ADVANCED TOPICS IN ALGEBRAIC GEOMETRY

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Outline of talk:

My goal is to introduce a few more advanced topics in algebraic geometry but not to go into too much detail. This will be a survey of

(1) Elimination Theory and review from last time (BRIEFLY)
   (a) History and goals
   (b) Geometric Extension Theorem
(2) Invariant Theory
(3) Dimension Theory
   (a) Krull Dimension
   (b) Hilbert Polynomial
   (c) Dimension of a variety
(4) Syzygies
   (a) Definition
   (b) Free resolutions
   (c) Hilbert's Syzygy Theorem
(5) Intersection Theory
   (a) Bezout's Theorem
   (b) Discriminant
   (c) Sylvester Matrix
   (d) Resultants
   (e) Grassmannians

Recall basics of algebraic geometry: sets in affine space correspond to functions in $k[x_1, \ldots, x_n]$ which vanish on those sets. This is given by $I \to V(I)$ (an affine algebraic set) and by $V \to I(V)$ (an ideal). It was inclusion-reversing and had other friendly properties like $I(V(I)) = I$. An affine variety was an affine algebraic set which could not be written as a union of two other non-trivial affine algebraic sets.

Almost all the familiar algebraic geometry from affine space holds over to projective space but we deal with homogeneous polynomials (all monomials of the same degree) so that they will be well-defined on points. This is because for projective points $(x_0 : \cdots : x_n)$ is equivalent to $(\lambda x_0 : \cdots : \lambda x_n)$ for all $\lambda \in k \setminus \{0\}$. But for homogeneous polynomials this just corresponds to multiplication

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by a constant since $F(\lambda x_0 : \cdots : \lambda x_n) = \lambda^d F(x_0 : \cdots : x_n)$. As a projective point this is equal to $F(x_0 : \cdots : x_n)$ but we needed homogeneity to have the same constant applied to all monomials. Note that the point $(0 : 0 : \cdots : 0)$ is not in projective space.

To a projective algebraic set X we can associate an ideal (now homogeneous) $J(X)$ of polynomials vanishing on X. Similarly, to a homogeneous ideal J we can associate a projective algebraic set $X(J)$. The same basic facts from last semester hold for these operations. Note that the object $k[x_0, \ldots, x_n]/J(X)$ is a (finitely generated) k algebra called the coordinate ring of X and is sometimes denoted $R(X)$. It's also sometimes called the ring of regular functions and the functions in $R(X)$ are called regular. It's a useful object because it gets rid of the problem of two different functions being equal in X. We also get a surjection from $k[x_0, \ldots, x_n]$ to $R(X)$ with kernel $J(X)$.

I'll try to mention explicitly when we are in the projective setting, but sometimes my notation will do the talking for me. If you see homogeneous as a condition in a theorem then we're in the projective setting. If you see X's and J's then we're in the projective setting. All day today k is a field which will often be algebraically closed. When in doubt assume k is algebraically closed.

1. Elimination Theory

Elimination theory is the study of algorithmic approaches to eliminating variables and reducing problems in algebra and algebraic geometry done in several variables. Computational techniques for elimination are primarily based on **Grobner basis** methods. Recall from Becky's talks that a Grobner basis for an ideal I is a generating set $\{g_1, g_2, \ldots\}$ for I such that the leading terms of the g_i generate the leading term ideal of I.

Examples: **Gaussian elimination**, the Geometric Extension Theorem I mentioned in my last talk. The theory of **quantifier elimination** can be thought of as a subfield of elimination theory, and there is a beautiful connection to geometry in this example.

The goal of this theory is to solve problems by reducing the number of dimensions. The key fact is that **projection onto fewer coordinates preserves varieties**. Last time I discussed the Geometric Extension Theorem. This time I want to mention that intersection theory goes much further and in fact relates to a topological invariant called the intersection form defined on the nth cohomology group.

