LECTURE NOTES

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1. Motivation for Spectra

It is VERY hard to compute homotopy groups. We want to put as much algebraic structure as possible in order to make computation easier. We want to study the ring-like objects that arise in this category. "Ring-like" means ring-object, i.e. using the lens of category theory. They have no points, so you can't do traditional algebra. To measure complexity of these we'll use dimension.

2. Algebraic

The simplest rings are fields, which clearly have Krull dim zero because no ideals. Dimension is telling us about complexity. For us, Krull dim fails because no points or ideals.

Note: Krull dim is the max length of a chain of prime ideals $P_0 \subset P_1 \subset \dots$. Zero ideal not prime.

Key property of fields: all modules over F are free. Next simplest module after free is projective (they are direct summands of free)

We say module P is **projective** if:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

A **projective resolution** of M is $\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$, with all the P_i's projective.

Definition 1. ***Projective dimension*** $= pd(M) =$ *min. length of a projective resolution.*

Ex: If P is projective, $\mathrm{pd}(P) = 0$ since $\cdots \to 0 \to 0 \to P \to P \to 0$ is a projective resolution.

Ex: For $R = \mathbb{Z}$, $\mathrm{pd}(\mathbb{Z}/n) = 1$ since $\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ is minimal projective resolution, where the first map is mult by n and the second is quotient. THIS SHOWS $\mathrm{pd}(\mathbb{Z}/n) \leq 1$. TO SHOW IT'S NOT 0, NOTE THAT IT CAN'T BE A SUMMAND OF A FREE MODULE.

Definition 2 (Right Global Dimension)**.** *r.gl.dim(R)* $= \sup\{pd(M) \mid M \in R-mod\}$

Ex: $\text{r.gl.dim}(k[x_1, \dots, x_n]) = n$ because of the module $(x_1, \dots, x_n)$

Ex: $\text{r.gl.dim}(k[x]/(x^2)) = \infty$ because k is an R-module and the minimal projective resolution is an infinite chain $\cdots \to k[x]/(x^2) \to k[x]/(x^2) \to k \to 0$, where each map takes $x \to 0$ and $1 \to x$.

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r.gl.dim$(R) = 1$ implies submodules of projective modules are projective. This is the next simplest ring after a semisimple ring. Ex: all PIDs.

3. Definitions

Definition 3 (Spectrum)**.** *A spectrum X is a sequence (X_i) of topological spaces (path conn. CW-complexes) with maps from $\Sigma X_i \to X_{i+1}$ where Σ is reduced suspension. ΣX is shift.*

Example: For any space X, $Z = \Sigma^\infty X$ is the spectrum with $Z_i = \Sigma^i X$, and ϵ_i homeomorphism for all i. So we recover SPACES INSIDE RING SPECTRA

Example: the sphere spectrum $S = (S^n) = \Sigma^\infty S^0$. NOTE: We've erased dimension, but we have no points.

$$\begin{tikzcd} \Sigma X_i \arrow[r] \arrow[d] & X_{i+1} \arrow[d] \\ \Sigma Y_i \arrow[r] & Y_{i+1} \end{tikzcd} \qquad \begin{tikzcd} S \wedge X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[d] \\ S \wedge Y_n \arrow[r] & Y_{n+1} \end{tikzcd}$$

i.e. maps in the category f must play nicely with ϵ_n and $S \wedge -$.

$$\begin{tikzcd} R \times R \times R \arrow[r, "\mu \times 1_R"'] \arrow[d, "1_R \times \mu"] & R \times R \arrow[d, "\mu"] \\ R \times R \arrow[r, "\mu"] & R \end{tikzcd} \qquad \begin{tikzcd} (a,b,c) \arrow[r, mapsto, "\mu"] \arrow[d, mapsto, "\mu"] & (ab,c) \arrow[d, mapsto, "\mu"] \\ (a,bc) \arrow[r, mapsto, "\mu"] & abc \end{tikzcd}$$

$$\begin{tikzcd} \{e\} \times R \arrow[r, hook, "u \times 1"'] \arrow[dr, "proj"'] & R \times R \arrow[d, "\mu"] & R \times \{e\} \arrow[l, hook', "1 \times u"] \arrow[dl, "proj"] \\ & R & \end{tikzcd}$$

Definition 4 (Ring Spectrum)**.** *A **ring spectrum** E is a generalized cohomology theory with a cup product that is associative up to infinitely coherent homotopy. E comes with $\wedge : E \times E \to E$ and $u : S \to E$.*

$$\begin{tikzcd} E \times E \times E \arrow[r, "\wedge \times 1"'] \arrow[d, "1 \times \wedge"] & E \times E \arrow[d, "\wedge"] \\ E \times E \arrow[r, "\wedge"] & E \end{tikzcd} \qquad \begin{tikzcd} S \times E \arrow[r, "u \times 1"'] \arrow[dr, "proj"'] & E \times E \arrow[d, "\wedge"] & E \times S \arrow[l, "1 \times u"] \arrow[dl, "proj"] \\ & E & \end{tikzcd}$$

An S-algebra E is an S-module because we have $S \wedge E \to E$. In particular, $S^i \wedge (S^j \wedge E) \cong (S^i \wedge S^j) \wedge E \cong S^{i+j} \wedge E$.

