BOUSFIELD LOCALIZATION OF MONOIDAL MODEL CATEGORIES

DAVID WHITE

1. Outline

(1) Localization
(2) Categories
(3) Localization in Categories

I do my research in the context of model categories, sometimes specializing to the category Spectra in particular. I want to be able to apply algebraic ideas and methods to these categories, and I do so by replacing algebraic concepts by their categorical analogs. Every talk I give includes diagrams defining a ring object in a category. My research tends to be of the flavor: take some cool thing in algebra and figure out how to develop that theory in alg. top. So you need the right definition and proofs in algebra, then you need to define the analogous thing and try to lift the proofs.

Alg. Top	ring-objects	derived category	qual talk	thesis	today
Algebra	Rings	R-mod	homological dimension	ideals	localization

2. Localization

Localization is a way of studying an algebraic object **"at" a prime**. One may study an object by studying it at every prime (the "local question"), then piecing these together to understand the original object (the **"local-to-global question"**).

More abstractly, one studies a ring by localizing at a prime ideal, obtaining a local ring. One then often takes the completion. From the point of view of the **spectrum of a ring**, the primes are the points of a ring, and thus localization studies a ring (or similar algebraic object) at every point, then the local-to-global question asks to piece these together to understand the entire space. The failure of local solutions to piece together to form a global solution is a form of **obstruction theory**, and often yields cohomological invariants, as in sheaf cohomology.

In abstract algebra, localization is a systematic method of adding multiplicative inverses to a ring. Given a ring R and a subset S, one wants to construct some ring R^* and ring homomorphism from R to R^*, such that the image of S consists of units (invertible elements) in R^*. Further one wants R^* to be the 'best possible' or 'most general' way to do this in the usual fashion this should be expressed by a universal property. The localization of R by S is often denoted by $S^{-1}R$, or by R_I if S is the complement of a prime ideal I.

Another way to **describe the localization of a ring** R at a subset S is **via category theory**. If R is a ring and S is a subset, consider the set of all R-algebras A, so that, under the canonical homomorphism $R \to A$, every element of S is mapped to a unit. The elements of this set form the objects of a category, with R-algebra homomorphisms as morphisms. Then, the localization of R at S is the **initial object** of this category.

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Suppose $S \subset R$ is a multiplicative set, i.e. 1 is in S and for s and t in S we also have st in S. The construction proceeds as follows: on $R \times S$ define an equivalence relation $\sim$ by setting $(r1, s1) \sim (r2, s2)$ iff there exists $t \in S$ such that $t(r1s2 - r2s1) = 0$.

We think of the equivalence class of (r, s) as the "fraction" r/s and, using this intuition, the set of equivalence classes R^* can be turned into a ring with operations that look identical to those of elementary algebra: $a/s + b/t = (at + bs)/st$ and $(a/s)(b/t) = ab/st$. The map $j : R \to R^*$ which maps r to the equivalence class of (r,1) is then a ring homomorphism. (In general, this is not injective; if two elements of R differ by a zero divisor with an annihilator in S, their images under j are equal.)

The above mentioned **universal property** is the following: the ring homomorphism $j : R \to R^*$ maps every element of S to a unit in R^*, and if $f : R \to T$ is some other ring homomorphism which maps every element of S to a unit in T, then there exists a unique ring homomorphism $g : R^* \to T$ such that $f = g \circ j$. The above $S^{-1}R$ satisfies this universal property, because given f define $g(r/s) = f(r)f(s)^{-1}$. This is well defined because $r/s = r'/s'$ implies $x(s'r - r's) = 0$ so $f(s'r) = f(r's)$ so $f(r)f(s)^{-1} = f(r')f(s')^{-1}$. This g is a ring homomorphism because the operations on $S^{-1}R$ were defined with this in mind (they work just like in $\mathbb{Q}$). Also, $g(j(r)) = g(r/1) = f(r)f(1) = f(r)$.