Definition 1. *For a given ideal $I \subset k[x_0, \ldots, x_n]$, define I_t to be the t-th* ***elimination ideal*** *$I \cap k[x_{t+1}, \ldots, x_n]$. This eliminates the variables $x_1, \ldots, x_k$. Note that I_2 is the first elimination ideal of I_1.*

There is clearly a relationship between I_t and the projection of $Z(I)$ onto the last $n-t$ dimensions. The following theorem is about how to extend partial solutions in the projected space to full solutions in the original space. First note that given $I = \langle f_1, f_2, \ldots, f_s\rangle \subset k[x_0, \ldots, x_n]$. We can write f_i as $f_i(x) = x_1^{N_1} g_i(x_2, \ldots, x_n)+$ terms in which x_1 has lower degree with coefficients being polynomials in $x_2, \ldots, x_n$.

Theorem 1 (Geometric Extension Theorem)**.** *If the leading coefficients $g_i(x_2, \ldots, x_n)$ do not vanish simultaneously at the point $(a_2, \ldots, a_n)$ (i.e. $(\mathbf{a_2}, \ldots, \mathbf{a_n}) \notin \mathbf{Z}(\mathbf{g_1}, \ldots, \mathbf{g_s})$), then there exists some $a_1 \in k$ such that the partial solution $(a_2, \ldots, a_n)$ extends to the complete solution $(a_1, a_2, \ldots, a_n) \in Z(I)$.*

2. Invariant Theory

This is what my other talk is about. Invariant Theory studies the actions of groups on algebraic varieties and their corresponding polynomial equations. The point it to talk about the polynomials left invariant under action of a general linear group. We define the **ring of invariants** as $k[x_1, \ldots, x_n]^G$ the polynomials fixed by G. This is finitely generated by homogeneous polynomials when G is a finite matrix group. To prove this we define the **Reynold Operator** which tells the average effect of a group on a polynomial. Then Grobner bases allow us to discuss uniqueness of a representation in terms of these generators. We can then talk about G orbits and all our favorite techniques from algebra come in to give structure to the geometry. The invariant theory of finite groups has an intimate connection with Galois Theory. Related, but much harder, is geometric invariant theory.

3. Dimension Theory

The notion of dimension of a variety V should match up with its dimension in linear algebra (based on points, lines, planes, etc). But to define this properly we'll need to use the connection between varieties and algebra. We first define the dimension of a ring.

Krull Dimension

Let R be a ring and $p \subset R$ a prime ideal. We define the height of p as $h(p) = \sup\{s \in \mathbb{Z} \mid \exists\ p_0 \subset \cdots \subset p_s = p$ a chain of prime ideals in R ordered by strict containment$\}$. We then define $\dim_k R = \sup\{h(p) \mid p \subset R$ is a prime ideal$\}$

A useful fact is that if I is an ideal in $k[x_1, \ldots, x_n]$ then $\dim V(I) = \dim(k[x_1, \ldots, x_n]/I)$

Example: $V = \mathbb{C}^1$ has $I(V) = \langle 0 \rangle$. Clearly $\dim \mathbb{C}^1 = 1$ and $\dim_k \mathbb{C}[x]/\langle 0 \rangle = \dim_k \mathbb{C}[x] = 1$ because chains have maximum length 1.

Example: $I = \langle y - x^2 \rangle$ in $\mathbb{C}[x, y]$ has dimension 2 because $\mathbb{C}[x, y]/\langle y - x^2 \rangle \cong \mathbb{C}[x]$

Algebraic Dimension

There is another way to define dimension. Recall that the transcendence degree of a field extension L/K is the largest cardinality of an algebraically independent subset S of L over K (meaning the elements of S do not satisfy any non-trivial polynomial equations with coefficients in K). We'll denote the transcendence degree of an extension as $t\deg$.

Given a variety V, define $k(V)$ as the quotient field of $I(V)$. This means taking $\frac{f}{g}$ for all $f, g \in I(V), g \neq 0$. We may then define $\dim_a V = t\deg(k(V))$ where this extension is clearly to be considered over k since that's where the coefficients come from. This notion of dimension measures the number of independent rational functions that exist on V.

Example: $\dim_a(\mathbb{A}^n_k) = n$ because $I(\mathbb{A}^n_k) = k[x_1, \ldots, x_n] = k(\mathbb{A}^n_k)$. This field has $x_1, \ldots, x_n$ all algebraically independent.

