Dimension Theory of Rings and Ring Spectra

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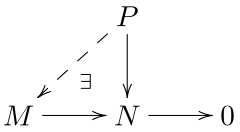
Based on work of Mark Hovey and Keir Lockridge

Dimension Measures Complexity

The simplest rings are fields F. Krull $\dim(F) = 0$.
Krull dim = sup of lengths of chains of prime ideals.

Key property of a field F: all F-modules are free.

Next simplest module after free is projective module P:



R is **semisimple** iff all modules over R are projective

Homological Dimension

Definition (Projective dimension)

A ***projective resolution*** *of* M *is*
$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$
$\text{pd}(M) =$ *min. length of a projective resolution.*

Ex: P projective $\Rightarrow \text{pd}(P) = 0$: $\cdots \to 0 \to P \to P \to 0$

Ex: $\text{pd}(\mathbb{Z}/n) = 1$: $\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$

Definition (Right Global Dimension)

$\text{r.gl.dim}(R) = \sup\{\text{pd}(M) \mid M \in \text{R-mod}\}$

Ex: $\text{r.gl.dim}(k[x_1, \ldots, x_n]) = n$. $\text{r.gl.dim}(k[t]/(t^2)) = \infty$

$\cdots \to k[t]/(t^2) \to k[t]/(t^2) \to k \to 0$, so $\text{pd}(k) = \infty$

Spectra

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Definition (Spectrum)

A ***spectrum*** *X is a sequence (X_i) of topological spaces (path conn. CW-complexes) with maps from $\epsilon_i : \Sigma X_i \to X_{i+1}$ where Σ is reduced suspension. $(\Sigma X)_i = X_{i+1}$*

A morphism is $f = (f_i : X_i \to Y_i)$ with

$f \circ \epsilon_i^X = \epsilon_i^Y \circ (\Sigma f) : \Sigma X_i \to Y_{i+1}$

Example: For any space X, $Z = \Sigma^\infty X$ is the spectrum with $Z_i = \Sigma^i X$, and ϵ_i homeomorphism for all i

Example: the sphere spectrum $S = (S^n) = \Sigma^\infty S^0$.

Rings via Categorical Lens

Associativity:

$$\begin{array}{ccc} R \times R \times R & \xrightarrow{\mu \times 1_R} & R \times R \\ \downarrow{\scriptstyle 1_R \times \mu} & & \downarrow{\scriptstyle \mu} \\ R \times R & \xrightarrow{\mu} & R \end{array} \qquad \begin{array}{ccc} (a,b,c) & \overset{\mu}{\longmapsto} & (ab,c) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ (a,bc) & \overset{\mu}{\longmapsto} & abc \end{array}$$

e is a left and right identity:

$$\begin{array}{ccccc} \{1\} \times R & \xhookrightarrow{u \times 1} & R \times R & \xhookleftarrow{1 \times u} & R \times \{1\} \\ & {\scriptstyle proj} \searrow & \downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle proj} & \\ & & R & & \end{array}$$

R-module M has $R \times M \to M$. $(r_1 r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$

Ring Spectra

Definition (Ring Spectrum)

*A **ring spectrum** E is a generalized cohomology theory with a cup product that is associative up to infinitely coherent homotopy. E comes with $\wedge : E \times E \to E$ and $u : S \to E$.*

$$\begin{array}{ccc} E \times E \times E & \longrightarrow & E \times E \\ \downarrow & & \downarrow \\ E \times E & \longrightarrow & E \end{array} \qquad \begin{array}{ccccc} S \times E & \longrightarrow & E \times E & \longleftarrow & E \times S \\ & \searrow & \downarrow & \swarrow & \\ & & E & & \end{array}$$

E is an S-module because we have $S \wedge E \to E \wedge E \to E$

$E_* = \pi_*(E) = [\Sigma^* S, E]$. Functor π_* :RingSpectra$\rightarrow$GrRing

The Derived Category

An E-module is a spectrum X with $E \wedge X \to X$

$\mathcal{D}(E)$ objects are E-modules, $\mathcal{D}(E)(X, Y) = \{X, Y\}[S^{-1}]$ for S the collection of weak homotopy equivalences

Definition (Projective E-module)

$X \in \mathcal{D}(E)$ is ***projective*** *iff X_* is a projective E_*-module. Define* $\text{pd}(X) = 0$.

Definition (Projective Dimension)

$\text{pd}(X) \leq n+1$ *iff* $Y \to P \to \tilde{X} \to \Sigma Y$ *with P projective,* $\text{pd}(Y) \leq n$, *and X a retract of $\tilde{X}$.*

Dimensions of Ring Spectra

Definition

$\mathrm{r.gl.dim}(E) = \sup\{\mathrm{pd}(X) \mid X \in \mathcal{D}(E)\}$

Example: Singular cohomology theory $H^n(-)$ is a spectrum. $H^n(X;R) \cong [X, K(R,n)]$ where $\pi_m(K(R,n)) = R$ iff $m = n$

HR has $(HR)_n = K(R,n)$ and $(HR)_* = [S, HR] \cong R$

$\mathrm{r.gl.dim}(HR) = \mathrm{r.gl.dim}(R)$ because $\mathcal{D}(HR) \cong \mathcal{D}(R)$

Always true: $\mathrm{r.gl.dim}(E) \leq \mathrm{r.gl.dim}(E_*)$

Theorem (Hovey-Lockridge)

If E is a commutative ring spectrum and E_* is Noetherian with $\mathrm{gl.dim}(E_*) < \infty$ then $\mathrm{r.gl.dim}(E) = \mathrm{r.gl.dim}(E_*)$

The Sphere Spectrum

Definition (Ghost)

*A map $f : X \to Y$ in $\mathcal{D}(E)$ is **ghost** if $f_* = 0$*

Proposition

1. *$X \in \mathcal{D}(E)$ is projective iff the natural map $\mathcal{D}(E)(X, Y) \to \mathrm{Hom}_{E_*}(X_*, Y_*)$ is iso for all Y*
2. *$\mathrm{pd}(X) \leq n$ iff $E_2^{s,t} = \mathrm{Ext}_{E_*}^{s,t}(X_*, Y_*) \Rightarrow \mathcal{D}(E)(X, Y)_{t-s}$ has $E_\infty^{s,*} = 0 \;\forall\; s > n$ iff any chain of ghosts with $\mathrm{Dom}(f_1) = X$ has $f_{n+1} \circ \cdots \circ f_1 = 0$*

Corollary

$\mathrm{r.gl.dim}(S) = \infty$

Pf sketch: Consider the S-module $Z = \Sigma^\infty(\mathbb{R}P^n)$. The Steenrod operations are ghosts.

References

1. Homological Dimensions of Ring Spectra, preprint 2010 by M. Hovey and K. Lockridge
2. Lectures on Modules and Rings, 1999 by T.Y. Lam
3. Ideals in Triangulated Categories: Phantoms, Ghosts and Skeleta, 1998 by Christiensen