Examples:

(1) Given a commutative ring R, we can consider the multiplicative set S of non-zerodivisors (i.e. elements a of R such that multiplication by a is an injection from R into itself.) The ring $S^{-1}R$ is called the total quotient ring of R, often denoted K(R). S is the largest multiplicative set such that the canonical mapping from R to $S^{-1}R$ is injective. When R is an integral domain, this is none other than the fraction field of R.

(2) Let $R = \mathbb{Z}$, and p a prime number. If $S = \mathbb{Z} - p\mathbb{Z}$, then $R^* = \mathbb{Z}_{(p)} = \{a/b : p \not| b\}$ is the localization of the integers at p. As a generalization of the previous example, let R be a commutative ring and let p be a prime ideal of R. Then R - p is a multiplicative system and the corresponding localization is denoted R_p. The unique maximal ideal is then p, so R_p is a local ring. Localization corresponds to restriction to the complement U in Spec(R) of the irreducible Zariski closed subset V(P) defined by the prime ideal P.

(3) If $R = \mathbb{Z}$ and $S = \langle 2 \rangle$ then $S^{-1}R$ is the dyadic rationals.

(4) If $R = k$ is a field and $0 \notin S$ then $S^{-1}R = k$ because all elements in S were already invertible. If $R = K[X]$ is the polynomial ring and $S = \{X\}$ then the localization produces the ring of Laurent polynomials $K[X, X^{-1}]$. In this case, localization corresponds to the embedding $U \to \mathbb{A}^1$, where $\mathbb{A}^1$ is the affine line and U is its Zariski open subset which is the complement of 0.

(5) $S^{-1}R = \{0\}$ if and only if S contains 0.

(6) Does localization always make the ring bigger? Equivalently, is $R \to S^{-1}R$ always injective? No! The ring homomorphism $R \to S^{-1}R$ is injective if and only if S does not contain any zero divisors. Consider $R = \mathbb{Q} \times \mathbb{Q}$ and $S = \langle (1, 0) \rangle$. Then $S^{-1}R \cong Q \times \{0\} \cong \mathbb{Q}$.

R-Mod: Let S a multiplicatively closed subset of R. Then the localization of M with respect to S, denoted $S^{-1}M$, is defined to be the following module: as a set, it consists of equivalence classes of pairs (m, s), where $m \in M$ and $s \in S$. Two such pairs (m, s) and (n, t) are considered equivalent if there is a third element u of S such that $u(sn - tm) = 0$. One interesting characterization of the equivalence relation is that it is the smallest relation (considered as a set) such that cancelation laws hold for elements in S. That is, it is the smallest relation such that rs/us = r/u for all s in S.

Universal Property: There is a module homomorphism $j : M \to S^{-1}M$ s.t. for any S-local T with $M \to T$ there exists a unique module homomorphism $S^{-1}M \to T$ making the triangle commute. Again, $j(m) = m/1$ and again, this need not be injective. Key fact: $S^{-1}M = M \otimes_R S^{-1}R$, by the very definition of "extension of scalars." Also: $S^{-1}R$ is a flat module over R.

Categories don't have an operation, so there's no a priori way to localize. However, another way to think about localization of rings is as formally inverting *maps*. In particular, to invert $s \in S$ you take the ring generated by R and s^{-1}. Equivalently, simply insist that the multiplication by s map $\mu_s : R \to R$ be invertible.

Proposition 1. *Suppose R_* is a ring containing s on which μ_s is an isomorphism. Further, suppose there is a unique ring homomorphism $i : R \to R_*$ and for any $f : R \to T$ with $\mu_s : T \to T$ an isomorphism, there exists a unique $g : R_* \to T$ such that $g \circ i = f$. Then $R_* \cong s^{-1}R$.*