Example: $\dim(V(y^2 - x^3)) = 1$ because $k[x, y]/(y^2 - x^3) \cong k[t^2, t^3]$ and $k(V) \cong k(t)$. This field has t algebraically independent.

It's not easy to show but it turns out that Krull Dimension agrees with Algebraic Dimension.

Dimension of a variety

We now return to the question of dimension of a variety. Let us first consider the case where V is affine, i.e. contained in $\mathbb{A}^n_k$. A **coordinate subspace** will be one generated by standard normal vectors e_j. For every monomial ideal $I \subset k[x_1, \ldots, x_n]$ we get $V(I)$ as a union of finitely many coordinate subspaces.

Example: $V(xz, yz) = V(x, y) \cup V(z)$ is the xy-plane and the z-axis. It's dimension is 2.

All of this holds over to the projective case, but we need the ideal to be homogeneous and we need to grade by degree. The ring $k[x_0, \ldots, x_n]$ is naturally graded by degree as $\oplus_{d\geq 0} k[x_0, \ldots, x_n]_d$ where each $k[x_0, \ldots, x_n]_d$ is the vector space of homogeneous polynomials of degree d (the monomials of degree d form a basis). The dimension of $k[x_0, \ldots, x_n]_d$ is the number of monomials of degree d up to scaling by constants (which don't matter in the projective setting). This is the number of $(n+1)$-tuples such that $\alpha_0 + \cdots + \alpha_n = d$ which is $\binom{n+d}{n}$.

The above grading holds over to the case after we mod out by a homogeneous ideal, but there is no longer a simple way to count the dimension for different d. Indeed, we get a function in d, called **Hilbert's Function** $HF_J(d) = \dim((k[x_0, \ldots, x_n]/J)_d) = \dim(k[x_0, \ldots, x_n]_d) - \dim(J_d)$ (last equality due to Rank-Nullity Theorem). The Hilbert function of a variety X is $HF_{J(X)}(d)$. It is a non-trivial theorem (due to Hilbert's) that when $d >> 0$ the Hilbert function is a polynomial. We call this polynomial **Hilbert's Polynomial** and denote it $H_J(d)$. This polynomial only takes on integers values, but the polynomial does not necessarily have integer coefficients.

We use this to define the dimension of a variety X: $\dim X = \deg H_{J(X)}$. This also holds for an affine variety: $\dim V = \deg H_{I(V)}$

Proposition 1. *(1) Given homogeneous J, $H_J(d)$ is an invariant (of morphisms of $X(J)$ for instance). So is its constant term and the leading coefficient.*

(2) In the affine setting, $HF_I(d) = HF_{LT(I)}(d)$ for all d. This then holds over to the Hilbert Polynomial case.

(3) If k is algebraically closed and I is an ideal in $k[x_1, \ldots, x_n]$, then $\dim V(I) = \deg H_I(d)$

(4) If $V_1 \subset V_2$ then $\dim V_1 \leq \dim V_2$

(5) $\dim V = 0$ *iff V is a finite point-set*

(6) If W and V are varieties then $\dim(W \cup V) = \max(\dim V, \dim W)$. *This is especially useful for irreducible decompositions.*

The Hilbert polynomial of a graded commutative algebra or graded module is a polynomial in one variable that measures the rate of growth of the dimensions of its homogeneous components. Same invariants as above. The idea for calculating H_M is by approximating M with free modules.

The Hilbert polynomial of a graded commutative algebra $S = \oplus_{d\geq 0} S_d$ over a field k that is generated by the finite dimensional space S_1 is the unique polynomial $H_S(t)$ with rational coefficients such that $H_S(n) = \dim_k S_n$ for all but finitely many positive integers n. Similarly, one can define the Hilbert polynomial H_M of a finitely generated graded module M, at least, when the grading is positive.

