MODEL CATEGORIES AND OPERADS

DAVID WHITE

1. Abstract

Abstract: I'll motivate the definition of a model category and give examples. I'll then discuss monoidal model categories, the types of objects one likes to study inside them, and how operads come into the picture. I'll review what's known about the relationship between model categories and operads, and then state a new theorem of mine in this vein. If there's time at the end I'll talk about how this result connects to the larger problem I've been working on of understanding when Bousfield localization preserves strict commutative monoids.

2. Review of last GSS talk

A **category** $C = (Ob, Mor)$ is a class of objects and a class of morphisms (containing id_A and $f \circ g$). The idea is to contain an entire theory of math. For example:

(1) (Groups, Group Homo's)
(2) Abelian Groups
(3) Top
(4) Top_*
(5) HoTop_*
(6) Graphs
(7) R-mod
(8) Ch(R)

The **yoga of category theory** is that one must study maps between objects to study the objects. Applying this to categories themselves leads you to **functors** $F : CD$, i.e. maps from objects to objects and morphisms to morphisms compatible with id_A and $f \circ g$. For example, Forget:$AbGp \to Gp$ or Abelianization: $Gp \to AbGp$.

Thinking of **localization as "formally inverting maps** then we want to pick a set S of morphisms and create a **universal functor** $C \to C[S^1]$ where those morphisms land in the class of isomorphisms, i.e. $F(f)$ is an iso for all $f \in S$. For example, $Top \to HoTop$.

This doesn't say $C[S^{-1}]$ exists, and trying to construct it leads to **set-theoretic issues** (taking equiv relation requires you to have a set). The solution is to invent model categories, i.e. categories with distinguished classes of maps $\mathcal{W}, Q, \mathcal{F}$ satisfying axioms similar to those in Top. So again, the motivation is that we have some class $\mathcal{W}$ of maps we wish were isomorphisms and we're trying to find conditions on the category $\mathcal{M}$ which allows us to force them to be isomorphisms (via a universal functor $\mathcal{M} \to \mathcal{M}[\mathcal{W}^{-1}]$ taking those maps to isomorphisms).

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A **model category** $\mathcal{M}$ is the most general place one can do homotopy theory. $\mathcal{W}$ is the class to invert, e.g. homotopy equivalences. But topology cares about more than just maps like this, so to actually construct $Ho(\mathcal{M})$ requires two other classes of maps. Cofibrations $\mathcal{Q}$ are like gluing on cells, and it lets you build complicated objects from simple ones. If you think of $\mathcal{Q}$ as monomorphisms that's fine. Fibrations $\mathcal{F}$ are like covering spaces, or fiber bundles. It lets you take quotients. For model categories, the localization $\mathcal{M}[\mathcal{W}^{-1}]$ exists and we denote it by $Ho(\mathcal{M})$.

Examples of model categories:

(1) Top

(2) sSet (like simplicial complexes)

(3) Spectra (X_n) used in stable homotopy theory. There are many structures here, e.g. Symmetric Spectra, S-modules, orthogonal spectra, G-spectra

(4) Ch(R) - this leads to Andre-Quillen cohomology, for which Quillen won the Fields

(5) DGA (graded algebra equipped with a map $d\colon A \to A$ which is degree -1 and has $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$)

(6) StMod - Daniel researched this

(7) EnlargedSchemes - used in Voevodsky's proof of Milnor Conjecture, won Fields

3. Monoidal Categories

Idea: let's add more algebra to the situation, to make things easier. A **monoidal category** is one with a bifunctor $\otimes : C \times C \to C$ which is associative ($\otimes(\otimes \times 1) = \otimes(1 \times \otimes)$) and has a unit $S \in C$ together with isomorphisms $\lambda_A : S \times A \to A, \rho_A : A \times S \to A$ for all $A \in C$. We also need coherence diagrams, e.g. for 4-fold associativity and for associativity and unit

$$\begin{tikzcd} A \otimes (S \otimes B) \arrow[rr, "\alpha"] \arrow[dr, "\lambda"'] & & (A \otimes S) \otimes B \arrow[dl, "\rho"] \\ & A \otimes B & \end{tikzcd}$$

