LOCALIZATION AND RING OBJECTS IN MODEL CATEGORIES

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THANK THE ORGANIZERS, comment about Mike Hill's talk motivating this work

1. Localization in algebra

Localization is a **systematic way of adding multiplicative inverses to a ring**, i.e. given commutative R and multiplicative $S \subset R$ (contains 1, closed under $*$, doesn't contain 0), localization constructs $S^{-1}R$ and a ring homomorphism $j : R \to S^{-1}R$ that takes elements in S to units in $S^{-1}R$. It's universal w.r.t. this property. Recall that $S^{-1}R$ is just $R \times S/\sim$ where (r, s) is really r/s and $\sim$ says you can reduce to lowest terms without leaving the equivalence class. Ring just as $\mathbb{Q}$ is. The map j takes $r \mapsto r/1$, and given f you can set $g(r/s) = f(r)f(s)^{-1}$.

Examples: $(\mathbb{Z}, \langle 2 \rangle) \mapsto \mathbb{Z}[\frac{1}{2}]$. $(\mathbb{Z}, \mathbb{Z} - p\mathbb{Z}) \mapsto \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \not| b\}$

This is NOT the right definition to a category theorist (no operation, so what's a "multiplicative inverses?"). Better: **systematic way of formally inverting maps**. We can't do this for all maps, but we can do it for maps of the form $\mu_s : R \to R$ which take $r \mapsto s \cdot r$ for an element s, and that's equivalent to inverting the element s.

2. Localization of Categories

Thinking of localization as "formally inverting maps" then we want to pick a set T of morphisms and create a universal functor $\mathcal{C} \to \mathcal{C}[T^{-1}]$ where those morphisms land in the class of isomorphisms, i.e. $F(f)$ is an iso for all $f \in T$. Universal means we add the smallest number of new morphisms possible (i.e. just those generated by T^{-1} via composition).

Example: If $\mathcal{C}$ is Top and $T = \{\text{homotopy equivalences}\}$, then $\mathcal{C}[T^{-1}]$ is the homotopy category HoTop.

To do this in general, note that given $f : X \to Y$ in T and $g : X \to Z$, we get $g \circ f^{-1} : Y \to Z$, i.e. we have to generate new morphisms based on the inverses I added. So what are the morphisms $\mathcal{C}[T^{-1}]$ between X and Y? You can get there by any **zig-zag**, so you want to define $\mathcal{C}[T^{-1}](X, Y) = \{X \leftarrow \bullet \to \bullet \cdots \bullet \to Y\}/\sim$ where this relation at least allows us to add in pairs of identities or compose two when it's allowed. PROBLEM: the collection of zigzags $X \leftarrow \bullet \to \bullet \cdots \bullet \to Y$ is not a set, so you can't mod out by an equivalence relation. Even in the category Set we can pick any set to be $\bullet$, and there's more than a set worth of choices there.

We've faced this problem before. HoTop is **not concretizable** because you can have a class of morphisms between two objects. So to get around our problem here, we need to make C look more like Top and make T look like homotopy equivalences.

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3. Model Categories

To get around this you are **forced into homotopy theory** again. You need restrictions on the types of T you can invert. It worked for Top, so let's generalize the properties we had there. This leads to the concept of a Model Category (Quillen 1967). The idea is you have a special class of maps $\mathcal{W}$ called the weak equivalences which satisfy some rules and generalize the homotopy equivalences above. But **algebraic topology is about more than just homotopy equivalences**.

For instance, we care about vector bundles $E \to X$ (where the fibers are vector spaces) and more general fiber bundles $F \to E \to X$. For these, the map $E \to X$ is a **fibration**. For example, $O(n) \to O(n)/O(n-1)$. More generally, the quotient of any two Lie groups.

Another thing topology studies is when one space X can be built from another A by adjoining cells. We use this for example to write $H_n(X, A) \cong H_n(X/A)$. Call such a map $A \to X$ a **cofibration**.