Singularities

With dimension well-defined we can talk about singularities. We'll call a variety X **non-singular** at p if $\dim \text{Span}_k\{\bigtriangledown f_1(p), \ldots, \bigtriangledown f_m(p)\} = n - \dim X$

A ring R is **regular** (i.e. non-singular) if for each maximal ideal $m \subset R$ we have $\dim_k m/m^2 = \dim R$ where $k = R/m$. Of course, this all has connections to singularities in affine space and it's another reason why the coordinate ring of a variety is so useful.

4. Syzygies

Definition 2. *If $f_1 \dots, f_r \in k[x_1, \dots, x_n]$ then a **syzygy** is any r-tuple $(h_1, \dots, h_r)$ where $h_i \in k[x_1, \dots, x_n]$ for all i and $h_1 f_1 + \cdots + h_r f_r = 0$. The set of syzygies of $f_1, \dots, f_r$ forms a $k[x_1, \dots, x_n]$-module*

More generally, a syzygy is a relation between the generators of a module M. The set of all such relations is called the "first syzygy module of M". The set of relations between generators of the first syzygy module is called the "second syzygy module of M"...... The syzygy modules of M are not unique, for they depend on the choice of generators at each step.

Syzygies appear when studying invariant theory as a way to define the **ideal of relations** which consists of polynomials which give relations among the polynomials which generate the ring of invariants. Syzygies also pop up in Buchberger's Algorithm because they account for the bad cancellations that mess up the division algorithm. It is a fact that if M is finitely generated over $k[x_1, \dots, x_n]$, this process terminates after a finite number of steps; i.e., eventually there will be no more syzygies. This is why the algorithm terminates. The generalizes version of this statement is Hilbert's Syzygy Theorem, which is really a result of commutative algebra and homological algebra. To understand it we need **Free Resolutions** mentioned by Daniel in his last talk.

Definition 3. *Given a (finite, graded S-)module, M, a free resolution of M is an exact sequence (possibly infinite) of modules*

$$\cdots \to F_n \to \cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

with all the F_i's are free (S-)modules (i.e. they all have a free basis or a linearly independent generating set). The exactness condition means $\ker d_i = \operatorname{im} d_{i+1}$ where $d_i : F_i \to F_{i-1}$

This is most interesting for us when M is a graded module then each d_i is a degree 0 map.

Theorem 2 (Hilbert's syzygy theorem)**.** *Let k be a field and M a finitely generated module over the polynomial ring $k[x_0, \dots, x_n]$. Then there exists a free resolution of M of length at most $n+1$.*

A free resolution is **minimal** if for each i, a basis of F_{i-1} maps onto a minimal set of generators for coker d_i under the quotient map. Minimal free resolutions are heavily studied. It is a fact that they are unique up to isomorphism, and the theorem above says that they always have finite length. Interestingly, Grobner bases generalize to the situation of finite modules defined over polynomial rings and so Buchberger's Algorithm gives a way to compute the minimal free resolution of a finite S-module M.

5. Intersection Theory

Bezout's Theorem

Recall from my last GSS talk:

Theorem 3 (Bezout's Theorem)**.** *Let X and Y are two plane projective curves defined over a field F (i.e. they are in $\mathbb{P}^2_F$) that **do not have a common component**. Then the total number of intersection points of X and Y with coordinates in an algebraically closed field E containing F, counted with their multiplicities, is equal to $\deg(X)\deg(Y)$.*

(Draw a picture of two curves with an overlap and explain that this is what is meant by common component)

Discriminant

The discriminant of a polynomials is an invariant which is equal to zero if and only if the polynomial has a multiple root in its splitting field. All day today let $p(x) = a_n x^n + \cdots + a_0$ and suppose this has roots $r_1, r_2, \ldots, r_k$ in the splitting field. Then the discriminant is $a_n^{2n-2} \prod_{i<j} (r_i - r_j)^2$. This has been generalized far beyond polynomials and turns out to be a useful invariant all over algebraic and analytic number theory. The discriminant of a quadratic $ax^2 + bx + c$ is $b^2 - 4ac$, which is what appears under the radical in the quadratic formula and what determines if the quadratic has two real roots, one real root, or no real roots. The discriminant of a cubic $x^3 + px + q$ is $-4p^3 - 27q^2$ which appears in the cubic equation.