Proof. First, $s^{-1}R$ contains s and has μ_s an isomorphism (it's inverse if $\mu_{s^{-1}}$. Thus, the map $j : R \to s^{-1}R$ yields a unique map $g : R_* \to s^{-1}R$ such that $g \circ i = j$. Next, R_* is a ring where s is invertible because $\mu_s^{-1}(1) \cdot s = \mu_s^{-1}(1) \cdot \mu_s(1) = (\mu_s^{-1} \circ \mu_s)(1) = 1$. So the universal property of localization implies there's a unique map $h : s^{-1}R \to R_*$ such that $h \circ j = i$:

$$\begin{tikzcd} & R \arrow[dl, "i"'] \arrow[d, "j"] \arrow[dr, "i"] & \\ R_* \arrow[r, "g"] & R^* \arrow[r, "h"] & R_* \end{tikzcd}$$

The bottom composition must be the identity on R_* because the triangles are the same. Draw a similar picture to prove $g \circ h$ is the identity on R^*. □

Going off this idea, let's say M is S**-local** if $\mu_s : M \to M$ is an isomorphism for all $s \in S$. A map $f : M \to N$ is an S**-equivalence** if $f^* : \mathrm{Hom}(N, T) \to \mathrm{Hom}(M, T)$ is an isomorphism for all S-local T.

Proposition 2. *$j : R \to R[S^{-1}]$ is an S-equivalence and $R[S^{-1}]$ is S-local.*

Proof. We already proved $R[S^{-1}]$ is S-local. Let T be S-local. Then $\mathrm{Hom}(R[S^{-1}], T) \to \mathrm{Hom}(R, T) \cong T$ sends f to $f \circ j$ to $(f \circ j)(1)$, i.e. to $f(1/1)$. This is 1-1 because if $f(1_{R[S^{-1}]}) = 0$ then f must be the zero map. This is a homomorphism because $(f + g)(1) = f(1) + g(1)$ and $(f \times g)(1) = f(1)g(1)$. This is onto because for any $t \in T$ we simply define f to take $1/1$ to t. We needed T to be S-local to even form $\mathrm{Hom}(R[S^{-1}], T)$, i.e. for these maps to be well-defined. □

Further Examples... **Abelian Groups** If $H \leq G$ then $H^{-1}G \cong G$ because all $h \in H$ already have inverses in G, so the smallest group which G maps into and in which all elements of H have inverses is G. You again have the universal property.

Rings Need some conditions on R for it to exist. The right Ore condition for a domain R, and any pair a, b of non-zero elements, is the requirement that the sets aR and bR should intersect in more than the element 0. The left Ore condition is defined similarly. A domain that satisfies the right Ore condition is called a right Ore domain. For every right Ore domain R, there is a unique (up to natural R-isomorphism) division ring D containing R as a subring such that every element of D is of the form rs^{-1} for r in R and s nonzero in R. Such a division ring D is called a ring of right fractions of R, and R is called a right order in D.

Topological Spaces We let A be a subring of the rational numbers, and let X be a simply connected CW complex. Then there is a simply connected CW complex Y together with a map from X to Y such that Y is A-local (all its homology groups are modules over A) and the map is **universal** for (homotopy classes of) maps from X to A-local CW complexes. This space Y is unique up to homotopy equivalence, and is called the localization of X at A.

3. Categories

A category is an algebraic structure that comprises "objects" that are linked by "arrows". A category has two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. In general, the objects and arrows may be abstract entities of any kind, and the notion of category provides a fundamental and abstract way to describe mathematical entities and their relationships. This is the central idea of category theory, a branch of mathematics which seeks to generalize all of mathematics in terms of objects and arrows, independent of what the objects and arrows represent. Virtually every branch of modern mathematics can be described in terms of categories.

A category C consists of

- a class ob(C) of objects
- a class hom(C) of morphisms, or arrows, or maps, between the objects. Each morphism f has a unique source object a and target object b where a and b are in ob(C). We write $f : a \to b$, and we say "f is a morphism from a to b". We write C(a, b) to denote the class of all morphisms from a to b.
- for every three objects a, b and c, a binary operation $C(a, b) \times C(b, c) \to C(a, c)$ called composition of morphisms $((f, g) \to g \circ f$

Satisfying the axioms:

- (associativity) $ho(gof) = (hog)of$
- (identity) for every object x, there exists a morphism $1_x : x \to x$ called the identity morphism for x, such that for every morphism $f : a \to b$, we have $1_b of = f = fo1_a$.