Examples: see above list. For Ch(R) it's not just levelwise. $(A \otimes B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$

A **monoid** $R \in C$ has $\mu : R \otimes R \to R$ associative and $\eta : S \to R$ with

$$\begin{tikzcd} R \otimes S \arrow[r, "\eta"] \arrow[dr, "\rho"'] & R \otimes R \arrow[d, "\mu"] & S \otimes R \arrow[l, "\eta"'] \arrow[dl, "\lambda"] \\ & R & \end{tikzcd}$$

Think about the category of rings. We have $\eta : \mathbb{Z} \to R$ for any R, and that picks out the identity element in R via the image of 1. Think of μ as taking $(a, b) \mapsto a \cdot b$.

We can restrict focus to the subcategory of monoids in C. Call this $Mon(C)$. It's objects are monoids and its morphisms are monoid homomorphisms, i.e. $f : R \to R'$ such that $f \circ \mu = \mu \circ f$. Passage from C to $Mon(C)$ is just like passage from Set to $Group$. It's a passage that we have to make in order to "do algebra" in C. Just having products on the category is not enough without having morphisms containing the information of multiplication on monoids R.

A **commutative monoid** is a monoid along with a twist isomorphism $\tau : R \otimes R \to R \otimes R$ which commutes with μ, i.e. $a \cdot b = b \cdot a$.

4. Monoidal Model Categories

Now let's suppose we want to do homotopy theory on a category like this, i.e. that C above is really a model category $\mathcal{M}$. Well, we'd want to know that $Ho(\mathcal{M})$ was also a monoidal category. Sadly, this does not come for free. You need a coherence condition between the monoidal structure on $\mathcal{M}$ and the model structure.

Given $f : A \to B$ and $g : X \to Y$, define the **pushout product** $f \square g$ to be the corner map in

$$\begin{tikzcd} A \otimes X \arrow[r] \arrow[d] & A \otimes Y \arrow[d] & \\ B \otimes X \arrow[r] & Q_2 \arrow[dr] & \\ & & B \otimes Y \end{tikzcd}$$

Pushout product axiom: if $f, g \in \mathcal{Q}$ then $f \square g \in \mathcal{Q}$. Additionally, if either is in $\mathcal{W}$ then $f \square g \in \mathcal{W}$.

Unit Axiom: If Z is cofibrant then $QS \otimes Z \to S \times Z \cong Z$ is a weak equivalence.

These axioms assure you that $Ho(\mathcal{M})$ is a monoidal category. Now suppose we want to study the monoids in $\mathcal{M}$ using methods from homotopy theory, i.e. we want $Mon(\mathcal{M})$ to be a model category. Informally, we stacked algebra onto topology by shifting focus from model categories to monoidal model categories, and now we're stacking more topology onto that. We'll need another axiom on $\mathcal{M}$ to make this work.

Monoid Axiom: For all Z, transfinite compositions of pushouts of maps in $(id_Z \otimes \mathcal{Q} \cap \mathcal{W})$ are weak equivalences. Note that this follows if $id_Z \otimes \mathcal{Q} \cap \mathcal{W}$ is a trivial cofibration.

This was proven by Schwede and Shipley in 2000. It applies to all the examples above. The proof involves an analysis of pushouts in $Mon(\mathcal{M})$ of maps of the form $TX \to TY$ where $X \to Y$ is a trivial cofibration in $\mathcal{M}$ and T is the free monoid functor. The question of when CommMon($\mathcal{M}$) is a model category is much more subtle, because the free commutative monoid functor Sym is harder to analyze. In the 2000's several people found examples of such $\mathcal{M}$, most famously Shipley's positive model structure on Symmetric Spectra and EKMM's on S-modules. But no one had an axiom like the monoid axiom to work in general until recently.