Sylvester Matrix

The resultant is used to find common roots of polynomials. Let $q(x) = b_m x^m + \cdots + b_0$. Then the Sylvester matrix associated to p and q is then the $(m+n) \times (m+n)$ matrix obtained as follows:

The first row is $\begin{pmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \end{pmatrix}$

The second row is the first row, shifted one column to the right; the first element of the row is zero. The following $(m-2)$ rows are obtained the same way, still filling the first column with a zero.

The $(m+1)$-th row is $\begin{pmatrix} b_m & b_{m-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \end{pmatrix}$. The following rows are obtained the same way as before using left shifts.

Thus, if we put $n = 4$ and $m = 3$, the matrix is:

$$S_{p,q} = \begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{pmatrix}$$

This matrix alone is worth studying. For a taste of how interesting it is note that $\deg(\gcd(p,q)) = n + m - \operatorname{rank} S_{p,q}$. But we are interested in the Sylvester Matrix only to get the resultant.

Resultant

We define the resultant of p and q as the determinant of the Sylvester Matrix:

$$res(p,q) = a_n^m b_m^n \left(\prod_{(x,y):\, p(x)=0,\, q(y)=0} (x - y) \right)$$

Obviously computing resultants is not something you want to do by hand or by computer. But having resultants does make your life easier because they have some cool properties:

(1) The resultant is zero iff $p(x)$ and $q(x)$ have a non-constant common factor.

(2) When q is separable (i.e. all of its irreducible factors have distinct roots $\overline{k}$), $res(p,q) = \prod_{p(x)=0} q(x)$

(3) $res(p,q) = (-1)^{nm} \cdot res(q,p)$

(4) $res(p \cdot r, q) = res(p,q) \cdot res(r,q)$

(5) $res(p(-x), q(x)) = res(q(-x), p(x))$

(6) If $f = p + r * q$ and $\deg f = n = \deg p$, then $res(p, q) = res(f, q)$

(7) If f,g,p,q have the same degree and $f = a_{00} \cdot p + a_{01} \cdot q, g = a_{10} \cdot p + a_{11} \cdot q$, then

$$res(f, g) = \det \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}^n \cdot res(p, q)$$

(8) When f and g are curves in $\mathbb{A}^2$ viewed as polynomials in x with coefficients in y (i.e. viewed in $(k[x])[y]$) then $res(f, g)$ is a polynomial in y whose roots are the y-coordinates of the intersection of the curves.

The resultant also gives an alternate way to define the discriminant of p. First, take the derivative p', then find the resultant of p and p'. This entails taking the determinant of the matrix

$$\begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & 0\dots & \dots & 0 \\ 0 & a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & 0\dots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & \dots & 0 & a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \\ na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & 1a_1 & 0 & \dots & \dots & 0 \\ 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & 1a_1 & 0 & \dots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & \dots & 0 & na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \dots & 1a_1 \end{pmatrix}$$

$$\text{We then define } D(p) = (-1)^{\frac{1}{2}n(n-1)} \frac{1}{a_n} R(p, p')$$

Many of these invariants are useful for solving problems.

$p(x) = ax^2 + bx + c$ so $p'(x) = 2ax + b$ and we have $n = 2, m = 1, n + m = 3$. Thus, the Sylvester Matrix is 3×3:

$$\begin{pmatrix} a & b & c \\ 2a & b & 0 \\ 0 & 2a & b \end{pmatrix}$$

The determinant is $ab^2 + 0 + 4a^2c - 0 - 2ab^2 - 0 = 4a^2c - ab^2$. Thus, $D(p) = (-1)^{\frac{1}{2}2(2-1)} \frac{1}{a} R(p, p') = (-1)(\frac{1}{a})(4a^2c - ab^2) = b^2 - 4ac$ as desired.

Grassmannians

Define $G(r, n) = \{r$ dimensional subspaces of $\mathbb{A}^n\}$. Define $\mathbb{G}_r\mathbb{P}^n = \{r$-planes in $\mathbb{P}^n\} = G(r+1, n+1)$.