An isomorphism is a morphism $f : a \to b$ s.t. there exists $g : b \to a$ and $f \circ g = 1_b$, $g \circ f = 1_a$.

Examples:

- Set, the category of sets and set functions. Isomorphisms are bijections.
- Grp, the category of groups and group homomorphisms. Isomorphisms.
- Ab, the category of abelian groups and group homomorphisms. Isomorphisms.
- Ring, the category of rings and ring homomorphisms
- CRing, the category of commutative rings and ring homomorphisms
- R-Mod, the category of R-modules and module homomorphisms
- Top, category of topological spaces and continuous maps. Homeomorphisms
- Top_*, category of topological spaces with a distinguished choice of basepoint and continuous basepoint-preserving maps. Homeomorphisms

- Banach spaces and bounded linear maps...but no localization here because no operation. One case for non-commutative rings where localization has a clear interest is for rings of differential operators. It has the interpretation, for example, of adjoining a formal inverse D^{-1} for a differentiation operator D. This is done in many contexts in methods for differential equations.

The point is, the morphisms always preserve the structure of the objects. What if we want to create a category of categories? Well, first we need to avoid some set-theoretic problems. A category C is called small if both ob(C) and hom(C) are actually sets and not proper classes. There is a category with small categories as objects, but what are the morphisms? They are **functors**, i.e. $F : C \to D$ which associates to each object $X \in C$ an object $F(X) \in D$ and associates to each morphism $f : X \to Y \in C$ a morphism $F(f) : F(X) \to F(Y) \in D$ such that

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object $X \in C$
- $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$. Note that a contravariant functor has $F(g \circ f) = F(f) \circ F(g)$.

That is, functors must preserve identity morphisms and composition of morphisms. Just like you need to understand group homomorphisms to understand groups, you need functors to understand categories. Functors preserve isomorphisms because $Ff \circ Fg = F(f \circ g) = F(1_b) = 1_{Fb}$ and $Fg \circ Ff = F(g \circ f) = F(1_a) = 1_{Fa}$.

Examples

- Forgetful functor: Grp$\to$ Set.
- Free functor: Set$\to$ Grp. Or free-abelian functor from Set$\to$ Ab
- Abelianization: Grp$\to$ Ab.

You can also have morphisms between the morphisms, and in Cat these are called **natural transformations**. Formally, if F and G are functors from C to D then for each $X \in C$ you have $\eta_X : FX \to GX$ in D s.t.:

$$\begin{array}{ccc} F(X) & \xrightarrow{Ff} & F(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{Gf} & G(Y) \end{array}$$

Example: If K is a field, then for every vector space V over K we have a "natural" injective linear map $V \to V^{**}$. These maps are "natural" in the following sense: the double dual operation is a functor, and the maps are the components of a natural transformation from the identity functor to the double dual functor. If V is finite dimensional this is a natural isomorphism. In linear algebra, that "natural" just meant independent of choice of basis, but this way is even stronger because it relates the two as abstract objects in a category where no one said anything at all about a basis.

4. Localization on Categories

Using Prop 1: perhaps a better way to think of localization is "formally inverting maps." This we can do in category theory, and it makes a class of morphisms into isomophisms. You then need to put in composites which use the new morphisms, just like you have to generate using R and s^{-1} above. For example, let $\mathcal{C}$ be Top$_*$ and consider the class of morphisms $\mathcal{S}$ which are homotopy equivalences (i.e. $f : X \to Y$ s.t. there exists $g : Y \to X$ and $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$).

Then $\mathcal{C}[\mathcal{S}^{-1}]$ is $Ho\mathcal{C}$ the category of topological spaces up to homotopy equivalence. Another example is the definition of a derived category, which I've discussed before (you invert so-called quasi-isomorphisms).