Quillen's brilliant idea was to focus on just these three types of maps, pick out their most important properties, and use these properties to make a definition. A **Model Category** is a category $\mathcal{M}$ with distinguished classes $\mathcal{W}, \mathcal{F}, \mathcal{Q}$ satisfying those properties. The localization described above for spaces works on any model category, i.e. you get a concrete way to make a <u>universal</u> functor $\mathcal{M} \to \operatorname{Ho}\mathcal{M}$ taking $\mathcal{W}$ to isomorphisms. So functors $\mathcal{M} \to \mathcal{C}$ which do this induce functors $\operatorname{Ho}\mathcal{M} \to \mathcal{C}$

Model categories let you apply ideas of homotopy theory much more generally. This transforms algebraic topology from the study of topological spaces into a general tool **useful in many areas of mathematics**. If you saw **Krzystof Kapulkin's talk**, then you should know that model categories were lurking in the background. An object X is fibrant if the map $X \to *$ is a fibration. Similarly, in Mike Hill's talk the categories of spectra and equivariant spectra have model category structures.

This viewpoint let's you do homotopy theory in algebraic geometry, e.g. on the category of Schemes. Voevodsky won a fields medal in 2002 by creating the **motivic stable homotopy category** from a model category structure on an enlargement of Schemes to resolve the **Milnor Conjecture**. The **∞-categories** of Joyal, much studied by Jacob Lurie, are a way to study categories of categories. Homotopy plays a motivating role: model categories are one way to think about $(\infty, 1)$-categories, and they are very helpful for doing computations and constructions. Results in model categories are prized because they suggest things which should be true for (∞, n)-categories.

4. Bousfield Localization

The localization above always lands in a homotopy category and always takes exactly the zig-zags of weak equivalences to isomorphisms. What if we want to invert some map which is not a weak equivalence? Let f be a map in $\mathcal{M}$. Because the homotopy category is nice (admits a calculus of fractions), we can do it. **Bousfield's Theorem** (1978) says you can do this and you still get a model category structure, but it's technical and you have to be careful with how you generate $\mathcal{W}'$ from f. The category $L_f\mathcal{M}$ is called the **Bousfield Localization of $\mathcal{M}$ with respect to f**. You might think we've lost something by just considering one map f, but we haven't. If you have a set of maps T you wish to localize, it's enough to localize $f = \coprod_{\alpha \in T} \alpha$.

5. What does L preserve?

The localization functor L_f preserves the properties of being: Left Proper, Combinatorial (sometimes), Cellular , Enriched, Simplicial, Almost finitely generated (sometimes)

Last year I gave a talk on ring objects in the category of symmetric spectra. My advisor, Mark Hovey, wrote one of two standard references on Model Categories. It should not come as a surprise then that I'm thinking about ring objects in monoidal model categories. One question I want to answer is when L_f preserves ring objects and commutative ring objects.

Mike Hill has an example in G-spectra which shows that strict commutative ring objects need not be preserved by Bousfield localization. He also has a **theorem** giving some conditions on a localization functor which force it to preserve strict commutative ring objects.

6. Monoidal Model Categories and Commutative Ring Objects

In a category with a **product bifunctor** $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative (i.e. $\otimes \circ (\otimes \times 1) \cong \otimes \circ (1 \times \otimes)$) and has a unit object S (i.e. $S \otimes E \cong E \cong E \otimes S$), we can discuss **objects** E **which act like rings,** i.e. have maps $\mu : E \times E \to E$ and $\eta : S \to E$ satisfying some diagrams, e.g. **associativity and unit**. A **commutative ring object** is one which also has a **twist map** $\tau : E \otimes E \to E \otimes E$ and more diagrams. Note that every ring R has a map $\mathbb{Z} \to R$ and every function $f : X \to X$ has a map from the identity to that function by applying 1 on domain and f on codomain.

If you only care about diagrams commuting up to infinitely coherent homotopy, then the answer has been known for a long time: **localization preserves A_∞ and E_∞ ring objects**. Formally, going from $[E] \in \operatorname{Ho} \mathcal{M}$ to $[E'] \in \operatorname{Ho} L_f \mathcal{M}$ and then to $[E''] \in \operatorname{Ho} \mathcal{M}$ you'll have a representing object for $[E'']$ which is A_∞ or E_∞ as $[E]$ is.

strict ring object , ***strict commutative ring object***

7. Preservation of Monoidal Model Categories

Standard hypotheses: **(cocomplete, cofibrantly generated, left proper, almost finitely generated, can choose domains and codomains of generating (trivial) cofibrations to be cofibrant)** and also when one is in a monoidal situation (pushout product axiom, unit axiom, monoid axiom, cofibrant objects flat). To work in this area and get results with homotopy theoretic meaning, we need

(1) & (2) $\Rightarrow$ Ho($\mathcal{M}$) is monoidal ($\otimes$ is a Quillen bifunctor)
(3) implies the monoids Mon($\mathcal{M}$) form a model category.