KRULL DIM FAILS HERE BECAUSE NO POINTS. SO NEED HOMOLOGICAL DIM.

An E-module X has $E \wedge X \to X$ satisfying the usual action rule.

4. Derived Category

The correct category to study modules over an S-algebra E is $\mathcal{D}(E)$. Objects are E-modules, and $\mathcal{D}(E)(X,Y) = \{X,Y\}[S^{-1}]$ for S the collection of weak homotopy equivalences

CORRECT CATEGORY because triangulated. It's also compactly generated and has derived tensor products and derived Hom objects.

$X \in \mathcal{D}(E)$ is **projective** iff X_* is a projective E_*-module. Define $\mathrm{pd}(X) = 1$. Projective E_*-modules are realizable.

Definition 5. $\mathrm{pd}(X) \leq n+1$ *iff* $Y \to P \to \widetilde{X} \to \Sigma Y$ *with* P *projective,* $\mathrm{pd}(Y) \leq n$*, and* X *a retract of* $\widetilde{X}$*.*

YOU PROBABLY WANT AN EXAMPLE HERE, DON'T YOU? SO DO I! I'D LIKE A NON-TRIVIAL EXAMPLE WITH $\mathrm{pd}(X) = 1$, even.

5. Dimensions of Ring Spectra

Definition 6. *$pd(X) \leq n+1$ iff $Y \to P \to \widetilde{X} \to \Sigma Y$ with P projective, $pd(Y) \leq n$, X a retract of $\widetilde{X}$*

Definition 7. *$r.gl.dim(E) = \sup\{pd(X) \mid X \in \mathcal{D}(E)\}$*

Example: Singular cohomology theory $H^n(-)$ is a spectrum. $H^n(X;\Lambda) \cong [X, K(\Lambda,n)]$

$$\pi_m(K(\Lambda,n)) = \begin{cases} \Lambda & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

For all R, the Eilenberg-MacLane spectrum HR has $(HR)_n = K(R,n)$. Well-known: $K(R,n-1) \xrightarrow{\simeq} \Omega K(R,n)$. This gives $\Sigma K(R,n-1) \to K(R,n)$.

Example: Any cohomology theory is a ring spectrum, e.g. $H\mathbb{Q}$.

$(HR)_* = [S, HR] \cong R$, so we recover RINGS INSIDE RING SPECTRA

For RINGS R with identity e and mult μ:

$\mathrm{r.gl.dim}$ generalizes r.gl.dim of a ring in that $\mathrm{r.gl.dim}(HR) = \mathrm{r.gl.dim}(R)$. We have a categorical equivalence: $\mathcal{D}(HR) \cong \mathcal{D}(R)$

$\mathrm{r.gl.dim}(E) \leq \mathrm{r.gl.dim}(E_*)$ and equal in nice cases

Theorem 1. *If E is a commutative S-algebra and E_* is Noetherian with $gl.dim(E_*) < \infty$ then $gh.dim(E)$ =$r.gl.dim(E) = r.gl.dim(E_*)$*

Proof: depth = gl.dim so the chain of inequalities collapses to equalities. Let $R = E_*$. Then $\mathrm{gl.dim}(R) < \infty \Rightarrow R$ is regular (all localizations $R_{\mathcal{P}}$ are regular local rings). For regular rings, Krull dim = depth (Regular $\Rightarrow$ Cohen-Macaulay) and Krull dim = gl.dim

We need to have the Noetherian condition on E_* because without IDEALS we have no definition for E to be Noetherian.

E_* semisimple $\Rightarrow E$ semisimple. Converse fails because of a DGA with homology $k[x]/(x^2)$. DGA is equiv to an $H\mathbb{Z}$-algebra and these are types of spectra. $HoDGA \leftrightarrow \mathcal{D}(H\mathbb{Z})$

6. Sphere Spectrum

Proposition 1. *$X \in \mathcal{D}(E)$ is projective iff the natural map $\mathcal{D}(E)(X,Y) \to \mathrm{Hom}_{E_*}(X_*, Y_*)$ is iso for all Y*

We use this in practice all the time, especially to show when ghosts are null.

Proposition 2. *$pd(X) \leq n$ iff every composite of $n+1$ ghosts $f_{n+1} \circ \cdots \circ f_1$ is null where $\mathrm{Dom}(f_1) = X$. This holds iff $E_2^{s,t} = \mathrm{Ext}_{E_*}^{s,t}(X_*, Y_*) \Rightarrow \mathcal{D}(E)(X,Y)_{t-s}$ has $E_\infty^{s,*} = 0 \; \forall \; s > n$*

Here we have algebra on the E_2 term converging to topology on the E_∞ term.

EXAMPLE: $\mathrm{gh.dim}(S) = \infty$. Suppose it's $n < \infty$. Then you need an S-module X with $\mathrm{pd}(X) \geq n+1$, i.e. find a chain of n ghosts out of X which is non-null. Any spectrum is an S-module. Turns out you can take $X = \Sigma^\infty \mathbb{R}P^k$ for large k and use the Steenrod Squares, which are well-studied maps that turn out to be ghost.