This viewpoint focuses on when a ring morphism $R \to R'$ is really one of those structure maps $j : R \to S^{-1}R$. As in the case of rings above, localization in categories gives a functor $F : \mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ such that if there is any other functor $\mathcal{C} \to \mathcal{D}$ sending morphisms in S to isomorphisms, then there exists a unique functor $\mathcal{C}[\mathcal{S}^{-1}] \to \mathcal{D}$ making the triangle commute. F is **universal** w.r.t. the property that it takes $s \in \mathcal{S}$ to an isomorphism.

To construct $\mathcal{C}[\mathcal{S}^{-1}]$ we want to allow morphisms to be equivalence classes of zigzags, i.e. $\mathcal{C}[\mathcal{S}^{-1}](a,b) = \{a \leftarrow \bullet \to \bullet \leftarrow \bullet \cdots \bullet \to b\}/\sim$. Sadly, this is a proper class in general. But it worked above with Top_*, and it will work again if we simply generalize that example. This leads to the notion of a **Model Category**, i.e. a category $\mathcal{M}$ along with three classes of morphisms called weak equivalences ($\mathcal{W}$), fibrations ($\mathcal{F}$), and cofibrations ($\mathcal{C}$), satisfying some complicated axioms.

Think of this as the most general place you can do homotopy theory. For topological spaces, F is Serre fibrations, W is weak homotopy equivalences, and C is harder to describe, but is determined by the other two. Invented by Quillen, Hovey wrote the book on them, Daniel's thesis defense had him verifying those axioms all over the place. It's not easy to get a flavor for what F and C are in general, but W is the class you're going to invert, so it's always your choice of what a "homotopy equivalence" should be. On the category Set there are 9 valid choices for W, F, C which give different model category structures. In one, C are injections and F are surjections. In another that's switched. The point is: it gets pretty crazy. What model categories are good for is doing this process and getting homotopy categories, plus doing constructions at the point-set level which you know will carry over to the homotopy level.

Ok, so we know why we care about model categories. The problem is that the localization above takes you out of the category $\mathcal{M}$ you'd been working in. It's somehow not of the same flavor as the ring localizations above. What if we want to study a space or a spectrum localized at p? How can we reduce studying π_* to a single prime using the localization above? The answer is: you can't. This is why we have Bousfield Localization. The idea here is to add to the class of weak equivalences in a model category, knowing that these will BECOME isomorphisms in the homotopy category. So suppose we have a class $\mathcal{S}$ of maps we'd like to turn into weak equivalences.

Following the equivalent formulation of localization from Prop 2, define $M \in \mathcal{M}$ to be $\mathcal{S}$-local if M is fibrant and for all $s : X \to Y \in \mathcal{S}$, $s^* : \mathcal{M}(Y, M) \to \mathcal{M}(X, M)$ is a weak equivalence. An object M is fibrant if the map $M \to *$ is a fibration (here $*$ is the terminal object). A map $f : A \to B$ is an $\mathcal{S}$-local equivalence if for all $\mathcal{S}$-local M, $f^* : \mathcal{M}(B, M) \to \mathcal{M}(A, M)$ is a weak-equivalence.

The **(left) Bousfield localization** of $\mathcal{M}$ w.r.t. $\mathcal{S}$ is a new model category structure on $\mathcal{M}$ with the same cofibrations as $\mathcal{M}$ and with weak equivalences equal to $\mathcal{S}$-local equivalences. Denote this model structure $L_{\mathcal{S}}\mathcal{M}$. Note that weak equivalences of $\mathcal{M}$ are still weak equivalences, but now there are more of them. The identity functor gives a Quillen adjoint pair:

$1 : \mathcal{M} \rightleftarrows L_{\mathcal{S}}\mathcal{M} : 1$. Here LX of fibrant X are the L-locals, and $X \simeq LX$. This yields:

$F : Ho\mathcal{M} \rightleftarrows HoL_{\mathcal{S}}\mathcal{M} : U$ and F takes the images in $Ho\mathcal{M}$ of maps in $\mathcal{S}$ into isomorphisms in $HoL_{\mathcal{S}}\mathcal{M}$, and $L_{\mathcal{S}}\mathcal{M}$ is the smallest model category with this property, i.e. if there's another $\mathcal{N}$ then we get a unique left Quillen functor: $L_{\mathcal{S}}\mathcal{M} \to \mathcal{N}$.