Σ_n-Equivariant Monoid Axiom: If h is a (trivial) cofibration then $g^{\square n}/\Sigma_n = * \otimes_{\Sigma_n} g^{\square n}$ is a (trivial) cofibration.

Theorem: If $\mathcal{M}$ is a monoidal model category satisfying the monoid axiom and the Σ_n-Equivariant Monoid Axiom then $CommMon(\mathcal{M})$ is a model category.

Something like this axiom first appeared in 2010 in Lurie's work. The theorem also appeared, but in less generality. The examples were not fully worked out. **My contribution** is making it work with a weaker version of the axiom, generalizing the theorem to hold for more model categories, proving that it's sufficient to check this axiom on the generating (trivial) cofibrations, and working out the examples.

Examples: sSet, positive, DGA in characteristic 0, EKMM?, G-spectra?

Of course, a commutative monoid is just an element of $CAlg(S)$. One could also do $CAlg(R)$ for R commutative, and I have. Furthermore, $R \simeq T$ induces a Quillen equivalence. Furthermore, if $\mathcal{M} \simeq \mathbb{N}$ is a Quillen equivalence then $CAlg_{\mathcal{M}}(R) \simeq CAlg_{\mathbb{N}}(F(R))$

5. Operads

The passage from $\mathcal{M}$ to $Mon(\mathcal{M})$ and from $\mathcal{M}$ to $CommMon(\mathcal{M})$ are both examples of passage to algebras over an operad. An **operad** is a gizmo which keeps track of operations on $\mathcal{M}$, i.e. keeps the data of what rules those operations have to satisfy. Formally, it's a sequence of sets $O(n)$ with identity element in $O(1)$, associative composition rule $O(n) \times O(k_1) \times \cdots \times O(k_n) \to O(k_1 + \cdots + k_n)$, and Σ_n actions on $O(n)$ which are compatible with composition. Think of $O(n)$ as the n-ary operations on $\mathcal{M}$

All we care about with operads is that they can act on objects in $\mathcal{M}$ and this results in categories of O-algebras that we like. An **O-algebra** is $A \in \mathcal{M}$ equipped with coherent maps $O(n) \times A^n \to A$. If $O(n)$ is viewed as n-ary operations, then the map $O(n) \times A^n \to A$ takes $(\mu, a_1, \ldots, a_n)$ to $\mu(a_1, \ldots, a_n)$. The fact that O acts on A is what makes A have those maps (and satisfy those diagrams) to make it a monoid, commutative monoid, etc.

Note that the operad itself doesn't know about A or even $\mathcal{M}$. It's a single object which keeps track of "operations of type O" in any category. That's why the theory of operads is also called **Universal Algebra**.

Examples:

1. For any $A \in C$, we have an operad $End(A)$ whose n-th space really is n-ary maps
2. Similarly, we can do the linear isometries operad where the n-th space is linear isometries $\mathbb{R}^\infty \times \cdots \times \mathbb{R}^\infty \to \mathbb{R}^\infty$.
3. $O = Ass$ gives $O - alg(\mathcal{M}) = Mon(\mathcal{M})$
4. $O = Com$ gives $O - alg(\mathcal{M}) = CommMon(\mathcal{M})$. Note that $Com_n = *$ for all n, so it's levelwise cofibrant.
5. $O = A_\infty$ gives homotopy coherent monoids, which always form a model category even without the monoid axiom. This is a cofibrant operad
6. $O = E_\infty$ gives homotopy coherent commutative monoids, which always form a model category even without the equivariant monoid axiom.
7. $O = Lie$ gives Lie Algebras. L_∞ is the homotopy coherent version.
8. O to give Poisson algebras, Gerstenhaber algebras, Leibniz, Zinbiel, Batalin-Vilkovisky, etc

6. What's known about model structures and operads

The **category of operads** is a model category. So we have a notion of cofibrant operad. Independently of this, we can view a C-valued operad as sitting inside C^Σ and so if C is a model category this carries the projective model structure and lets us define the notion of Σ-cofibrant operad.

