}$ (for $m = 0, 1, 3$). So really it's measuring something about mapping spheres into manifolds or into other spheres and you can reduce the problem to one about the existence of certain elements in the stable homotopy groups of spheres. This is how it was eventually solved.

5. New Results

Theorem 1. *Suppose $L_f\mathcal{M}$ is a monoidal model category such that CMon($L_f\mathcal{M}$) and CMon($\mathcal{M}$) inherit model structures in the usual way. Then the Bousfield localization preserves strict commutative monoids.*

Proof. Consider $U_f^H : \mathrm{Ho}\, L_f\mathcal{M} \to \mathcal{M}$, the right derived functor of id: $L_f\mathcal{M} \to \mathcal{M}$. In general, $U_f^H([X]) = [R_f X]$ where R_f is fibrant replacement in $L_f\mathcal{M}$. Given $[E]$ as above, $U_f^H(L_f^H([E])) = [R_f QE]$. We'll show this object can be represented by the commutative monoid $R_{f,m}Q_mE$ where Q_m is cofibrant replacement in $CMon(\mathcal{M})$ and $R_{f,m}$ be fibrant replacement in $CMon(L_f\mathcal{M})$.

The map $Q_mE \to E$ is a weak equivalence in $CMon(\mathcal{M})$, hence in $\mathcal{M}$. The map $QE \to E$ is also a weak equivalence in $\mathcal{M}$ and lifting gives a map (necessarily a weak equivalence) from $QE \to Q_mE$.

Since Q_mE is a commutative monoid in $\mathcal{M}$ it must also be a commutative monoid in $L_f\mathcal{M}$, since the monoidal structure of the two categories is the same. We may therefore do fibrant replacement on it in CMon($L_f\mathcal{M}$) and construct a lift:

$$\begin{tikzcd}
Q_mE \arrow[r] \arrow[d, hook] & R_{f,m}Q_mE \arrow[d, two heads] \\
R_fQ_mE \arrow[r] \arrow[ur, dotted] & \star
\end{tikzcd}$$

Using this lift we can draw a much more complicated diagram where all the arrows are weak equivalences in $L_f\mathcal{M}$ and those in the triangle of fibrant replacements are weak equivalences in $\mathcal{M}$ because those objects are local:

$$\begin{tikzcd}
QE \arrow[rr] \arrow[dd] & & Q_mE \arrow[dl] \arrow[dd] \\
& R_fQ_mE \arrow[dr] & \\
R_fQE \arrow[ur] \arrow[rr] & & R_{f,m}Q_mE
\end{tikzcd}$$

The triangle commutes because the bottom map is defined as the composite. The square commutes in $\mathrm{Ho}\,\mathcal{M}$ and demonstrates that R_fQE is isomorphic in $\mathrm{Ho}\,\mathcal{M}$ to the commutative monoid $R_{f,m}Q_mE$. This proves Bousfield localization preserves strict commutative monoids. □

It's a bit unfair to just assume CMon($L_f\mathcal{M}$) is a model category. After all, it can be very difficult to get your hands on $L_f\mathcal{M}$. We'd rather have hypotheses on $\mathcal{M}$ and f to make sure this situation happens. Recall from last time that there is an axiom on $L_f\mathcal{M}$ which will guarantee CMon($L_f\mathcal{M}$) is a model category:

Σ_n-Equivariant Monoid Axiom: If h is a (trivial) cofibration then $g^{\square n}/\Sigma_n = * \otimes_{\Sigma_n} g^{\square n}$ is a (trivial) cofibration.

Theorem 2. *If C is a monoidal model category satisfying the monoid axiom and the Σ_n-Equivariant Monoid Axiom then $CommMon(C)$ is a model category.*

Thus, we need to prove $L_f\mathcal{M}$ is a monoidal model category satisfying the monoid axiom and the Σ_n-Equivariant Monoid Axiom.

There are standard hypotheses on a model category $\mathcal{M}$ when one is working with Bousfield localization (cocomplete, cofibrantly generated, left proper, almost finitely generated, can choose domains and codomains of generating (trivial) cofibrations to be cofibrant) and also when one is in a monoidal situation (pushout product axiom, cofibrant objects flat, monoid axiom). Assume these from now on.

5.1. $L_f(\mathcal{M})$ is a monoidal model category.