It's not that surprising that localization in algebra is a special case of this, since we defined it in complete analogy. What is amazing is that completion in algebra is also a special case. Algebraic geometers often need to localize and then complete, and they are unrelated operations. In algebraic topology the situation is often far more complicated than that for algebra, but in this one case it's simpler. Completion often takes the form in algebra of an inverse limit. There has to be a topology running around in order for complete to make sense (Cauchy sequences converge).

Formally, a localization is an idempotent and coaugmented functor $L : C \to C$. A coaugmented functor is a pair (L, i) where $L : C \to C$ is an endofunctor and $i : Id \to L$ is a natural transformation from the identity functor to L (called the coaugmentation). A coaugmented functor is idempotent if, for every X, both maps $L(iX), iL(X) : L(X) \to LL(X)$ are isomorphisms. It can be proven that in this case, both maps are equal. **Universal Property**: $\eta_X : X \to LX$ is initial among morphisms from X to L-local objects, and terminal among L-equivalences with domain X. Localizations of more complicated categories must preserve more structure. For Model Categories j must be left Quillen s.t. the left derived Lj takes images in HoM of elements in S to isomorphisms in HoL_SM. Furthermore, for any N with this property (i.e. $M \to N$ and the Lf property), there's a map from $L_SM \to N$ making the triangle commute.

People seek to understand subcategories S which are $\ker L$. Bousfield invented all this to localize with respect to a given homology theory, i.e. $X \to L_E X$ where E is a homology theory. It works:

- If $E = H\mathbb{Q}$ then a spectrum X is E-local iff $\pi_* X$ are rational vector spaces.
- In chain complexes, say a projective A_* is $\mathbb{Z}/p\mathbb{Z}$ acyclic if each $H_n(A_*)$ is a $\mathbb{Z}[1/p]-$module. Say B_* is $\mathbb{Z}/p\mathbb{Z}$-local if every map from an A_* as above is nullhomotopic. Then the $\mathbb{Z}/p\mathbb{Z}$-localization of a projective X_* is $\operatorname{holim}_{\leftarrow} X_* \otimes \mathbb{Z}/p^n\mathbb{Z}$, i.e. the homotopy completion.

5. My Work

To really generalize algebra, you need a notion of a ring in a category. A category is said to be **monoidal** if there is a bifunctor $\otimes : C \times C \to C$ which is associative $(\otimes(\otimes \times 1) = \otimes(1 \times \otimes)$ and has a unit object e along with $\lambda_a : e \otimes a \to a$ and $\rho_a : a \otimes e \to a$ s.t.

$$\begin{tikzcd}
a \otimes (e \otimes c) \arrow[rr, "\alpha"] \arrow[dr, "\lambda"] & & (a \otimes e) \otimes c \arrow[dl, "\rho"] \\
& a \otimes c &
\end{tikzcd}$$

You also need coherence diagrams for 4-fold associativity. A **ring object** $R \in C$ has $\mu : R \otimes R \to R$ which is associative and $\eta : e \to R$ s.t.

$$\begin{tikzcd}
e \otimes c \arrow[r, "\eta"] \arrow[dr, "\lambda"] & c \otimes c & c \otimes e \arrow[l, "\eta"'] \arrow[dl, "\rho"] \\
& c &
\end{tikzcd}$$

My task: find conditions on $\mathcal{M}$ and on the functor L (equiv: on the class $\mathcal{S}$) such that a commutative ring object $R \in \mathcal{M}$ goes to a commutative ring object in $LR \in L_{\mathcal{S}}\mathcal{M}$. We know already that $Ho(R) \mapsto Ho(LR)$, but not on the level of model categories. This really comes down to understanding those functors F and U from before, and using the fact that they are derived functors of much nicer functors. It also involves proving $L_{\mathcal{S}}\mathcal{M}$ is monoidal, and that the category of (commutative) ring objects forms a model category.