Berger and Moerdijk get a model structure on **O-alg when O is cofibrant** and $\mathcal{M}$ has a cofibrant unit, a symmetric monoidal fibrant replacement functor, and a commutative Hopf interval. This is using the fibrant side of things, so the proofs are very simple, but the hypotheses on $\mathcal{M}$ a bit restrictive.

John Harper gets a model structure on **O-alg for O a cofibrant operad** and $\mathcal{M}$ satisfying the monoid axiom. This is using the cofibrant side of things, so the proofs involve complicated analyses of pushouts of $O(X) \to O(Y)$ in O-alg.

Lesson: some cofibrancy hypothesis on O and some hypotheses on $\mathcal{M}$ are necessary to get a model structure on O-alg. There are counter-examples for $\mathcal{M}$ not satisfying hypotheses.

John Harper goes beyond his previous work and says that if all symmetric sequences and symmetric arrays in $\mathcal{M}$ are cofibrant in the projective model structure then O-alg gets a model structure for all O. This is an insanely strong hypothesis on $\mathcal{M}$ and appears to only be satisfied by rational DGAs. Still, it shows you can get away with less cofibrancy on O if you're willing to put more on $\mathcal{M}$

7. New Result regarding levelwise cofibrant operads

Σ_n-Equivariant Monoid Axiom: If h is a (trivial) cofibration then $g^{\square n}/\Sigma_n = * \otimes_{\Sigma_n} g^{\square n}$ is a (trivial) cofibration.

Operad version: If h is a (trivial) cofibration and $Z \in \mathcal{M}^{\Sigma_n}$ is a — object then $Z \otimes_{\Sigma_n} g^{\square n}$ is a (trivial) cofibration. See below regarding the blank.

Note that $\mathcal{M}^{\Sigma_n}$ is the category whose objects are objects in $\mathcal{M}$ equipped with a Σ_n action. For example, the n-th power of any object in $\mathcal{M}$. Or, if we want a free action, the $n!$-power.

We can fill in the blank in the operad version to make it a strong axiom or a weak axiom. If it's a strong axiom then it gives a homotopy theory on O-alg for very many O. If it's a weak axiom it gives a homotopy theory for a much smaller class of O. Some of these are known but some are new, and the applications are new. Here is a table, where in the right-hand column the hypotheses written in any box include those in the boxes above, i.e. the right-hand column is increasing in strength as you go down the table whereas the left-hand column is decreasing in strength. Technical point: sometimes you don't get a full model structure but only a semi-model structure. But once you forget to ∞-categories this difference vanishes, so you still have a homotopy theory in either case. There is an additional axiom which can be added to get from those semi-model structures to model structures but I haven't checked it on any examples yet so it might be overly restrictive.

Hyp. on O	Hyp on $\mathcal{M}$
Cofibrant	Cofibrantly generated, monoidal (i.e. Operad axiom for $Z \in \mathcal{M}^{\Sigma_n}$ cofibrant in $\mathcal{M}^{\Sigma_n}$)
Σ-cofibrant	monoid axiom
Levelwise cofibrant	Operad axiom for $Z \in \mathcal{M}^{\Sigma_n}$ cofibrant in $\mathcal{M}$
Special case: Com	Operad axiom for $Z = *$, i.e. Σ_n-equivariant monoid axiom
Arbitrary	Operad axiom for all Z

The proofs of these rely on complicated analyses of pushouts. The key trick is shifting from O to the enveloping operad O_A for $A \in O$-alg and then using the hypotheses on $O_A[n]$. We create a dictionary between cofibrancy hypotheses O and cofibrancy of $O_A[n]$. The bottom line is a weakening of Harper's hypothesis, and is easier to check.

