Localization and Ring Objects in Model Categories

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March 31, 2012

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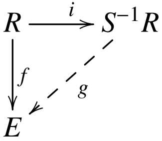

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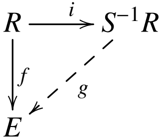

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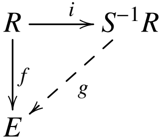

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Inverting s is the same as inverting the map $\mu_s(r) = s \cdot r$

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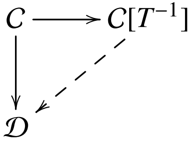

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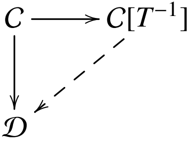

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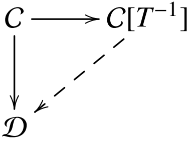

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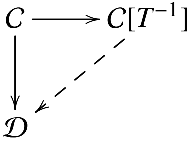

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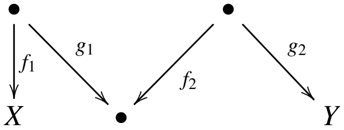

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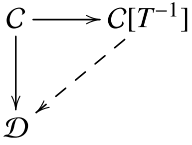

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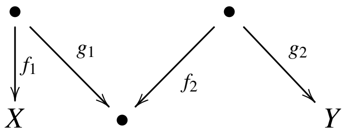

Oops! Zigzags is not a set

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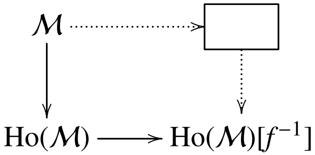

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Note: localizing a set T of maps is the same as localizing $f = \coprod_{g \in T} g$, so it's fine to look at just L_f

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Commutative E also has twist $\tau : E \otimes E \to E \otimes E$.

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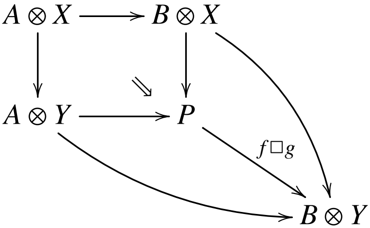

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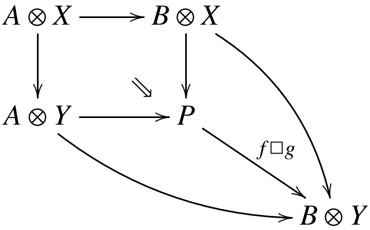

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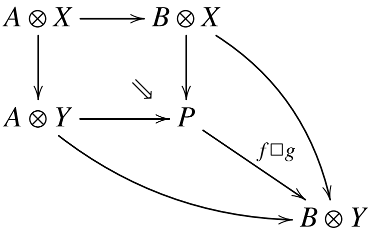

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3. Monoid Axiom: Transfinite compositions of pushouts of maps in {Trivial-Cofibrations $\otimes id_X$} are weak equivalences.

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Localization *preserves strict monoids* if the composition Ho($\mathcal{M}$) $\to$ Ho($L_f\mathcal{M}$) $\to$ Ho($\mathcal{M}$) takes X to a strict monoid

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$L_f\mathcal{M}$ can fail Pushout Product Axiom: $\mathcal{M} = \mathbb{F}_2[\Sigma_3]$-mod and $f : \mathbb{F}_2 \to \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$ taking 1 to $(1, 1, 1)$

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Theorem

Assuming further that $\mathcal{M}$ is weakly finitely generated, that f has SSet-small (co)domain, and a technical condition on $Q \otimes -$, then $L_f \mathcal{M}$ ***satisfies the monoid axiom.***

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If $L_f\mathcal{M}$ is a monoidal model category with CommMon$(L_f\mathcal{M})$ a model category, then L_f preserves strict commutative monoids

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Next: L_f preserves Σ_n-equivariant monoid axiom

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After that: Applying results to examples, especially G-spectra.

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