Theorem 1. *If $L_f \mathcal{M}$ satisfies (1)-(3) then L_f preserves strict monoids*

I found that even under all these hypotheses, $L_f \mathcal{M}$ could fail to be a monoidal model category, though it is always a model category and monoidal. The coherence fails if $\mathcal{M} = \mathbb{F}_2[\Sigma_3]$-mod and $f : \mathbb{F}_2 \to \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2$ taking 1 to $(1, 1, 1)$

If we place an assumption on the map f to be inverted (just like Quillen had to do), we can get around this.

Theorem 1. *Suppose M is a cofibrantly generated, left proper, monoidal model category in which cofibrant objects are flat. Let I and J denote the generating cofibrations and generating trivial cofibrations respectively. Suppose that the domains of maps in $I \cup J$ are cofibrant. If f* ***is a cofibration such that $f \otimes K$ is an f-local equivalence as K runs through the domains and codomains of maps in $I \cup J$,*** *then the Bousfield localization $L_f M$ is a cofibrantly generated, left proper, monoidal model category in which cofibrant objects are flat. Furthermore, the domains of the generating cofibrations and generating trivial cofibrations in M are cofibrant.*

Similarly, the monoid axiom can fail, so we add a hypothesis about how the cofibrations behave (which makes $\mathcal{M}$ a little bit more like Top):

Definition 1. *A homotopical cofibration is a map $g : A \to B$ such that every pushout square with g at the top (i.e. g pushed out by some map $A \to W$) is a homotopy cofiber square, i.e. the map from $Z' \to Z$ is a weak equivalence in the following diagram:*

$$\begin{tikzcd}
QA \arrow[rr, hook] \arrow[dr] \arrow[dd] & & QB \arrow[dr] \arrow[dd] & \\
& A \arrow[rr, "g"] \arrow[dd] & & B \arrow[dd] \\
QW \arrow[rr] \arrow[dr] & & Z' \arrow[dr, dotted] & \\
& W \arrow[rr] & & Z
\end{tikzcd}$$

Theorem 2. *Suppose $\mathcal{M}$ is a cofibrantly generated, left proper, weakly finitely generated, monoidal model category in which cofibrant objects are flat. Suppose* ***"cofibrations $\otimes X \subset$ homotopical cofibrations for any X"****. Suppose that f* ***has SSet-small domain and codomain****. Then $L_f\mathcal{M}$ satisfies the monoid axiom.*

This gives a model category structure on $\mathrm{Mon}(\mathcal{M})$ and $\mathrm{Mon}(L_f\mathcal{M})$. We can thus ask the question of when $L_T(E)$ is a (commutative) ring object.

Theorem 3. *Under the hypotheses from the two theorems above, L_f preserves ring objects (of course, this really means L_f then U_f going back to $\mathcal{M}$)*

8. Commutative Monoids

Theorem 2. *If $L_f\mathcal{M}$ is a monoidal model category with CommMon$(L_f\mathcal{M})$ a model category, then L_f preserves strict commutative monoids*

We are trying to find a model category structure on $\mathrm{CommMon}(\mathcal{M})$ and on $\mathrm{CommMon}(L_f\mathcal{M})$. If we can get this, then the proof above will generalize and tell us that L_f preserves commutative ring objects. **Σ_n-equivariant monoid axiom:** Transfinite compositions of pushouts of maps $J^{\square n} \otimes_{\Sigma_n} X \subset \mathcal{W}$ for all X

This implies pushouts of maps $\overline{Q}_n \wedge X \to L^{\wedge n} \wedge X$ are trivial cofibrations for all X. Some debate with Hovey. Might need pushouts of cubes with X's interspersed, e.g. $X \wedge K \wedge X \wedge L \cdots \wedge X$. This would require a slightly different Σ_n-equivariant monoid axiom.

Next: L_f **preserves Σ_n-equivariant monoid axiom**. Then: Generalize to **coloured operads**. After that: Applying results to examples, especially **G-spectra.**