(1) $G(1, n+1) = \mathbb{P}^n$

(2) $G(2, 3) = \mathbb{G}_1\mathbb{P}^2 = (\mathbb{P}^2)^* \cong \mathbb{P}^2 = \{$lines in 2-space$\}$.

(3) $\mathbb{G}_{n-1}\mathbb{P}^n = \{$hyperplanes in $\mathbb{P}^n\} = (\mathbb{P}^n)^* \cong \mathbb{P}^n$

A line in $G(a, b)$ can be written as an $a \times b$ matrix of basis vector rows. A line L can be represented by an $r \times n$ matrix in $G(r, n)$. This allows us to represent L by MA where M is an $r \times r$ invertible matrix.

In this way $G(r, n)$ can be thought of as an $r(n-r)$-dimensional manifold via $G(r, n) = \{r \times n$ matrices of A of rank $r\}/(A \sim MA)$

Plücker Embedding

We can embed $G(r,n)$ as a projective algebraic set in $\mathbb{P}^N$ where $N = \binom{n}{r} - 1$. The idea here is to represent L by its list of $r \times r$ minor determinants:

(1) Fix a matrix representative for each $L \in G(r,n)$

(2) For column indices $1 \leq j_1 < \cdots < j_r \leq n$ define $L_{j_1,\ldots,j_r}$ as an $r \times r$ submatrix formed by columns $j_1, \ldots, j_r$. Take all possible determinants of these submatrices (this gives $\binom{n}{r}$ determinants)

(3) Each determinant becomes a coordinate in $\mathbb{P}^N$, starting with the zero coordinate. Call this point $\Lambda(L)$

It is a fact (but I won't show it) that choosing a different representative for L only changes $\Lambda(L)$ by a scalar multiple, i.e. this process is well-defined.

For each $I : 1 \leq i_1 < \cdots < i_{r-1} \leq n$ and each $J : 1 \leq j_1 < \cdots < j_{i+1} \leq n$ define

$$P_{I,J} = \sum_{\lambda=1}^{r+1} (-1)^\lambda x(i_1, \ldots, i_{r-1}, j_\lambda) x(j_1, \ldots, \widehat{j_\lambda}, \ldots, j_{r+1})$$

Where $\widehat{j_\lambda}$ means omit j_λ. This thing will be called a Plücker relation and it's a polynomial. The variety associated to the collection of these $P_{I,J}$ is exactly the image of $G(r,n)$ under the Plücker Embedding.

More generally, Schubert Calculus deals with solving various counting problems of projective geometry (part of enumerative geometry). Schubert cells are locally closed sets in a Grassmannian defined by conditions of incidence of a linear subspace in projective space with a given flag $A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_r$. A Schubert variety is a certain subvariety of a Grassmannian, usually with singular points. Intersection theory of these cells answers questions like

"How many lines meet four given lines in $\mathbb{R}^3$?" (answer: consider surface of lines meeting L_1, L_2, L_3. It's a quadric hypersurface and has 2 points of intersection with L_4 corresponding to two lines)

Examples:

The Plücker embedding of $G(1,2)$ is $\mathbb{P}^1$ because $N = \binom{2}{1} - 1 = 2 - 1 = 1$ and because by definition $G(1,2) = \mathbb{G}_0\mathbb{P}^1$ the space of points in $\mathbb{P}^1$. The Plücker embedding of $G(1,n)$ is $\mathbb{P}^{n-1}$ because $N = \binom{n}{1} - 1 = n - 1$ and because by definition this space is all lines through the origin in $\mathbb{A}^n$, i.e. all points in $\mathbb{P}^{n-1}$.

For $G(2,4)$ we have $N = \binom{4}{2} - 1 = 5$. We can label the points $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$ (there are six because $\mathbb{P}^5$ have six coordinates). Then the Plücker relation is $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$ and the image sits in $\mathbb{P}^5$. Moreover, any line $L \in G(2,4)$ can be written as

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

The Plücker embedding gives $(a_0b_1 - a_1b_0, a_0b_2 - a_2b_0, a_0b_3 - a_3b_0, a_1b_2 - a_2b_1, a_1b_3 - a_3b_1, a_2b_3 - a_3b_2)$. The surface is a 4-dimensional manifold.