Further things proved: for all but the last line we show you can check the hypothesis just on the generators. Also, we proved that a Quillen equivalence $O \simeq \mathbb{P}$ implies O-alg $\simeq \mathbb{P}$-alg is a Quillen equivalence.

Examples: for any O, O-alg is a model category for $\mathcal{M} = sSet$, $\mathcal{M}$ = positive stable on symmetric spectra, $\mathcal{M}$ = DGA over characteristic zero.

Hope: for any levelwise cofibrant O, O-alg is a model cat for $\mathcal{M} = S-$modules, orthogonal spectra, and G-spectra (perhaps with some hypothesis like in Hill's theorem). Seems to be true, but not sure if it follows from this method.

Hope: all O-alg for any excellent model category

8. Relation to Thesis

We wanted to know when Bousfield localization preserves strict commutative monoids

Theorem 1. *If CommMon($\mathcal{M}$) and CommMon($L_f\mathcal{M}$) are model categories then localization preserves strict commutative monoids*

There are standard hypotheses on a model category $\mathcal{M}$ when one is working with Bousfield localization (cocomplete, cofibrantly generated, left proper, almost finitely generated, can choose domains and codomains of generating (trivial) cofibrations to be cofibrant) and also when one is in a monoidal situation (pushout product axiom, cofibrant objects flat, monoid axiom). I found that even under all these hypotheses, $L_f\mathcal{M}$ could fail to be a monoidal model category, though it is always a model category and monoidal. The coherence fails. If we place an assumption on the map f to be inverted (just like Quillen had to do), we can get around this.

Theorem 2. *Under the standing hypotheses above, if for* ***all domains and codomains*** *K* ***of*** *$I \cup J$*, ***maps in*** *$f \otimes id_K$* ***are f-local equivalences****, then $L_f\mathcal{M}$ is a monoidal model category.*

This is really the right hypothesis, as can be seen from old work of Kelly. Similarly, the monoid axiom can fail, so we add a hypothesis about how the cofibrations behave (which makes $\mathcal{M}$ a little bit more like Top):

Definition 3. *A* ***homotopical cofibration*** *is a map $g : A \to B$ such that every pushout square with g at the top (i.e. g pushed out by some map $A \to W$) is a homotopy cofiber square, i.e. the map from $Z' \to Z$ is a weak equivalence in the following diagram:*

$$\begin{tikzcd}
QA \arrow[rr, hook] \arrow[dr] \arrow[dd] & & QB \arrow[dr] \arrow[dd] & \\
& A \arrow[rr, "g"] \arrow[dd] & & B \arrow[dd] \\
QW \arrow[rr] \arrow[dr] & & Z' \arrow[dr, dotted] & \\
& W \arrow[rr] & & Z
\end{tikzcd}$$

Hypothesis: **"cofibrations $\otimes X \subset$ homotopical cofibrations for any X."**

Theorem 4. *Under the standing hypotheses on $\mathcal{M}$ and f, $L_f\mathcal{M}$ satisfies the monoid axiom.*

Now we add a hypothesis to preserve the equivariant monoid axiom, namely that $\mathrm{Sym}^n(f) : A^n/\Sigma_n \to B^n/\Sigma_n$ is an f-local equivalence

Theorem 5. *Under the standing hypotheses on $\mathcal{M}$ and f, $L_f\mathcal{M}$ satisfies the Σ_n-equivariant monoid axiom.*

Theorem 6. *Under the hypotheses above, L_f preserves strict commutative monoids (of course, this really means L_f then U_f going back to $\mathcal{M}$)*

Future work: recover Hill's theorem as a special case of this. Work out some examples for categories and maps of interest.

Future work: figure out an axiom on the map f so that L_f preserves the operad version of the Σ_n-equivariant monoid axiom. Then we'll know when $\mathcal{O}$-alg in $L_f(\mathcal{M})$ is a model category, and when localization preserves $\mathcal{O}$-algebras. Need to check that the theorem about preservation holds in this generality.