I found that even under all the standing hypotheses, $L_f\mathcal{M}$ could fail to be a monoidal model category, though it is always a model category and monoidal (i.e. the coherence fails). If we place an assumption on the map f to be inverted (just like Quillen had to do), we can get around this.

Theorem 3. *Under the standing hypotheses above, if for* ***all domains and codomains*** K ***of*** $I \cup J$***, maps in*** $f \otimes id_K$ ***are*** f***-local equivalences****, then* $L_f\mathcal{M}$ *is a monoidal model category.*

Proof. It's sufficient to check the pushout product axiom on generating cofibrations. So suppose $h : X \to Y$ is an $L_f\mathcal{M}$ trivial cofibration and $g : K \to L$ is a generating cofibration in $L_f\mathcal{M}$. We must show $h\square g$ is an $L_f\mathcal{M}$ trivial cofibration.

The hypothesis that the generating (trivial) cofibrations have cofibrant domain is preserved by localization, so we can assume K and L are cofibrant. Because h is a cofibration, $K \otimes h$ and $L \otimes h$ are cofibrations.

The hypothesis that cofibrant objects are flat says if α is a weak equivalence and X is cofibrant then $X \otimes \alpha$ is a weak equivalence. This property is preserved by localizations satisfying the hypothesis of the theorem. Incidentally, this property implies the unit axiom, so now we have the unit axiom on $L_f\mathcal{M}$. Because cofibrant objects are flat in L_fM, $K \otimes h$ and $L \otimes h$ are also weak equivalences.

$$\begin{tikzcd}
K \otimes X \arrow[r, hook, "\simeq"] \arrow[d] & K \otimes Y \arrow[d] \arrow[ddr, bend left] & \\
L \otimes X \arrow[r, "\simeq"] \arrow[drr, bend right, "\simeq"'] \arrow[ur, phantom, "\Searrow"] & (K \otimes Y) \coprod_{K \otimes X} (L \otimes X) \arrow[dr, "h\square g"] & \\
 & & L \otimes Y
\end{tikzcd}$$

□

5.2. Monoid Axiom.

Next we deal with the **Monoid Axiom**: For all Z, transfinite compositions of pushouts of maps in $(id_Z \otimes Q \cap \mathcal{W})$ are weak equivalences.

We add a hypothesis about how the cofibrations behave (which makes $\mathcal{M}$ a little bit more like Top):

Definition 4. *A* ***homotopical cofibration*** *is a map $g : A \to B$ such that every pushout square with g at the top (i.e. g pushed out by some map $A \to W$) is a homotopy cofiber square, i.e. the map from $Z' \to Z$ is a weak equivalence in the following diagram:*

$$\begin{tikzcd}
QA \arrow[rr, hook] \arrow[dr] \arrow[dd] & & QB \arrow[dr] \arrow[dd] & \\
 & A \arrow[rr, "g"] \arrow[dd] & & B \arrow[dd] \\
QW \arrow[rr] \arrow[dr] & & Z' \arrow[dr, dotted] & \\
 & W \arrow[rr] & & Z
\end{tikzcd}$$

Hypothesis: **"cofibrations** $\otimes X \subset$ **homotopical cofibrations for any** X**."**

Theorem 5. *Under the standing hypotheses on $\mathcal{M}$ and f, $L_f\mathcal{M}$ satisfies the monoid axiom.*

I'd like to find a better hypothesis–one which deals with f and is easier to check. However, for the examples of interest this is either known to be true or unnecessary because a different proof shows the monoid axiom is preserved. So we have it for sSet and Top, and we have it for all monoidal categories of spectra as long as the localization is really L_E for some object E.

5.3. Equivariant Monoid Axiom.

Now we add a hypothesis to preserve the equivariant monoid axiom, namely that $\mathrm{Sym}^n(f) : A^n/\Sigma_n \to B^n/\Sigma_n$ is an f-local equivalence

Theorem 6. *Under the standing hypotheses on $\mathcal{M}$ and f, $L_f\mathcal{M}$ satisfies the Σ_n-equivariant monoid axiom.*

6. Future Work

Recover Hill's theorem as a special case of this. Work out some examples for categories and maps of interest.

Figure out an axiom on the map f so that L_f preserves the operad version of the Σ_n-equivariant monoid axiom. Then we'll know when $\mathcal{O}$-alg in $L_f(\mathcal{M})$ is a model category, and when localization preserves $\mathcal{O}$-algebras. Need to check that the theorem about preservation holds in this generality.